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DEPARTMENT OF MATHEMATICS
U. S. NAVAL ACADEMY
Annapolis, MD

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Topological games and products I

by

Yukinobu Yajima (Yokohama)

Abstract. Our main purpose in this paper is to show the following result: If a paracompact space X has a σ-closure-preserving cover by compact sets and Y is a paracompact space, then the inequality $\dim X\times Y\leqslant \dim X+\dim Y$ holds. As a matter of fact, we shall prove it in the more generalized form. The main tool of its proof is the topological game (in the sense of R. Telgársky).

§ 1. Introduction. R. Telgársky [12] introduced and studied the concept of topological game $G(K, X)$. Moreover, making use of it, he showed that the topological product of paracompact spaces one of which has a σ-closure-preserving cover by compact sets is paracompact (cf. [12, Theorem 14.7]). In fact, he obtained this result by proving the form which is generalized in terms of topological game (cf. [12, Theorem 14.6]). In § 2, we prove our main theorem. It is another generalization of the above result. Besides, the product inequality of covering dimension simultaneously holds in it, which is given by proving that a locally finite open cover of product space has a locally finite refinement by cozero rectangles. Here, the product space with this topological property is named to be strongly rectangular. In § 3, we apply the technique used in the proof of the above theorem to the product of Hurewicz spaces. In § 4, furthermore, we investigate what kind of a topological product is strongly rectangular. In § 5, we state several questions unanswered.

Throughout this paper, each space is assumed to be a Hausdorff space. However, for a topological product $X\times Y$, we shall mainly discuss in the case $X\times Y$ is normal and we assume either X or Y is non-empty. Non-negative integers are denoted by the letters i, j, k, m, n etc, and μ denotes an infinite cardinal number.

The descriptions and the details of the topological game $G(K, X)$ are found in [12]. Let us note that a sequence $(E_n\colon n\geqslant 0)$ of closed subsets of X is a play for $G(K, X)$ if and only if each finite subsequence $(E_0, \ldots, E_n)$ of it is admissible for $G(K, X)$. In particular, we consider as K, in this paper, the following two classes of spaces:

DC — the class of all spaces which can be decomposed into a discrete collection by compact sets.

DC_μ — the class of all spaces which can be decomposed into a discrete collection by μ-compact sets [1].

The topological game $G(DC, X)$ $(G(DC_\mu, X))$ is closely related to a space which has a σ-closure-preserving closed cover by $(\mu$-$)$compact sets. These facts are known from the following results.

PROPOSITION 1.1. *If a space X has a σ-closure-preserving cover by compact sets, then Player I has a winning strategy in $G(DC, X)$.*

PROPOSITION 1.2. *If a space X has a σ-closure-preserving closed cover by μ-compact sets, then Player I has a winning strategy in $G(DC_\mu, X)$.*

Proposition 1.1 is given in [12, Corollary 10.2]. However, this result is essentialy due to H. B. Potoczny [7]. Proposition 1.2 follows from [12, Theorem 4.7] and [13, Lemma 5].

A subset of a topological product $X\times Y$ of the form $A\times B$ is called a *rectangle.* For a rectangle E in $X\times Y$, E' and E'' denote the projections of E into X and Y respectively. So we have $E = E'\times E''$. A rectangle E in $X\times Y$ is said to be a *cozero* (*zero*, *open* and *closed*) *rectangle* if E' and E'' are cozero (zero, open and closed, respectively) in X and Y respectively.

§ 2. Strongly rectangular products.

At first, we state the following definition.

DEFINITION. A topological product $X\times Y$ is said to be *strongly rectangular* if each locally finite open cover of $X\times Y$ has a locally finite refinement by cozero rectangles.

PROPOSITION 2.1. *The following conditions are equivalent.*

(a) *$X\times Y$ is strongly rectangular.*

(b) *Each finite open cover of $X\times Y$ has a locally finite refinement by cozero rectangles.*

(c) *For each closed subset F and each open set U of $X\times Y$ with $F\subset U$, there is a locally finite collection $\mathfrak{A}$ by cozero rectangles such that $F\subset \bigcup\mathfrak{A}\subset U$.*

(d) *$X\times Y$ is normal and for each zero-set F and each cozero-set U of $X\times Y$ with $F\subset U$ there is a locally finite collection $\mathfrak{A}$ by cozero rectangles such that $F\subset\bigcup\mathfrak{A}\subset U$.*

(e) *For each pair F_0, F_1 of disjoint closed subsets in $X\times Y$, there is a continuous function $f\colon X\times Y\to[0, 1]$ such that $f(F_0) = 0$, $f(F_1) = 1$ and*

$$f(x, y) = \sum_{t\in T} g_t(x)h_t(y),$$

where $g_t\colon X\to[0, 1]$ and $h_t\colon Y\to[0, 1]$ are continuous and the sum is locally finite.

This proposition has been pointed out by R. Telgársky. Its proof is left to the reader.

Our main theorem is as follows, where let χ be the cardinal function denoting character.

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[1] A space is said to be *μ-compact* if each its open cover of power $\leqslant\mu$ has a finite subcover.

THEOREM 2.1. *Let X be a collectionwise normal space and Player I has a winning strategy in $G(DC_\mu, X)$ and let Y be a paracompact space with $\chi(Y)\leqslant\mu$. Then the product space $X\times Y$ is collectionwise normal, μ-paracompact* [2] *and strongly rectangular.*

In order to prove Theorem 2.1, we need the following two lemmas.

LEMMA 2.1. *Let F be a μ-compact closed subset of a normal space X and let Y be a completely regular space with $\chi(Y)\leqslant\mu$ and $y\in Y$. If $\mathfrak{G}$ is a finite collection of open sets in $X\times Y$ such that $\mathfrak{G}$ coverse $F\times\{y\}$, then there exists a finite collection $\mathfrak{U} = \{U_i\times V\colon i = 1, \ldots, k\}$ by cozero rectangles in $X\times Y$ such that $F\subset\bigcup\{U_i\colon i\leqslant k\}$, $y\in V$ and $\mathfrak{U}$ refines $\mathfrak{G}$.*

LEMMA 2.2. *Let $\{Z_\alpha\colon \alpha\in\Omega\}$ be a locally finite collection by zero-sets in a space X such that there is a locally finite collection $\{H_\alpha\colon \alpha\in\Omega\}$ by cozero-sets in X with $Z_\alpha\subset H_\alpha$ for each $\alpha\in\Omega$. Then $\bigcup\{Z_\alpha\colon \alpha\in\Omega\}$ is a zero-set in X.*

Lemma 2.1 is well-known and easy to prove. Lemma 2.2 is given in [4].

Proof of Theorem 2.1. Let s be a winning strategy of Player I in $G(DC_\mu, X)$. Let $\mathfrak{G}$ be any open cover of $X\times Y$, satisfying the following condition;

(1.0) for each μ-compact closed subset F in $X\times Y$, $\mathfrak{G}$ contains a finite subcollection $\tilde{\mathfrak{G}}$ which covers F.

