A Whitehead theorem in CG-shape

by

Thomas J. Sanders* (Annapolis, Md.)

Abstract. With each σ-compact, locally compact metric space is associated a sequence of bigraded homotopy groups. These bigraded homotopy groups are used in a Whitehead theorem for compactly generated shape theory.

1. Introduction. Analogues of the classical theorem of J. H. C. Whitehead have been proven in shape theory by both M. Moszyńska [Mos] and S. Mardešić [Mar], using the homotopy pro-groups. The purpose of this paper is to give an analogue of this theorem in CG-shape theory [R–S], using homotopy bi-groups. The notion of shape for compact Hausdorff spaces is to be that given by Mardešić and Segal [M–S. 1, 2].

2. Category ind$(\mathscr{K})$. Let $\mathscr{K}$ be a category and consider the category dir$(\mathscr{K})$ whose objects are direct systems $X^* = [X^\alpha, p^{\alpha\alpha'}, A]$ in $\mathscr{K}$ where for $\alpha' \leqslant \alpha$, $p^{\alpha\alpha'}\colon X^{\alpha'} \to X^\alpha$ is a $\mathscr{K}$-morphism satisfying the usual commutative diagrams (cf. [Sp]). A dir$(\mathscr{K})$-morphism $f^* = (f, f^\alpha)\colon X^* \to Y^* = [Y^\beta, q^{\beta\beta'}, B]$ consists of an increasing cofinal function $f\colon A \to B$ and a collection of $\mathscr{K}$-morphisms $f^\alpha\colon X^\alpha \to Y^{f(\alpha)}$ such that for $\alpha' \leqslant \alpha$, there is an index $\beta \geqslant f(\alpha'), f(\alpha)$ with $q^{\beta, f(\alpha)} f^\alpha p^{\alpha\alpha'} = q^{\beta, f(\alpha')} f^{\alpha'}$. Compositions and identities are defined in the usual manner. A dir$(\mathscr{K})$-morphism is termed *special* if $A = B$ and $f = 1\colon A \to A$.

Two dir$(\mathscr{K})$-morphism $f^*, g^* = (g, g^\alpha)\colon X^* \to Y^*$ are said to be *equivalent*, $f^* \simeq g^*$, if for each $\alpha \in A$ there is a $\beta \in B$, $\beta \geqslant f(\alpha), g(\alpha)$, with $q^{\beta, f(\alpha)} f^\alpha = q^{\beta, g(\alpha)} g^\alpha$. This is a morphism equivalence relation and the resulting quotient category, dir$(\mathscr{K})/\simeq$, is dual to the pro-category (cf. [Mos] or [Mar]). Thus dir$(\mathscr{K})/\simeq$ has the same objects as dir$(\mathscr{K})$ and morphisms are equivalence classes $F = [f^*]$ of morphisms in dir$(\mathscr{K})$.

Consider $\mathscr{K}^*$, the full subcategory of dir$(\mathscr{K})$ consisting of direct systems in $\mathscr{K}$ whose indexing sets are N, the set of natural numbers. The quotient category, $\mathscr{K}^*/\simeq$, will be termed ind$(\mathscr{K})$. The following are dual to results given by Moszyńska and may be proven by dualizing the proofs she gave (cf. § 1, (3.1), (3.2), (3.3) of [Mos]).

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(2.1) *If* $f^* = (f^k)$ *is a special* $\mathscr{K}^*$*-morphism, then*
(a) *if each* f^k *is a monomorphism in* $\mathscr{K}$, *then* $[f^*]$ *is a monomorphism in* $\operatorname{ind}(\mathscr{K})$,
(b) *if each* f^k *is an epimorphism in* $\mathscr{K}$, *then* $[f^*]$ *is an epimorphism in* $\operatorname{ind}(\mathscr{K})$.

(2.2) *If for each* $k \in N$, $\omega^k\colon X^k \to Y^k$ *is a zero-morphism in* $\mathscr{K}$, *then* $\Omega = [1, \omega^k]\colon X^* \to Y^*$ *is a zero-morphism in* $\operatorname{ind}(\mathscr{K})$.

(2.3) *If* $f^*\colon X^* \to Y^*$ *is a* $\mathscr{K}^*$*-morphism and* $\operatorname{Ker}(f^k) = (N^k, j^k)$ *for all* $k \in N$, *then there is a collection of* $\mathscr{K}$*-morphisms,* $\{\eta^{kk'}\}_{k' \leqslant k}$, *such that* $N^* = [N^k, \eta^{kk'}]$ *is an object in* $\mathscr{K}^*$, $j^* = (1, j^k)$ *is a* $\mathscr{K}^*$*-morphism, and* $\operatorname{Ker}[f^*] = (N^*, [j^*])$ *in* $\operatorname{ind}(\mathscr{K})$.