First, we shall construct a sequence $\{\mathfrak{U}_n\colon n\geqslant 0\}$ of collections by cozero rectangles, an inverse system $\{\langle\mathfrak{Z}_n, \varphi^n_m\rangle\colon n\geqslant 0, m\leqslant n\}$ of a sequence of collections $\mathfrak{Z}_n$ by zero rectangles with the bonding maps φ^n_m of $\mathfrak{Z}_n$ into $\mathfrak{Z}_m$, a sequence $\{\mathfrak{H}_n\colon n\geqslant 0\}$ of collections by cozero rectangles and a countable collection $\{V(n, m)\colon n, m\geqslant 0\}$ of cozero-sets in $X\times Y$, where $\mathfrak{U}_0 = \{\emptyset\}$, $\mathfrak{Z}_0 = \mathfrak{H}_0 = \{X\times Y\}$ and $V(0, m) = X\times Y$ for each $m\geqslant 0$, satisfying the following conditions (1.1_n)–(1.11_n) for each $n\geqslant 1$:

(1.1_n) $\mathfrak{U}_n$ is locally finite in $X\times Y$.

(1.2_n) $\mathfrak{Z}_n$ is locally finite in $X\times Y$.

(1.3_n) Each $U\times V\in\mathfrak{U}_n$ is contained in some $G\in\mathfrak{G}$.

(1.4_n) $Z\subset\varphi^n_{n-1}(Z)$ for each $Z\in\mathfrak{Z}_n$.

(1.5_n) If $p\in Z_{n-1}\in\mathfrak{Z}_{n-1}$ and $p\notin\bigcup\mathfrak{U}_n$, then there exists some $Z_n\in\mathfrak{Z}_n$ such that $p\in Z_n$ and $\varphi^n_{n-1}(Z_n) = Z_{n-1}$.

(1.6_n) If $\langle Z_0, \ldots, Z_n\rangle\in\prod_{i=0}^{n}\mathfrak{Z}_i$ satisfies $\varphi^i_j(Z_i) = Z_j$ for each $j\leqslant i\leqslant n$, then the finite sequence $(E_0, \ldots, E_{2n})$ defined by $E_0 = X$, $E_{2i} = Z_i'$ and $E_{2i-1} = s(E_0, \ldots, E_{2i-2})$ for each $1\leqslant i\leqslant n$, which are well-defined, is admissible for $G(DC_\mu, X)$.

(1.7_n) $\mathfrak{H}_n = \{H(Z)\colon Z\in\mathfrak{Z}_n\}$ such that $Z\subset H(Z)$ for each $Z\in\mathfrak{Z}_n$.

(1.8_n) $\mathfrak{H}_n$ is locally finite in $X\times Y$.

(1.9_n) $\bigcup\mathfrak{Z}_n = \bigcap\{V(n, m)\colon m\geqslant 0\}$, where $V(n, 0) = X\times Y$.

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[2] A space is said to be *μ-paracompact* if each its open cover of power $\leqslant\mu$ has a locally finite open refinement.

(1.10_n) $\mathrm{Cl}\,V(n,m)\subset V(n,m-1)\cap V(n-1,m)$ for each $m\geqslant 1$.

(1.11_n) $\bigcup\mathfrak{U}_n\subset V(n-1,n-1)$.

Assume $\{\mathfrak{U}_i\colon i\leqslant n\}$, $\{\langle\mathfrak{Z}_i,\varphi^i_m\rangle\colon m\leqslant i\leqslant n\}$, $\{\mathfrak{H}_i\colon i\leqslant n\}$ and $\{V(i,m)\colon i\leqslant n, m\geqslant 0\}$, satisfying (1.1_i)–(1.11_i) for each $i\leqslant n$, have been already constructed. Now, fix $Z\in\mathfrak{Z}_n$. By (1.6_n), we can define $E_0=X$, $E_{2i}=(\varphi^n_i(Z))'$ and $E_{2i-1}=s(E_0,\ldots,E_{2i-2})$ for $1\leqslant i\leqslant n$ and the finite sequence $(E_0,\ldots,E_{2n})$ is admissible for $G(\boldsymbol{DC}_\mu,X)$. So $s(E_0,\ldots,E_{2n})$ is the union of a discrete collection $\{F_\alpha\colon\alpha\in\Omega(Z)\}$ by μ-compact closed subsets of Z'. We choose two discrete (in X) collections $\{W_\alpha\colon\alpha\in\Omega(Z)\}$ and $\{C_\alpha\colon\alpha\in\Omega(Z)\}$ such that W_α is a cozero-set in X, C_α is a zero-set in X and $F_\alpha\subset W_\alpha\subset C_\alpha$ for each $\alpha\in\Omega(Z)$. Then, applying Lemma 2.1 to Z, it follows from (1.0), (1.7_n), (1.9_n) and paracompactness of Y that there exists a locally finite collection $\mathfrak{U}_\alpha(Z)=\{U_{\lambda,i}\times V_\lambda\colon i=1,\ldots,k_\lambda$ and $\lambda\in\Lambda(\alpha)\}$ by cozero rectangles for each $\alpha\in\Omega(Z)$, satisfying the following conditions (i)–(v):

(i) $F_\alpha\subset U_\lambda=\bigcup\{U_{\lambda,i}\colon i\leqslant k_\lambda\}\subset W_\alpha$ for each $\lambda\in\Lambda(\alpha)$.

(ii) $\{V_\lambda\colon\lambda\in\Lambda(\alpha)\}$ is a locally finite collection by cozero-sets in Y, covering Z''.

(iii) Each $U_{\lambda,i}\times V_\lambda$ is contained in some $G\in\mathfrak{G}$.

(iv) $U_\lambda\subset H(Z)'$ and $V_\lambda\subset H(Z)''$ for each $\lambda\in\Lambda(\alpha)$.

(v) $U_\lambda\times V_\lambda\subset V(n,n)$ for each $\lambda\in\Lambda(\alpha)$.

Here, we set $\mathfrak{U}_{n+1}=\bigcup\{\mathfrak{U}_\alpha(Z)\colon Z\in\mathfrak{Z}_n$ and $\alpha\in\Omega(Z)\}$. From the above conditions (i)–(v) and (1.8_n), we can see that the conditions (1.1_{n+1}), (1.3_{n+1}) and (1.11_{n+1}) are satisfied. For each $Z\in\mathfrak{Z}_n$ and $\alpha\in\Omega(Z)$, we take a locally finite cover $\mathfrak{B}_\alpha(Z)=\{B_\lambda\colon\lambda\in\Lambda(\alpha)\}$ of Z'' by zero-sets in Y such that $B_\lambda\subset V_\lambda\cap Z''$ for each $\lambda\in\Lambda(\alpha)$. Let $W(Z)=\bigcup\{W_\alpha\cap Z'\colon\alpha\in\Omega(Z)\}$. So we put $\tilde Z=(Z'\setminus W(Z))\times Z''$. Then $\tilde Z$ is a zero rectangle. Next, we put $Z(\alpha,\lambda)=((C_\alpha\cap Z')\setminus U_\lambda)\times B_\lambda$ for each $\alpha\in\Omega(Z)$ and $\lambda\in\Lambda(\alpha)$. Then each $Z(\alpha,\lambda)$ is also a zero rectangle. Here, we set

$$\mathfrak{Z}_{n+1}=\{\tilde Z\colon Z\in\mathfrak{Z}_n\}\cup\{Z(\alpha,\lambda)\colon Z\in\mathfrak{Z}_n,\ \alpha\in\Omega(Z)\text{ and }\lambda\in\Lambda(\alpha)\}.$$

It is easy to see the condition (1.2_{n+1}) being satisfied. The bonding map $\varphi^{n+1}_n\colon\mathfrak{Z}_{n+1}\to\mathfrak{Z}_n$ is defined by $\varphi^{n+1}_n(\tilde Z)=Z$ and $\varphi^{n+1}_n(Z(\alpha,\lambda))=Z$ for each $Z\in\mathfrak{Z}_n$, $\alpha\in\Omega(Z)$ and $\lambda\in\Lambda(\alpha)$. Then the condition (1.4_{n+1}) is clearly satisfied and $\varphi^{n+1}_m=\varphi^n_m\circ\varphi^{n+1}_n$ for $m\leqslant n$. Assume $p=(x,y)\in Z_n\in\mathfrak{Z}_n$ and $p\notin\bigcup\mathfrak{U}_{n+1}$. In case of $x\notin W(Z_n)$, we have $p\in\tilde Z_n\in\mathfrak{Z}_{n+1}$ and $\varphi^{n+1}_n(\tilde Z_n)=Z_n$. So, moreover, assume $x\in W(Z_n)$. We can take some $\alpha_0\in\Omega(Z_n)$ with $x\in W_{\alpha_0}$ and take some $\lambda_0\in\Lambda(\alpha_0)$ with $y\in B_{\lambda_0}\in\mathfrak{B}_{\alpha_0}(Z_n)$. Since $U_{\lambda_0}\times V_{\lambda_0}$ is contained in $\bigcup\mathfrak{U}_{n+1}$, we have $x\notin U_{\lambda_0}$. Hence

$$(x,y)\in((W_{\alpha_0}\cap Z'_n)\setminus U_{\lambda_0})\times B_{\lambda_0}\subset Z_n(\alpha_0,\lambda_0)\in\mathfrak{Z}_{n+1}$$

and $\varphi^{n+1}_n(Z_n(\alpha_0,\lambda_0))=Z_n$. Thus the condition (1.5_{n+1}) is satisfied. Take any $\langle Z_0,\ldots,Z_n,Z_{n+1}\rangle\in\prod_{i=0}^{n+1}\mathfrak{Z}_i$ such that $\varphi^i_j(Z_i)=Z_j$ for $j\leqslant i\leqslant n+1$. If we put $E_0=X$, $E_{2i}=Z'_i$ and $E_{2i-1}=s(E_0,\ldots,E_{2i-2})$ for $1\leqslant i\leqslant n$, then they are well-defined and the sequence $(E_0,\ldots,E_{2n})$ is admissible for $G(\boldsymbol{DC}_\mu,X)$ because of (1.6_n). Moreover, we put $E_{2n+1}=s(E_0,\ldots,E_{2n})$ and $E_{2n+2}=Z'_{n+1}$. In order to show that the sequence $(E_0,\ldots,E_{2n+2})$ is admissible for $G(\boldsymbol{DC}_\mu,X)$, it is enough to show

$E_{2n+2}\subset E_{2n}$ and $E_{2n+1}\cap E_{2n+2}=\varnothing$. Note that $\varphi^{n+1}_n(Z_{n+1})=Z_n$ implies $Z_{n+1}=\tilde Z_n$ or $Z_{n+1}=Z_n(\alpha_0,\lambda_0)$ for some $\alpha_0\in\Omega(Z_n)$ and $\lambda_0\in\Lambda(\alpha_0)$. By $Z_{n+1}\subset Z_n$, $E_{2n+2}\subset E_{2n}$ is clear. Since $E_{2i}=(\varphi^i_n(Z_n))'$ for $0\leqslant i\leqslant n$, we have

$$s(E_0,\ldots,E_{2n})\subset W(Z_n)\cap Z'_n$$

and $s(E_0,\ldots,E_{2n})\cap C_{\alpha_0}=F_{\alpha_0}\subset U_{\lambda_0}$. So we can verify that $s(E_0,\ldots,E_{2n})$ is disjoint from both $\tilde Z'_n$ and $(Z_n(\alpha_0,\lambda_0))'$. Hence E_{2n+1} and E_{2n+2} are disjoint in either case for Z_{n+1}. Thus the condition (1.6_{n+1}) is satisfied. Now, we put $H(\tilde Z)=H(Z)$ for each $Z\in\mathfrak{Z}_n$. From (1.7_n), note that $Z'\subset H(Z)'$ and $Z''\subset H(Z)''$. Since the collection $\{C_\alpha\cap Z'\colon\alpha\in\Omega(Z)\}$ is discrete in X, we can take a discrete collection $\{L_\alpha\colon\alpha\in\Omega(Z)\}$ of cozero-sets in X such that $C_\alpha\cap Z'\subset L_\alpha\subset H(Z)'$ for each $\alpha\in\Omega(Z)$. So we put $H(Z(\alpha,\lambda))=L_\alpha\times V_\lambda$ for each $\alpha\in\Omega(Z)$ and $\lambda\in\Lambda(\alpha)$. Here we can set $\mathfrak{H}_{n+1}=\{H(Z)\colon Z\in\mathfrak{Z}_{n+1}\}$. Thus we have defined the collection $\mathfrak{H}_{n+1}$ by cozero rectangles. It is easy to verify that the remaining conditions (1.7_{n+1}) and (1.8_{n+1}) are satisfied. By (1.2_n), (1.7_n), (1.8_n) and Lemma 2.2, $\bigcup\mathfrak{Z}_{n+1}$ is a zero-set in $X\times Y$. Note that $\bigcup\mathfrak{Z}_{n+1}\subset\bigcup\mathfrak{Z}_n\subset V(n,m)$ for each $m\geqslant 0$. So, we can inductively obtain a countable collection $\{V(n+1,m)\colon m\geqslant 0\}$ of cozero-sets in $X\times Y$ satisfying the conditions (1.9_{n+1}) and (1.10_{n+1}). From the facts mentioned above, we have inductively constructed the desired collections $\{\mathfrak{U}_n\}$, $\{\mathfrak{Z}_n,\varphi^n_m\}$, $\{\mathfrak{H}_n\}$ and $\{V(n,m)\}$ satisfying the conditions (1.1_n)–(1.11_n).

Now, we set $\mathfrak{U}=\bigcup\{\mathfrak{U}_n\colon n\geqslant 0\}$. We shall show that $\mathfrak{U}$ is a cover of $X\times Y$.

Claim 1. *For each* $\langle Z_n\colon n\geqslant 0\rangle\in\varprojlim\{\mathfrak{Z}_n,\varphi^n_m\}$, *we have* $\bigcap\{Z_n\colon n\geqslant 0\}=\varnothing$.