Also, one can show that:

(2.4) *If* Z *is a zero-object in* $\mathscr{K}$, *then* $Z^* = [Z^k, \omega^{kk'}]$, *where each* $Z^k = Z$ *and each* $\omega^{kk'}$ *is the unique* $\mathscr{K}$*-morphism, is a zero object in* $\operatorname{ind}(\mathscr{K})$.

Combining (2.3) and (2.4) one has that:

(2.5) *If* $\mathscr{K}$ *has kernels and zero-objects, then so does* $\operatorname{ind}(\mathscr{K})$.

Mardešić has shown (cf. p. 57 of [Mar]):

(2.6) *Let* $\mathscr{K}$ *be a category with zero-objects and kernels and let*

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \xrightarrow{f_4} A_5$$

be a exact sequence. If f_1 *is an epimorphism and* f_4 *is a monomorphism, then* A_3 *is a zero-object.*

If X^* is a zero-object in $\operatorname{ind}(\mathscr{K})$, it must be equivalent to Z^*. This yields the following:

(2.7) *If* X^* *is a zero object in* $\operatorname{ind}(\mathscr{K})$, *then for each* $k \in N$, *there is a* $k' \geqslant k$ *such that* $p^{k'k}$ *is the zero-morphism.*

With a proof analogus to one given by Mardešić (cf. (4.4) of [Mar]) one obtains:

(2.8) *Let* $\mathscr{K}$ *be a category with zero-objects and kernels and let* $f^*\colon X^* \to Y^*$ *and* $g^*\colon Y^* \to W^*$ *be special* $\mathscr{K}^*$*-morphisms with the property that*

$$X^k \xrightarrow{f^k} Y^k \xrightarrow{g^k} W^k$$

is exact at Y^k *in* $\mathscr{K}$. *Then the sequence*

$$X^* \xrightarrow{[f^*]} Y^* \xrightarrow{[g^*]} W^*$$

is exact in $\operatorname{ind}(\mathscr{K})$.

3. CG-shape. Let S, S_0 and S_0^2 denote the shape category of compact metric spaces, the shape category of pointed compact metric spaces, and the shape category of pairs of pointed compact metric spaces, respectively. If X is a Hausdorff space, let $c(X)$ denote the set of all compact subsets of X. Recall [R-S] that a cover $\mathscr{F}$ of X is called CS-*cofinal* if there is a function $g\colon c(X) \to \mathscr{F}$ satisfying

(i) $F \subset g(F)$, for all $F \in \mathscr{F}$, and

(ii) if $F \subset F'$, then $g(F) \subset g(F')$.

If X is a metric space, the direct system $[F, i_{FF'}, c(X)]$ in S^* (where for $F' \subset F$, $i_{FF'}\colon F' \to F$ is the inclusion compact shape map) is used to determine the CG-shape of X (cf. [R-S]). Theorem 4.1 of [R-S] may be restated as:

(3.1) *If* $\mathscr{F}$ *is a compact cover of* X *that is* CS*-cofinal, then* $[F, i_{FF'}, c(X)]$ *and* $[F, i_{FF'}, \mathscr{F}]$ *are equivalent objects in* $\operatorname{dir}(S)$.

A Hausdorff space is termed *σ-compact* (cf. [Dug]) if it is locally compact and can be expressed as the union of at most countably many compact spaces. Thus X is a σ-compact space iff there is a sequence X_n of compact subsets of X with $X = \bigcup_1^\infty X_n$ and $X_n \subset \operatorname{int}(X_{n+1})$. Here $\operatorname{int}(\cdot)$ denotes the interior taken in the space X. If X and $Y = \bigcup_1^\infty Y_n$ are σ-compact metric spaces, then [R-S] there is a one-to-one functorial correspondence between CG-shape maps $F\colon X \to Y$ and $\operatorname{ind}(S)$-morphism from $X^* = [X^n, i^{nn'}]$ to $[Y^n, j^{nn'}]$ where $X^n = X_n$, $Y^n = Y_n$, and $i^{nn'}\colon X^{n'} \to X^n$ and $j^{nn'}\colon Y^{n'} \to Y^n$ denote the inclusion compact shape maps. If $\{n_k\}$ is a subsequence of $\{n\}$, then $Y = \bigcup_{k=1}^\infty Y_{n_k}$ is a valid representation of Y as a σ-compact set. Thus we have:

(3.2) *To each* CG *shape map* $F\colon X \to Y$ *and each representation of* $X = \bigcup_1^\infty X_n$, *there is a representation of* $Y = \bigcup_1^\infty Y_n$ *and a corresponding special* S^**-morphism* $(f^n)\colon [X^n, i^{nn'}] \to [Y^n, j^{nn'}]$. *This correspondence is a functorial correspondence between* CG*-shape maps* $F\colon X \to Y$ *and* $\operatorname{ind}(S)$*-morphisms. Similar correspondences exist for pointed and pairs of pointed spaces.*

4. Ind-pro-homotopy category. Let $\mathscr{W}$ denote the category whose objects are topological spaces having the homotopy type of a CW-complex and whose morphisms are homotopy classes of maps. Let $\mathscr{W}_0$ and $\mathscr{W}_0^2$ denote the corresponding categories of pointed and pairs of pointed spaces.

Following Morita [Mor. 2], an inverse system $X = \{X_\alpha, [p_{\alpha\alpha'}], A\}$ in $\mathscr{W}$ is said to be *associated with* a topological space X if there are continuous functions $p_\alpha\colon X \to X_\alpha$, $\alpha \in A$, such that

(1) if $\alpha \leqslant \alpha'$, then $[p_{\alpha\alpha'}][p_{\alpha'}] = [p_\alpha]$,

(2) for any map $f\colon X \to Q$ with $Q \in \operatorname{Ob}(\mathscr{W})$, there is an $\alpha \in A$ and a map $f_\alpha\colon X_\alpha \to Q$ with $[f] = [f_\alpha][p_\alpha]$,

(3) for $\alpha \in A$ and for two maps $f, g\colon X_\alpha \to Q$ with $Q \in \operatorname{Ob}(\mathscr{W})$ and $[f][p_\alpha] = [g][p_\alpha]$, there is an $\alpha' \in A$, $\alpha \leqslant \alpha'$, such that $[f][p_{\alpha\alpha'}] = [g][p_{\alpha\alpha'}]$.

Morita has shown [Mor. 2] that with every topological space is associated the inverse system in $\mathscr{W}$ formed by the nerves of all locally-finite normal open coverings. If X is a compact metric space, let $\{\mathscr{U}_n \mid n \in N\}$ be a sequence of locally finite (normal) open covers of X such that, for each n, the mesh of $\mathcal{U}_n$ is smaller than $1/n$ and $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$. Let K_n denote the nerve of $\mathcal{U}_n$ and let $p_n\colon X \to K_n$ denote canonical mappings $(p_n^{-1}(\mathrm{St}(U, K_n)) \subset U$ for all $U \in \mathcal{U}_n)$. Since the sequence $\{\mathcal{U}_n\}$ is cofinal in the set of all open coverings, the inverse sequence $\{K_n, [p_{nn'}]\}$, where $p_{nn'}\colon K_{n'} \to K_n$ are canonical projections whenever $n \leqslant n'$, is associated with the compact metric space X.

Morita has also shown [Mor. 2] that if X and Y have associated ANR-systems $\mathbf{X}$ and $\mathbf{Y}$, respectively, then there is a functorial bijection between shape maps from X to Y and pro($\mathcal{W}$)-morphisms from $\mathbf{X}$ to $\mathbf{Y}$.

Let M be a Hausdorff space. For each compact subset X of M, let $\mathbf{X}$ be an inverse system in $\mathcal{W}$ associated with X. If $X' \subset X$ is compact then there is a pro($\mathcal{W}$)-morphism $\boldsymbol{i}^{XX'}\colon \mathbf{X}' \to \mathbf{X}$ corresponding to the inclusion compact shape map from X' into X. If $\mathcal{F}$ is any compact cover of M that is CS-cofinal, then the direct system $X^* = [\mathbf{X}, \boldsymbol{i}^{XX'}, \mathcal{F}]$ in pro($\mathcal{W}$) will be said to be *associated* with M.

(4.1) *Any two direct systems in* pro($\mathcal{W}$) *associated with M are equivalent objects in* $\mathrm{dir}(\mathrm{pro}(\mathcal{W}))/_\simeq$.

(4.2) *If X^* and Y^* are direct systems in* pro($\mathcal{W}$) *associated with M and N, respectively, then there is a functorial bijection between* CG-*shape maps from X to Y and morphisms of* $\mathrm{dir}(\mathrm{pro}(\mathcal{W}))/_\simeq$ *from X^* to Y^*.*

Similar results hold for pointed and pairs of pointed spaces.