Proof. We put $E_0=X$, $E_{2n}=Z'_n$ and $E_{2n+1}=s(E_0,\ldots,E_{2n})$ for each $n\geqslant 1$. By (1.6_n), they are well-defined and the infinite sequence $(E_n\colon n\geqslant 0)$ is a play for $G(\boldsymbol{CD}_\mu,X)$. From the definition of s, we have $\bigcap\{E_{2n}\colon n\geqslant 0\}=\bigcap\{Z'_n\colon n\geqslant 0\}=\varnothing$. Hence $\bigcap\{Z_n\colon n\geqslant 0\}=\varnothing$ holds. Claim 1 is proved.

Assume $p_0\in X\times Y\setminus\bigcup\mathfrak{U}$. Since $p_0\in Z_0=X\times Y\in\mathfrak{Z}_0$ and $p_0\notin\bigcup\mathfrak{U}_n$ for each $n\geqslant 0$, by (1.5_n), we can inductively choose a sequence $\{Z_n\colon n\geqslant 0\}$ of zero rectangles such that $p_0\in Z_n\in\mathfrak{Z}_n$ and $\varphi^n_{n-1}(Z_n)=Z_{n-1}$ for each $n\geqslant 1$. So we have $\langle Z_n\colon n\geqslant 0\rangle\in\varprojlim\{\mathfrak{Z}_n,\varphi^n_m\}$ and $p_0\in\bigcap\{Z_n\colon n\geqslant 0\}$. This contradicts to Claim 1. Hence $\mathfrak{U}$ is a cover of $X\times Y$. Thus, by (1.1_n) and (1.3_n), $\mathfrak{U}$ is a σ-locally finite refinement of $\mathfrak{G}$ by cozero rectangles. Moreover, we shall show that $\mathfrak{U}$ is locally finite in $X\times Y$.

Claim 2. $\bigcap\{\bigcup\mathfrak{Z}_n\colon n\geqslant 0\}=\varnothing$.

Proof. Assume $q\in\bigcap\{\bigcup\mathfrak{Z}_n\colon n\geqslant 0\}$. Let $\mathfrak{Z}_n(q)$ be $\{Z\in\mathfrak{Z}_n\colon q\in Z\}$ for each $n\geqslant 0$. It follows from (1.2_n) and (1.4_n) that each $\mathfrak{Z}_n(q)$ is a non-empty finite subcollection of $\mathfrak{Z}_n$ such that $\varphi^n_{n-1}(\mathfrak{Z}_n(q))\subset\mathfrak{Z}_{n-1}(q)$. By König's lemma, there exists some $\langle Z_n\colon n\geqslant 0\rangle\in\varprojlim\{\mathfrak{Z}_n(q),\varphi^n_m|\mathfrak{Z}_n(q)\}$. Then we have $\langle Z_n\colon n\geqslant 0\rangle\in\varprojlim\{\mathfrak{Z}_n,\varphi^n_m\}$ and $q\in\bigcap\{Z_n\colon n\geqslant 0\}$. This contradicts to Claim 1. Thus Claim 2 is proved.

Let p be any point of $X\times Y$. By Claim 2, we can take some $n_1>0$ such that $p\notin\bigcup\mathfrak{Z}_{n_1}$. By (1.9_n) and (1.10_n), we can also take some $n_0\geqslant n_1$ such that $p\notin\mathrm{Cl}\,V(n_1,n_0)$. Hence $p\notin\mathrm{Cl}\,V(n_0,n_0)$. From (1.10_n) and (1.11_n), we have p

$\notin \mathrm{Cl}(\bigcup\{\bigcup \mathfrak{U}_n: n>n_0\})$. Since $\bigcup\{\mathfrak{U}_n: n\leqslant n_0\}$ is locally finite in $X\times Y$, $\mathfrak{U}$ is locally finite at p. Thus, we can obtain the following conclusion;

(*) each open cover $\mathfrak{G}$ of $X\times Y$ satisfying the condition (1.0) has a locally finite refinement $\mathfrak{U}$ by cozero rectangles.

Let $\{D_\xi: \xi\in\Xi\}$ be any discrete collection of closed subsets in $X\times Y$. Put $\mathfrak{G} = \{X\times Y\setminus\bigcup\{D_\eta: \eta\neq\xi\}: \xi\in\Xi\}$. Since each countably compact closed subset in $X\times Y$ intersects at most finite many elements of $\{D_\xi: \xi\in\Xi\}$, $\mathfrak{G}$ satisfies the condition (1.0). It follows from the conclusion (*) that $\mathfrak{G}$ is a normal cover of $X\times Y$. So it is easy to show that there exists a disjoint collection of open sets separating $\{D_\xi: \xi\in\Xi\}$ in $X\times Y$. Hence $X\times Y$ is collectionwise normal.

Let $\mathfrak{G}$ be any open cover of $X\times Y$ of power $\leqslant\mu$. Since $\mathfrak{G}$ clearly satisfies the condition (1.0), it follows from the conclusion (*) that $X\times Y$ is μ-paracompact.

Let $\mathfrak{G}$ be any locally finite open cover of $X\times Y$. Since each countably compact closed subset in $X\times Y$ is covered by some finite subcollection of $\mathfrak{G}$, $\mathfrak{G}$ satisfies the condition (1.0). So, it follows from the conclusion (*) that $X\times Y$ is strongly rectangular. Thus, the proof of Theorem 2.1 is complete.

B. A. Pasynkov [6] defined rectangular product and stated that the inequality $\dim X\times Y\leqslant\dim X+\dim Y$ holds if $X\times Y$ is rectangular. A strongly rectangular product $X\times Y$ is clearly rectangular. From this result and Theorem 2.1, we obtain the following.

COROLLARY 2.1. *Let X be a collectionwise normal space and Player I has a winning strategy in $G(DC_\mu, X)$ and let Y be a paracompact space with $\chi(Y)\leqslant\mu$. Then the inequality*

$$(*)\qquad \dim X\times Y\leqslant\dim X+\dim Y$$

holds.

Remark. We can immediately prove Corollary 2.1 using neither the above Pasynkov's result nor the fact $X\times Y$ is rectangular. In fact, let $\dim X\leqslant n$ and $\dim Y\leqslant m$. A given continuous map from a closed subset of $X\times Y$ into $(n+m)$-sphere S^{n+m} can be extended to $X\times Y$ by using the same technique as in the proof of Theorem 2.1. Since Theorem 2.1 guarantees $X\times Y$ is normal, this implies $\dim X\times Y\leqslant n+m$.

From Theorem 2.1 and Proposition 1.2, we obtain the following generalization of [12, Theorem 14.7].

THEOREM 2.2. *Let X be a collectionwise normal space which has a σ-closure-preserving closed cover by μ-compact sets and Y be a paracompact space with $\chi(Y)\leqslant\mu$. Then the product space $X\times Y$ is collectionwise normal, μ-paracompact and strongly rectangular.*

From Theorem 2.2, we have the following corollaries.

COROLLARY 2.2. *If X is a paracompact space which has a σ-closure-preserving cover by compact sets and Y is a paracompact space, then the product space $X\times Y$ is paracompact and the above inequality (*) holds.*

COROLLARY 2.3. *If X is a collectionwise normal space which has a σ-closure-preserving closed cover by countably compact sets and Y is a paracompact first countable space, then the product space $X\times Y$ is collectionwise normal, countably paracompact and the above inequality (*) holds.*

§ 3. Hurewicz spaces.

A regular space X is said to be a *Hurewicz space* [1] if for each sequence $\{\mathfrak{G}_n: n\geqslant 1\}$ of open covers of X there exists a cover $\mathfrak{H} = \bigcup\{\mathfrak{H}_n: n\geqslant 1\}$ of X, where $\mathfrak{H}_n$ is a finite subcollection of $\mathfrak{G}_n$ for each $n\geqslant 1$. Note that each Hurewicz space has the Lindelöf property and each σ-compact space has the Hurewicz property.

A. Lelek [1] showed that the product of two Hurewicz spaces need not be normal under continuum hypothesis. On the other hand, R. Telgársky [11] proved that the product of a Hurewicz (Lindelöf) C-scattered space and a Hurewicz space is a Hurewicz space. Being encouraged by this positive result, we can obtain another positive result concerning the Hurewicz property of product spaces.

First, we state the following lemma.

LEMMA 3.1. *Let F be a compact subset in a space X and R be a closed subset of a Hurewicz space Y. Furthermore, let $\{\mathfrak{G}_n: n\geqslant 1\}$ be a sequence of collections of open sets in $X\times Y$ such that each $\mathfrak{G}_n$ covers $F\times R$. Then there exists a countable collection $\mathfrak{U} = \bigcup\{\mathfrak{U}_n: n\geqslant 1\}$ by open rectangles, where each $\mathfrak{U}_n$ is a finite collection represented by the form*

$$\{U_{i,j}\times V_i: j = 1, \ldots, M(i) \text{ and } i = K(n-1)+1, \ldots, K(n)\}$$

such that $F\subset\bigcup\{U_{i,j}: j\leqslant M(i)\}$ for each $i\geqslant 1$, $R\subset\bigcup\{V_i: i\geqslant 1\}$ and $\mathfrak{U}_n$ refines $\mathfrak{G}_n$.

This is a modification of the result of R. Telgársky [11, Lemma 3.6] and the proof is also quite similar to his one.

THEOREM 3.1. *Let X be a Lindelöf space and Y be a Hurewicz space. If Player I has a winning strategy in $G(DC, X)$, then the product $X\times Y$ is a Hurewicz space.*

Proof. Let s be a winning strategy in $G(DC, X)$ and let $\{\mathfrak{G}_n: n\geqslant 1\}$ be a sequence of open covers of $X\times Y$.

We shall construct a sequence $\{\mathfrak{U}_n: n\geqslant 0\}$ of collections by open rectangles and an inverse system $\{\langle\mathfrak{R}_n, \psi_m^n\rangle: n\geqslant 0, m\leqslant n\}$ of a sequence of collections $\mathfrak{R}_n$ by closed rectangles with the bonding maps ψ_m^n of $\mathfrak{R}_n$ into $\mathfrak{R}_m$, where $\mathfrak{U}_0 = \{\emptyset\}$ and $\mathfrak{R}_0 = \{X\times Y\}$, satisfying for each $n\geqslant 1$

(2.1_n) $\mathfrak{U}_n$ is countable,

(2.2_n) $\mathfrak{R}_n$ is countable,

(2.3_n) for each $k\geqslant n$ there exists a finite subcollection $\mathfrak{H}_k^n$ of $\mathfrak{G}_k$ such that $\mathfrak{U}_n$ refines $\{\mathfrak{H}_k^n: k\geqslant n\}$,

and the three conditions (2.4_n)–(2.6_n) which are the same ones as (1.4_n)–(1.6_n) in the proof of Theorem 2.1 respectively provided that we replace $\mathfrak{Z}_n$, Z and φ_m^n by $\mathfrak{R}_n$, R and ψ_m^n respectively.

Assume that $\{\mathfrak{U}_i: i\leqslant n\}$ and $\{\langle\mathfrak{R}_i, \psi_m^i\rangle: m\leqslant i\leqslant n\}$ satisfying (2.1_i)–(2.6_i) for $i\leqslant n$ have been already constructed. By (2.2_n), we represent $\mathfrak{R}_n$ by $\{R_m: m\geqslant 1\}$. Now, fix $R_m\in\mathfrak{R}_n$. As the previous case, we can put $E_0 = X$, $E_{2i} = (\psi_i^n(R_m))'$ and $E_{2i-1} = s(E_0, \ldots, E_{2i-2})$ for $1\leqslant i\leqslant n$. Since R'_m has the Lindelöf property, each discrete collection of subsets of R'_m is at most countable. So $s(E_0, \ldots, E_{2n})$ is the union of a discrete countable collection $\{F_i: i\geqslant 1\}$ by compact sets in X. We take a discrete collection $\{W_i: i\geqslant 1\}$ of open sets in X such that $F_i\subset W_i$ for each $i\geqslant 1$. Applying Lemma 3.1 to F_i, R''_m and $\{\mathfrak{G}_j: j\geqslant n+m+i-1\}$, we can obtain a countable collection $\bigcup\{\mathfrak{U}^m_{i,j}: j\geqslant 1\}$ by open rectangles, where each $\mathfrak{U}^m_{i,j}$ is represented by the form

$$\{U^i_{r,t}\times V^i_r: t = 1, \ldots, M(r,i) \text{ and } r = K(j-1,i)+1, \ldots, K(j,i)\}$$

such that $F_i\subset U^i_r = \bigcup\{U^i_{r,t}: t\leqslant M(r,i)\}\subset W_i$ for each $r\geqslant 1$, $R''_m\subset\bigcup\{V^i_r: r\geqslant 1\}$ and $\mathfrak{U}^m_{i,j}$ refines $\mathfrak{G}_{n+m+i+j-2}$.

Here we set $\mathfrak{U}_{n+1} = \bigcup\{\mathfrak{U}^m_{i,j}: m, i, j\geqslant 1\}$. The condition (2.1_{n+1}) is clear. For each $k\geqslant n+1$, we put

$$\mathfrak{B}^{n+1}_k = \bigcup\{\mathfrak{U}^m_{i,j}: m, i, j\geqslant 1 \text{ and } n+m+i+j-2 = k\}.$$

Then we have $\mathfrak{U}_{n+1} = \bigcup\{\mathfrak{B}^{n+1}_k: k\geqslant n+1\}$ and each $\mathfrak{B}^{n+1}_k$ is a finite collection which refines $\mathfrak{G}_k$. So we can choose some finite subcollection $\mathfrak{H}^{n+1}_k$ of $\mathfrak{G}_k$ such that $\mathfrak{B}^{n+1}_k$ refines $\mathfrak{H}^{n+1}_k$. Then the condition (2.3_{n+1}) is satisfied. The construction of $\langle\mathfrak{R}_{n+1}, \psi^{n+1}_n\rangle$ is similar to the previous $\langle\mathfrak{Z}_n, \varphi^{n+1}_n\rangle$. Indeed, fix $R_m\in\mathfrak{R}_n$ again. We take a locally finite closed cover $\mathfrak{B}^m_i = \{B^i_r: r\geqslant 1\}$ of R''_m such that $B^i_r\subset V^i_r\cap R''_m$ for each $r\geqslant 1$. Let $W(m) = \bigcup\{W_i\cap R'_m: i\geqslant 1\}$ and let $\tilde{R}_m = (R'_m\setminus W(m))\times R''_m$. Moreover, let $R_m(i,r) = ((\mathrm{Cl}\,W_i\cap R'_m)\setminus U^i_r)\times B^i_r$ for each i, $r\geqslant 1$. Here we set

$$\mathfrak{R}_{n+1} = \{\tilde{R}_m: m\geqslant 1\}\cup\{R_m(i,r): m, i, r\geqslant 1\}.$$