5. Homotopy bi-groups. Let $\mathcal{G}$ denote the category of groups and homomorphisms, let $\mathcal{E}_0$ denote the category of pointed sets and base point preserving functions, and let $\pi_k(X, x_0)$ denote the kth homotopy group of a pointed space (X, x_0). With each inverse system $(\mathbf{X}, \boldsymbol{x}_0) = \{(X, x)_\alpha, [p_{\alpha\alpha'}], A\}$ in $\mathcal{W}_0$, there is a sequence of inverse systems $\pi_k(\mathbf{X}, \boldsymbol{x}_0) = \{\pi_k(X, x)_\alpha, (p_{\alpha\alpha'})_k, A\}$ in $\mathcal{G}$ (if $k = 0$, in $\mathcal{E}_0$). If (X, x_0) is a compact Hausdorff space and $(\mathbf{X}, \boldsymbol{x}_0)$ is an inverse system in $\mathcal{W}_0$ associated with (X, x_0), then the inverse system $\pi_k(\mathbf{X}, \boldsymbol{x}_0)$ is called the *k-th homotopy pro-group* of (X, x_0).

A pro($\mathcal{W}_0$)-morphism $\boldsymbol{f} = [f, f_\beta]\colon (\mathbf{X}, \boldsymbol{x}_0) \to (\mathbf{Y}, \boldsymbol{y}_0)$ induces a sequence of morphisms of pro-groups $\boldsymbol{f}_k = [f, (f_\beta)_k]\colon \pi_k(\mathbf{X}, \boldsymbol{x}_0) \to \pi_k(\mathbf{Y}, \boldsymbol{y}_0)$. This is a functorial correspondence so that if $(\mathbf{X}, \boldsymbol{x}_0)$ and $(\mathbf{Y}, \boldsymbol{y}_0)$ are equivalent in pro($\mathcal{W}_0$), then $\pi_k(\mathbf{X}, \boldsymbol{x}_0)$ and $\pi_k(\mathbf{Y}, \boldsymbol{y}_0)$ are equivalent pro-groups.

Dually, with each direct system $(X, x_0)^* = [(\mathbf{X}, \boldsymbol{x})^\mu, \boldsymbol{p}^{\mu\mu'}, M]$ in pro($\mathcal{W}_0$) there is a sequence of direct systems $\pi_k(X, x_0)^* = [\pi_k(\mathbf{X}, \boldsymbol{x})^\mu, \boldsymbol{p}_k^{\mu\mu'}, M]$ in pro($\mathcal{G}$) (in pro($\mathcal{E}_0$), if $k = 0$). If (X, x_0) is a Hausdorff space and $(X, x_0)^*$ is a direct system in pro($\mathcal{W}_0$) associated with (X, x_0), then the direct system $\pi_k(X, x_0)^*$ is called the *k-th homotopy bi-group* of (X, x_0).

A morphism $F = [f, \boldsymbol{f}^\mu]\colon (X, x_0)^* \to (Y, y_0)^*$ in $\mathrm{dir}(\mathrm{pro}(\mathcal{W}_0))/_\simeq$ induces a sequence of morphisms of bi-groups $F_k = [f, \boldsymbol{f}_k^\mu]\colon \pi_k(X, x_0)^* \to \pi_k(Y, y_0)^*$. This is a functorial correspondence so that if $(X, x_0)^*$ and $(Y, y_0)^*$ are equivalent in $\mathrm{dir}(\mathrm{pro}(\mathcal{W}_0))/_\simeq$ then $\pi_k(X, x_0)^*$ and $\pi_k(Y, y_0)^*$ are equivalent bi-groups.

Similarly, one defines relative homotopy bi-groups $\pi_k(X, A, x_0)^*$, $k \geqslant 1$, for direct systems in pro($\mathcal{W}_0^2$). They belong to $\mathrm{dir}(\mathrm{pro}(\mathcal{G}))/_\simeq$ for $k \geqslant 2$ and to $\mathrm{dir}(\mathrm{pro}(\mathcal{E}_0))/_\simeq$ for $k = 1$.