Then $\mathfrak{R}_{n+1}$ is a countable collection by closed rectangles. We define $\psi^{n+1}_n: \mathfrak{R}_{n+1}\to\mathfrak{R}_n$ as the previous φ^{n+1}_n. It clearly satisfies the condition (2.4_{n+1}). The verifications that the conditions (2.5_{n+1}) and (2.6_{n+1}) are satisfied are quite similar to the previous cases of (1.5_{n+1}) and (1.6_{n+1}) respectively, so they are omitted. Thus we have constructed the desired collections $\{\mathfrak{U}_n\}$ and $\{\mathfrak{R}_n, \psi^n_m\}$ satisfying the conditions (2.1_n)–(2.6_n).

Now, we set $\mathfrak{U} = \bigcup\{\mathfrak{U}_n: n\geqslant 1\}$. Then we can verify, as in the previous case, that $\mathfrak{U}$ is a cover of $X\times Y$. From (2.3_n), $\mathfrak{U}$ refines $\bigcup\{\mathfrak{H}^n_k: k\geqslant n\geqslant 1\}$. Put $\mathfrak{H}_k = \bigcup\{\mathfrak{H}^n_k: n = 1, \ldots, k\}$ for each $k\geqslant 1$ and put $\mathfrak{H} = \bigcup\{\mathfrak{H}_k: k\geqslant 1\}$. Then each $\mathfrak{H}_k$ is a finite subcollection of $\mathfrak{G}_k$ and $\mathfrak{H}$ is refined by $\mathfrak{U}$. Hence $\mathfrak{H}$ is a cover of $X\times Y$. The proof is complete.

Remark. We can see from Theorem 3.1 that if X is a Lindelöf space and Player I has a winning strategy in $G(\boldsymbol{DC}, X)$, then X is a Hurewicz space. This fact also follows from [9, Theorem 7]. Moreover, note that Theorem 3.1 is a generalization of [12, Theorem 14.12].

From Theorem 3.1 and Proposition 1.1, we obtain

COROLLARY 3.1. *If X is a Lindelöf space which has a σ-closure-preserving cover by compact sets and Y is a Hurewicz space, then the product $X\times Y$ is a Hurewicz space.*

§ 4. Other strongly rectangular products. Here, we shall deal with other well-known spaces than ones we have dealt with all this while. We consider when the products of those spaces are strongly rectangular.

THEOREM 4.1. *If X is a paracompact space which has a countable closed cover by C-scattered subsets and Y is a paracompact space, then the product space $X\times Y$ is strongly rectangular.*

The proof of Theorem 4.1 is obtained by the modification of that of [10, Theorem 2.3] and a few devices. The details of it is left to the reader as a exercise.

Remark. After the first version of this paper, R. Telgársky has informed the following result by letter:

For a paracompact space X, Player I has a winning strategy in $G(\boldsymbol{DC}, X)$ if and only if he has a winning strategy in $G(\boldsymbol{SC}, X)$, where $\boldsymbol{SC}$ denotes the class of all C-scattered spaces.

In truth, from this result, we can see the following facts; 1) [12, Theorem 14.6] is a consequence of our Theorem 2.1, 2) [11, Theorem 3.5] is a consequence of our Theorem 3.1 and 3) our Theorem 4.1 is a consequence of Theorem 2.1.

Let Y be a metric space. Let $\mathfrak{B} = \bigcup\{\mathfrak{B}_i: i\geqslant 1\}$ be an open basis of Y satisfying

(1) $\mathfrak{B}_i = \{V(\alpha_1, \ldots, \alpha_i): \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i\}$ is a locally finite open cover of Y,

(2) $\operatorname{mesh}\mathfrak{B}_i < 1/i$,

(3) $V(\alpha_1, \ldots, \alpha_i) = \bigcup\{V(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}): \alpha_{i+1}\in\Omega_{i+1}\}$.

Here, it should be noted that some of $V(\alpha_1, \ldots, \alpha_i)$ may be empty and every metric space has such an open basis.

Let X be a normal space. An open cover $\mathfrak{U}$ of $X\times Y$ is said to be a *basic cover* [3] if $\mathfrak{U}$ has the form

$$\mathfrak{U} = \{U(\alpha_1, \ldots, \alpha_i)\times V(\alpha_1, \ldots, \alpha_i): \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\}$$

and if $U(\alpha_1, \ldots, \alpha_i)$ are open sets in X such that

$$U(\alpha_1, \ldots, \alpha_i)\subset U(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}) \quad \text{for} \quad \alpha_1\in\Omega_1, \ldots, \alpha_{i+1}\in\Omega_{i+1}.$$

If, for a basic cover $\mathfrak{U}$ of $X\times Y$, there exists a collection

$$\{F(\alpha_1, \ldots, \alpha_i): \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\}$$

of closed subsets of X such that

$$F(\alpha_1, \ldots, \alpha_i)\subset U(\alpha_1, \ldots, \alpha_i),$$

$$\bigcup\{F(\alpha_1, \ldots, \alpha_i)\times V(\alpha_1, \ldots, \alpha_i): \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\} = X\times Y,$$

then we say that $\mathfrak{U}$ *has a special refinement* [3].

Lemma 4.1. *Let X be a normal space and Y a metric space. Then a basic cover of $X\times Y$ has a special refinement if and only if it is a normal cover.*

Lemma 4.2. *Let Y be a non-discrete metric space. If $X\times Y$ is normal, then $X\times Y$ is countably paracompact.*

Lemma 4.1 is due to K. Morita [3] and Lemma 4.2 is due to M. E. Rudin and M. Starbird [8].

J. Nagata [5] proved that the product of a normal P-space and a metric space is rectangular (i.e. a F-product), where P-space is in the sense of K. Morita [2]. Moreover, it follows from [6, Proposition 1. (3)] and above Lemma 4.2 that $X\times Y$ is rectangular if $X\times Y$ is normal and Y is metrizable. However, we can really obtain the following result.

Theorem 4.2. *If $X\times Y$ is normal and Y is metrizable, then $X\times Y$ is strongly rectangular.*

Proof. If Y is discrete, then nothing should be proved. So we can assume Y is non-discrete. Let $\mathfrak{G} = \{G_\lambda\colon \lambda\in\Lambda\}$ be any locally finite open cover of $X\times Y$. Let $\mathfrak{B} = \bigcup\{\mathfrak{B}_i\colon i\geqslant 1\}$ be the above open basis of Y satisfying the conditions (1)–(3). For $\alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i$ and $\lambda\in\Omega$, we define two open sets $U(\alpha_1, \ldots, \alpha_i\colon \lambda)$ and $U(\alpha_1, \ldots, \alpha_i)$ in X by

$$U(\alpha_1, \ldots, \alpha_i; \lambda) = \bigcup\{U'\colon U' \text{ is an open set in } X \text{ such that } U'\times V(\alpha_1, \ldots, \alpha_i)\subset G_\lambda\},$$

$$U(\alpha_1, \ldots, \alpha_i) = \bigcup\{U(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}.$$

Since we have $U(\alpha_1, \ldots, \alpha_i)\subset U(\alpha_1, \ldots, \alpha_i, \alpha_{i+1})$,

$$\mathfrak{U} = \{U(\alpha_1, \ldots, \alpha_i)\times V(\alpha_1, \ldots, \alpha_i)\colon \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\}$$

is a basic cover of $X\times Y$. Note that $\mathfrak{U}$ is a σ-locally finite open cover of $X\times Y$. By Lemma 4.2, $X\times Y$ is normal and countably paracompact. Hence $\mathfrak{U}$ is a normal cover. By Lemma 4.1, $\mathfrak{U}$ has a special refinement. So there exists a collection

$$\{F(\alpha_1, \ldots, \alpha_i)\colon \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\}$$

of closed subsets of X such that

$$F(\alpha_1, \ldots, \alpha_i)\subset U(\alpha_1, \ldots, \alpha_i),$$

$$\bigcup\{F(\alpha_1, \ldots, \alpha_i)\times V(\alpha_1, \ldots, \alpha_i)\colon \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } i\geqslant 1\} = X\times Y.$$

Now, fix $\alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i$ such that $V(\alpha_1, \ldots, \alpha_i)$ is non-empty. It is easily seen that the collection $\{U(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}$ is locally finite in X and for each $\lambda\in\Lambda$ we have

$$U(\alpha_1, \ldots, \alpha_i; \lambda)\times V(\alpha_1, \ldots, \alpha_i)\subset G_\lambda.$$

Since $\{U(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}$ covers $F(\alpha_1, \ldots, \alpha_i)$ and X is normal, there exists a locally finite collection $\{E(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}$ by cozero-sets in X such that

$$E(\alpha_1, \ldots, \alpha_i; \lambda)\subset U(\alpha_1, \ldots, \alpha_i; \lambda),$$

$$F(\alpha_1, \ldots, \alpha_i)\subset\bigcup\{E(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}.$$

Put $E(\alpha_1, \ldots, \alpha_i) = \bigcup\{E(\alpha_1, \ldots, \alpha_i; \lambda)\colon \lambda\in\Lambda\}$. Since $E(\alpha_1, \ldots, \alpha_i)$ is a cozero-set in X, we can choose two sequences

$$\{C_n(\alpha_1, \ldots, \alpha_i)\colon n\geqslant 1\} \quad\text{and}\quad \{Z_n(\alpha_1, \ldots, \alpha_i)\colon n\geqslant 1\}$$

of cozero-sets and zero-sets in X respectively such that

$$E(\alpha_1, \ldots, \alpha_i) = \bigcup\{C_n(\alpha_1, \ldots, \alpha_i)\colon n\geqslant 1\} = \bigcup\{Z_n(\alpha_1, \ldots, \alpha_i)\colon n\geqslant 1\},$$

$$C_n(\alpha_1, \ldots, \alpha_i)\subset Z_n(\alpha_1, \ldots, \alpha_i)\subset C_{n+1}(\alpha_1, \ldots, \alpha_i)$$

for each $n\geqslant 1$. Let $\lambda\in\Lambda$. We define

$$H(\alpha_1, \ldots, \alpha_i; \lambda) = E(\alpha_1, \ldots, \alpha_i; \lambda)\setminus\bigcup\{Z_i(\beta_1, \ldots, \beta_j)\colon j<i \text{ and } V(\alpha_1, \ldots, \alpha_i)\subset V(\beta_1, \ldots, \beta_j)\}.$$

Since $\bigcup\{\mathfrak{B}_j\colon j<i\}$ is locally finite in Y and $V(\alpha_1, \ldots, \alpha_i)$ is non-empty, $H(\alpha_1, \ldots, \alpha_i; \lambda)$ is a cozero-set in X. So, for each $\alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i$ and $\lambda\in\Lambda$ with $V(\alpha_1, \ldots, \alpha_i)\neq\emptyset$, we put

$$W(\alpha_1, \ldots, \alpha_i; \lambda) = H(\alpha_1, \ldots, \alpha_i; \lambda)\times V(\alpha_1, \ldots, \alpha_i).$$

In case $V(\alpha_1, \ldots, \alpha_i) = \emptyset$, we put $W(\alpha_1, \ldots, \alpha_i; \lambda) = \emptyset$ for each $\lambda\in\Lambda$. Then each $W(\alpha_1, \ldots, \alpha_i; \lambda)$ is a cozero rectangle in $X\times Y$ such that $W(\alpha_1, \ldots, \alpha_i; \lambda)\subset G_\lambda$. Here we set

$$\mathfrak{W}_i = \{W(\alpha_i, \ldots, \alpha_i; \lambda)\colon \alpha_1\in\Omega_1, \ldots, \alpha_i\in\Omega_i \text{ and } \lambda\in\Lambda\}$$

for each $i\geqslant 1$ and set $\mathfrak{W} = \bigcup\{\mathfrak{W}_i\colon i\geqslant 1\}$. Then each $\mathfrak{W}_i$ is locally finite in $X\times Y$ and $\mathfrak{W}$ refines $\mathfrak{G}$. Thus it is enough to show that $\mathfrak{W}$ is a locally finite cover of $X\times Y$.

Let $p = (x, y)$ be any point of $X\times Y$. We can choose some $\alpha_1\in\Omega_1, \ldots, \alpha_k\in\Omega_k$ such that $p\in F(\alpha_1, \ldots, \alpha_k)\times V(\alpha_1, \ldots, \alpha_k)$. So we have $x\in E(\alpha_1, \ldots, \alpha_k)$ and $y\in V(\alpha_1, \ldots, \alpha_k)$. So we can define a natural number k_0 by

$$k_0 = \min\{k\colon x\in E(\alpha_1, \ldots, \alpha_k) \text{ and } y\in V(\alpha_1, \ldots, \alpha_k) \text{ for some } \alpha_1\in\Omega_1, \ldots, \alpha_k\in\Omega_k\}.$$

Here we can take some $\gamma_1\in\Omega_1, \ldots, \gamma_{k_0}\in\Omega_{k_0}$ such that $x\in E(\gamma_1, \ldots, \gamma_{k_0})$ and $y\in V(\gamma_1, \ldots, \gamma_{k_0})$. Then, for each $\beta_1\in\Omega_1, \ldots, \beta_j\in\Omega_j$ and $j<k_0$ such that $V(\beta_1, \ldots, \beta_j)$ contains $V(\gamma_1, \ldots, \gamma_{k_0})$, we have $x\notin E(\beta_1, \ldots, \beta_j)$. We choose some $\lambda_0\in\Lambda$ such that $x\in E(\gamma_1, \ldots, \gamma_{k_0}; \lambda_0)$. Then we obtain $x\in H(\gamma_1, \ldots, \gamma_{k_0}; \lambda_0)$. Hence we have $p\in W(\gamma_1, \ldots, \gamma_{k_0}; \lambda_0)$. Thus $\mathfrak{W}$ is a cover of $X\times Y$.