With a direct system $(X, A, x_0)^* = [(\mathbf{X}, \mathbf{A}, \boldsymbol{x})^\mu, \boldsymbol{p}^{\mu\mu'}, M]$ in pro($\mathcal{W}_0^2$), there are direct systems $(A, x_0)^*$ and $(X, x_0)^*$ in pro($\mathcal{W}_0$) and morphisms $I\colon (A, x_0)^* \to (X, x_0)^*$, given by the inclusions $\boldsymbol{i}^\mu\colon (\mathbf{A}, \boldsymbol{x})^\mu \to (\mathbf{X}, \boldsymbol{x})^\mu$, and $J\colon (X, x_0, x_0)^* \to (X, A, x_0)^*$, given by the inclusions $\boldsymbol{j}^\mu\colon (\mathbf{X}, \boldsymbol{x}, \boldsymbol{x})^\mu \to (\mathbf{X}, \mathbf{A}, \boldsymbol{x})^\mu$. They induce morphisms of bi-groups $I_k\colon \pi_k(A, x_0)^* \to \pi_k(X, x_0)^*$ and $J_k\colon \pi_k(X, x_0, x_0)^* \to \pi_k(X, A, x_0)^*$, respectively. Finally, there is a boundary morphism

$$\partial_k = [\boldsymbol{\partial}_k^\mu]\colon \pi_k(X, A, x_0)^* \to \pi_{k-1}(A, X_0)^*$$

where each $\boldsymbol{\partial}_k^\mu\colon \pi_k(\mathbf{X}, \mathbf{A}, \boldsymbol{x}_0)^\mu \to \pi_{k-1}(\mathbf{A}, \boldsymbol{x}_0)^\mu$ is the boundary morphism for homotopy pro-groups determined by the boundary morphism for homotopy groups, $(\partial_\alpha^\mu)_k\colon \pi_k(X, A, x_0)_\alpha^\mu \to \pi_{k-1}(A, x_0)_\alpha^\mu$.

Since the homotopy sequence of pro-groups of $(\mathbf{X}, \mathbf{A}, \boldsymbol{x})^\mu$ is exact, we obtain from (2.8):

(5.1) *If $(X, A, x_0)^*$ is a direct system in pro($\mathcal{W}_0^2$), then the homotopy sequence of bi-groups*

$$\ldots \to \pi_k(A, x_0)^* \xrightarrow{I_k} \pi_k(X, x_0)^* \xrightarrow{J_k} \pi_k(X, A, x_0)^* \xrightarrow{\partial_k} \pi_{k-1}(A, x_0) \to \ldots$$
$$\ldots \xrightarrow{J_1} \pi_1(X, A, x_0)^* \xrightarrow{\partial_1} \pi_0(A, x_0)^* \xrightarrow{I_0} \pi_0(X, x_0)^*$$

is exact.

6. Nerves of coverings. For a metric space X, consider a sequence $\{\mathcal{U}_n |\ n \in N\}$ of locally finite (normal) open covers such that the mesh of $\mathcal{U}_n$ is smaller than $1/n$ and $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$. If $X = \bigcup_1^\infty X_m$ is a σ-compact metric space, let $\mathcal{U}_n(X_m) = \{U \cap X_m |\ U \in \mathcal{U}_n\}$ denote the open cover of X_m induced by $\mathcal{U}_n$ and let K_n^m denote its nerve. As in Section 4, the inverse sequence $\mathbf{X}^m = \{K_n^m, [P_{nn'}^m]\}$ is associated with the space X_m. For each n, there are inclusion maps $i_n^{m+1,m}\colon K_n^m \to K_n^{m+1}$ defined naturally by the correspondence $U \cap X_m \to U \cap X_{m+1}$. The inclusion shape maps $\boldsymbol{i}^{m+1,m}\colon X_m \to X_{m+1}$ have representatives the special map of systems $(i_n^{m+1,m})\colon \mathbf{X}^m \to \mathbf{X}^{m+1}$ and we may assume without loss that $P_{n,n+1}^{m+1} i_{n+1}^{m+1,m} = i_n^{m+1,m} p_{n,n+1}^m$. The direct sequence $X^* = [\mathbf{X}^m, [i_k^{m+1,m}]]$ in pro($\mathcal{W}$) is associated with the space X.

If A is a closed subset of X, $x_0 \in A$ and $A_m = A \cap X_m$, let L_n^m denote the nerve of $\mathcal{U}_n(A_m)$. The inverse sequence $(\mathbf{X}, \mathbf{A}, \boldsymbol{x})^m = \{(K_n^m, L_n^m, p_n), [p_{nn'}^m]\}$ in $\mathcal{W}_0^2$ is associated with the pair of pointed spaces (X_m, A_m, x_0) and the direct system $(X, A, x_0)^* = [(\mathbf{X}, \mathbf{A}, \boldsymbol{x})^m, [i_n^{m+1,m}]]$ in pro($\mathcal{W}_0^2$) is associated with the pair of pointed spaces (X, A, x_0).