Next, we choose some $n_1>k_0$ such that $x\in C_{n_1}(\gamma_1, \ldots, \gamma_{k_0})$. Moreover, we choose some $n_0\geqslant n_1$ and some open neighborhood V_0 of y in Y such that $\mathrm{St}(V_0, \mathfrak{B}_n)$

$\subset V(\gamma_1, \ldots; \gamma_{k_0})$ if $n \geqslant n_0$, where $\mathrm{St}(V_0, \mathfrak{V}_n) = \bigcup \{V \in \mathfrak{V}_n \colon V \cap V_0 \neq \emptyset\}$. We put $O_p = C_{n_0}(\gamma_1, \ldots, \gamma_{k_0}) \times V_0$. Then O_p is an open neighborhood of p in $X \times Y$. Let $W \in \mathfrak{W}_n$ and $n \geqslant n_0$. Then we represent

$$W = W(\beta_1, \ldots, \beta_n; \lambda) = H(\beta_1, \ldots, \beta_n; \lambda) \times V(\beta_1, \ldots, \beta_n)$$

for some $\beta_1 \in \Omega_1, \ldots, \beta_n \in \Omega_n$ and $\lambda \in \Lambda$. Assume V_0 intersects $V(\beta_1, \ldots, \beta_n)$. Then $V(\beta_1, \ldots, \beta_n)$ is contained in $V(\gamma_1, \ldots, \gamma_{k_0})$. From $k_0 < n$, this implies that $H(\beta_1, \ldots, \beta_n; \lambda)$ are disjoint from $Z_n(\gamma_1, \ldots, \gamma_{k_0})$. Since $Z_n(\gamma_1, \ldots, \gamma_{k_0})$ contains $C_{n_0}(\gamma_1, \ldots, \gamma_{k_0})$, O_p does not intersect W. Hence O_p does not intersect any elements of $\bigcup \{\mathfrak{W}_n \colon n \geqslant n_0\}$. Since each $\mathfrak{W}_i$ is locally finite in $X \times Y$, $\mathfrak{W}$ is locally finite at p. Thus we have shown that $\mathfrak{W}$ is a locally finite refinement of $\mathfrak{G}$ by cozero rectangles. The proof is complete.

LEMMA 4.3. *Let $f\colon X \to \tilde{X}$ and $g\colon Y \to \tilde{Y}$ be perfect maps of X and Y onto $\tilde{X}$ and $\tilde{Y}$ respectively. If $\tilde{X} \times \tilde{Y}$ is paracompact and strongly rectangular, then $X \times Y$ is strongly rectangular.*

The proof is quite standard, so it is omitted.

THEOREM 4.3. *If X is a paracompact P-space and Y is a paracompact M-space, then the product $X \times Y$ is strongly rectangular.*

Proof. From the assumption of Y, there exists a perfect map g of Y onto a metric space M. Then it follows from [2, Theorem 5.1] and our Theorem 4.2 that $X \times M$ is paracompact and strongly rectangular. So, using Lemma 4.3, we can see that $X \times Y$ is strongly rectangular.

LEMMA 4.4. *Let $\{Z_i \colon i = 1, \ldots, m\}$ be a finite collection by zero rectangles in a topological product $X \times Y$. Then the complement G of $\bigcup \{Z_i \colon i \leqslant m\}$ in $X \times Y$ is the finite union of cozero rectangles in $X \times Y$.*

Proof. Let Λ be the set of all finite subsets of $\{1, \ldots, m\}$. For each $\lambda \in \Lambda$ we put $U_\lambda = X \setminus \bigcup \{Z_i' \colon i \in \lambda\}$ and $V_\lambda = Y \setminus \bigcup \{Z_i'' \colon i \in \lambda\}$. Moreover, we put $W_{\lambda,\mu} = U_\lambda \times V_\mu$ for each $\lambda, \mu \in \Lambda$. Here we set

$$\mathfrak{W} = \{W_{\lambda,\mu} \colon \lambda, \mu \in \Lambda, \lambda \cap \mu = \emptyset \text{ and } W_{\lambda,\mu} \subset G\}.$$

For each $(x, y) \in G$, we put $\lambda_0 = \{i \leqslant m \colon X \times \{y\} \cap Z_i \neq \emptyset\}$ and $\mu_0 = \{i \leqslant m \colon U_{\lambda_0} \times Y \cap Z_i \neq \emptyset\}$. Then we have $(x, y) \in W_{\lambda_0, \mu_0} \in \mathfrak{W}$. Hence $\mathfrak{W}$ is the finite cover of G by cozero rectangles in $X \times Y$.

THEOREM 4.4. *If a product space $X \times Y$ has the Lindelöf property, then $X \times Y$ is strongly rectangular.*

Proof. Let $\mathfrak{G}$ be any (locally finite) open cover of $X \times Y$. From the Lindelöf property of $X \times Y$, we can take a countable cover $\{E_i \colon i \geqslant 1\}$ of $X \times Y$ by cozero rectangles, refining $\mathfrak{G}$. Then for each $i \geqslant 1$ there exist two sequences $\{C_{i,j} \colon j \geqslant 1\}$ and $\{Z_{i,j} \colon j \geqslant 1\}$ of cozero rectangles and zero rectangles respectively such that $E_i = \bigcup \{C_{i,j} \colon j \geqslant 1\} = \bigcup \{Z_{i,j} \colon j \geqslant 1\}$ and $C_{i,j} \subset Z_{i,j} \subset C_{i,j+1}$ for each $j \geqslant 1$. From Lemma 4.4, there exists a finite collection $\mathfrak{U}_i$ by cozero rectangles such that $\mathfrak{U}_1 = \{E_1\}$ and $\bigcup \mathfrak{U}_i = E_i \setminus \bigcup \{Z_{k,i-1} \colon k \leqslant i-1\}$ for each $i \geqslant 2$. Here we set $\mathfrak{U} = \bigcup \{\mathfrak{U}_i \colon i \geqslant 1\}$. Then it is easily seen that $\mathfrak{U}$ is a locally finite cover of $X \times Y$. Clearly, $\mathfrak{U}$ refines $\mathfrak{G}$. The proof is complete.

§ 5. Questions. We state several natural questions unanswered concerning strongly rectangular products.

5.1. Assume Y is normal (or paracompact) and rectangular. Is then $X \times Y$ strongly rectangular?

5.2. Assume X is a paracompact Σ-space (or σ-space) and Y is a paracompact P-space. Is then $X \times Y$ strongly rectangular?

Note that, from [6, Proposition 1.(5)], if 5.1 is affirmative then 5.2 is so.

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DEPARTMENT OF MATHEMATICS
KANAGAWA UNIVERSITY
Yokohama, Japan

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