If the dimension of X is $\leqslant K$, the sequence $\{\mathcal{U}_n\}$ of locally finite open covers can be chosen to have order $\leqslant K+1$. In this case, each nerve K_n^m will have dimension $\leqslant K$. Furthermore, if X is connected, each X_m may be chosen connected and each K_n^m will then be a connected simplicial complex.

7. CG-shape deformation retraction. In this section we shall prove the following:

(7.1) *Let $(X, A, x)^*$ and $(Y, B, y)^*$ be direct sequences in* $\mathrm{pro}(\mathscr{W}_0^2)$. *Let each $(X, A, x)_n^m$ and $(Y, B, y)_n^m$ be simplicial with $\dim X_n^m \leqslant K < \infty$, each Y_n^m connected, and each $(Y, B, y)_n^m$ a subcomplex of $(Y, B, y)_n^{m+1}$. If $\pi_k(Y, B, y)^* = 0$ for $1 \leqslant k \leqslant K+1$, then every morphism $F\colon (X, A, x)^* \to (Y, B, y)^*$ in* $\mathrm{ind}(\mathrm{pro}(\mathscr{W}_0^2))$ *admits a morphism $G\colon (X, x)^* \to (B, y)^*$ in* $\mathrm{ind}(\mathrm{pro}(\mathscr{W}_0))$ *such that*

$$JG = F\colon (X, x)^* \to (Y, y)^*, \qquad G|(A, x)^* = F|(A, x)^*\colon (A, x)^* \to (B, y)^*,$$

where $J\colon (B, y)^* \to (Y, y)^*$ is given by the inclusions $j_n^m\colon (B, y)_n^m \to (Y, y)_n^m$.

If one applies (7.1) to the identity morphism $I\colon (X, A, x)^* \to (X, A, x)^*$, one obtains

(7.2) *Let $(X, A, x)^*$ be a direct sequence in $\mathrm{pro}(\mathscr{W}_0^2)$ with each $(X, A, x)_n^m$ simplicial, each X_n^m connected, each $(X, A, x)_n^m$ a subcomplex of $(X, A, x)_n^{m+1}$, and $\dim X_n^m \leqslant K < \infty$. If $\pi_k(X, A, x)^* = 0$ for $1 \leqslant k \leqslant K+1$, then there is a morphism $R\colon (X, x)^* \to (A, x)^*$ in* $\mathrm{ind}(\mathrm{pro}(\mathscr{W}_0))$ *such that $JR = I$ and $RJ = R|(A, x)^* = I$. Consequently, the morphism $J\colon (A, x)^* \to (X, x)^*$ given by the inclusions $j_n^m\colon (A, x)_n^m \to (X, x)_n^m$ is an isomorphism in* $\mathrm{ind}(\mathrm{pro}(\mathscr{W}_0))$.

Note that by using (2.7) and a CS-cofinal subsequence, we may assume that $\pi_k(Y, B, y)^* = 0$ implies that each $(j^{m+1,m})_k\colon \pi_k(Y, B, y)^m \to \pi_k(Y, B, y)^{m+1}$ is the zero homomorphism in $\mathrm{pro}(\mathscr{G})$.

The following lemmas will be useful in proving (7.1).

(7.3) (cf. (6.2) of [Mar]). *Let (P, Q, p) be a simplicial pair with $\dim(P \setminus Q) \leqslant K+1$. Then for any $m \in N$, there is an increasing function $\sigma_m\colon N \to N$ such that, for all n, if $n^* = \sigma_m(n)$ and if $\varphi\colon (P, Q, p) \to (Y, B, y)_{n^*}^m$ is a map, then there is a map $\psi\colon (P, Q, p) \to (B, B, y)_n^{m+1}$ such that*

$$\varrho_{n,n^*}^{m+1,m}\varphi \simeq j_n^{m+1}\psi \quad \text{where} \quad \varrho_{n,n^*}^{m+1,m}\colon (Y, B, y)_{n^*}^m \to (Y, B, y)_n^{m+1}$$

is the composition of the projection map $q_{n,n^*}^m\colon (Y, B, y)_{n^*}^m \to (Y, B, y)_n^m$ and the inclusion map

$$j_n^{m+1,m}\colon (Y, B, y)_n^m \to (Y, B, y)_n^{m+1},$$

and

$$j_n^{m+1}\colon (B, B, y)_n^{m+1} \to (Y, B, y)_n^{m+1}$$

is the inclusion map.

Proof. Since $(j^{m+1,m})_k\colon \pi_k(Y, B, y)^m \to \pi_k(Y, B, y)^{m+1}$ is the zero-morphism, $1 \leqslant k \leqslant K+1$, for each $n \in N$ there is a chain $n_0 = n \leqslant n_1 \leqslant \ldots \leqslant n_{k+1} = \sigma_m(n)$ such that $(\varrho_{n_i, n_{i+1}}^{m+1,m})_k = 0$ for $0 \leqslant i \leqslant K$, $1 \leqslant k \leqslant K+1$. Choose a triangulation of (P, Q, p) such that Q is a full subcomplex of P. Let $L_k = (Q \cup P^k) \times I \cup (P \times 0)$, where P^k is the k-skeleton of P, $0 \leqslant k \leqslant K+1$.

As in the proof of (6.2) of [Mar], there is a sequence of maps $\chi_k\colon L_k \to Y_{n_{K+1-k}}^{m+1}$ such that

$$\begin{aligned} \chi_k(x, t) &= \varrho_{n_{K+1-k}, n_{K+1}}^{m+1,m}\varphi(x), && \text{if } (x, t) \in (Q \times I) \cup (P \times 0),\\ \chi_k(x, 1) &\in B_{n_{K+1-k}}^{m+1}, && \text{if } x \in P^k,\\ \chi_k(x, 1) &= y_{n_{K+1-k}}^{m+1}, && \text{if } x \in P^0 \setminus Q. \end{aligned}$$

Observe that $L_{K+1} = P \times I$ and consider $\chi_{K+1}\colon P \times I \to Y_n^{m+1}$. Setting $\psi(x) = \chi_{K+1}(x, 1)$ for all $x \in P$, we obtain a map $\psi\colon (P, Q, p) \to (B, B, y)_n^{m+1}$ that satisfies the required conditions. By induction, one can achieve that $\sigma_m(n) < \sigma_m(n+1)$ and $\sigma_m(n) < \sigma_{m+1}(n)$.

(7.4) (cf. (6.3) of [Mar]). *Let (P, p) be a simplicial complex, $\dim P \leqslant K$. Let $\varphi_0, \varphi_1\colon (P, p) \to (B, y)_{n*}^m$ be maps such that*

$$j_{n*}^m\varphi_0 \simeq j_{n*}^m\varphi_1\colon (P, p) \to (Y, y)_{n*}^m.$$

Then

$$\varrho_{n,n^*}^{m+1,m}\varphi_0 \simeq \varrho_{n,n^*}^{m+1,m}\varphi_1\colon (P, p) \to (B, y)_n^{m+1}.$$

Proof. As in (6.3) of [Mar], the homotopy between $j_{n*}^m\varphi_0$ and $j_{n*}^m\varphi_1$ gives a map

$$\varphi\colon (P \times I, P \times 0 \cup P \times 1, p \times I)/(p \times I) \to (Y, B, y)_{n*}^m.$$

By (7.3) there is a map $\psi\colon (P \times I, P \times 0 \cup P \times 1, p \times I)/(p \times I) \to (B, B, y)_n^{m+1}$ such that

$$\psi|(P \times 0 \cup P \times 1, p \times I)/(p \times I) \simeq \varrho_{n,n^*}^{m+1,m}\varphi|(P \times 0 \cup P \times 1, p \times I)/(p \times I)$$

in $(B, y)_n^{m+1}$ so that

$$\varrho_{n,n^*}^{m+1,m}\varphi_0 \simeq \psi|(P \times 0) \simeq \psi|(P \times 1) \simeq \varrho_{n,n^*}^{m+1,m}\varphi_1 \text{ in } (B, y)_n^{m+1}.$$

(7.5) (cf. (6.1) of [Mar]). *Let $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) = \{(X, A, x)_n, [P_{nn'}]\}$ be an inverse system in $\mathscr{W}_0^2$ with each $(X, A, x)_n$ simplicial and $\dim X_n \leqslant K < \infty$. Then every $\mathrm{pro}(\mathscr{W}_0^2)$-morphism $\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{x}) \to (\boldsymbol{Y}, \boldsymbol{B}, \boldsymbol{y})^m$ admits a $\mathrm{pro}(\mathscr{W}_0)$-morphism $\boldsymbol{g}\colon (\boldsymbol{X}, \boldsymbol{x}) \to (\boldsymbol{B}, \boldsymbol{y})^{m+2}$ such that*

$$\boldsymbol{j}^{m+2}\boldsymbol{g} = \boldsymbol{j}^{m+2,m}\boldsymbol{f}\colon (\boldsymbol{X}, \boldsymbol{x}) \to (\boldsymbol{Y}, \boldsymbol{y})^{m+2}$$

and

$$\boldsymbol{g}|(\boldsymbol{A}, \boldsymbol{x}) = \boldsymbol{j}^{m+2,m}\boldsymbol{f}|(\boldsymbol{A}, \boldsymbol{x})\colon (\boldsymbol{A}, \boldsymbol{x}) \to (\boldsymbol{B}, \boldsymbol{y})^{m+2}.$$

Proof. Let $n^* = \sigma_{m+1}(n)$ and $n^{**} = \sigma_m(n^*)$. Let $f_n\colon (X, A, x)_{f(n)} \to (Y, B, y)_n^m$ be a sequence of maps such that (f, f_n) is a map of inverse systems representative of $\boldsymbol{f}$. Let $g\colon N \to N$ be given by the composition $g(n) = f(n^{**})$. Consider $f_{n^{**}}\colon (X, A, x)_{g(n)} \to (Y, B, y)_{n^{**}}^m$. According to (7.3) there is a sequence of maps $\psi_n\colon (X, A, x)_{g(n)} \to (B, B, y)_{n^*}^{m+1}$ such that $\varrho_{n^*, n^{**}}^{m+1,m} f_{n^{**}} \simeq j_{n^*}^{m+1}\psi_n$. Define a sequence of maps $g_n\colon (X, x)_{g(n)} \to (B, y)_n^{m+2}$ by the composition $g_n = \varrho_{n,n^*}^{m+2,m+1}\psi_n$.

To show that (g, g_n) is a well defined map of inverse sequences, it suffices to show for all $n \in N$, $g_n p_{g(n),g(n+1)} \simeq q^{m+2}_{n,n+1} g_{n+1}$. First, note that

$$q^{m+2}_{n,n+1} \varrho^{m+2,m+1}_{n+1,(n+1)^*} = \varrho^{m+2,m+1}_{n,n^*} q^{m+1}_{n^*,(n+1)^*}.$$

Since

$$g_n p_{g(n),g(n+1)} = \varrho^{m+2,m+1}_{n,n^*} \psi_n p_{g(n),g(n+1)}$$

and

$$q^{m+2}_{n,n+1} g_{n+1} = q^{m+2}_{n,n+1} \varrho^{m+2,m+1}_{n+1,(n+1)^*} \psi_{n+1} = \varrho^{m+2,m+1}_{n,n^*} q^{m+1}_{n^*,(n+1)^*} \psi_{n+1}$$

by (7.4), it suffices to show that

$$j^{m+1}_{n^*} \psi_n p_{g(n),g(n+1)} \simeq j^{m+1}_{n^*} q^{m+1}_{n^*,(n+1)^*} \psi_{n+1}.$$

But,

$$\begin{aligned} j^{m+1}_{n^*} \psi_n p_{g(n),g(n+1)} &\simeq \varrho^{m+1,m}_{n^*,nn^{**}} f_{n^{**}} p_{g(n),g(n+1)} \\ &\simeq \varrho^{m+1,m}_{n^*,n^{**}} q^m_{n^{**},(n+1)^{**}} f_{(n+1)^{**}} \\ &\simeq q^{m+1}_{n^*,(n+1)^*} \varrho^{m+1,m}_{(n+1)^*,(n+1)^{**}} f_{(n+1)^{**}} \\ &\simeq q^{m+1}_{n^*,(n+1)^*} j^{m+1}_{(n+1)^*} \psi_{n+1} \\ &\simeq j^{m+1}_{n^*} q^{m+1}_{n^*,(n+1)^*} \psi_{n+1} \end{aligned}$$

and (g, g_n) is well defined.

The $\text{pro}(\mathscr{W}_0)$-morphism $\boldsymbol{g}\colon (X, x) \to (B, y)^{m+2}$ determined by (g, g_n) is such that

$$j^{m+2} \boldsymbol{g} = j^{m+2,m} \boldsymbol{f} \quad \text{and} \quad g|(A, x) = j^{m+2,m} f|(A, x).$$

Proof of (7.1). Let (f_m) be a special map of direct systems that is representative of F. By (7.5) there is a sequence of $\text{pro}(\mathscr{W}_0)$ morphism $\boldsymbol{g}_m\colon (X, x)^m \to (B, y)^{m+2}$ such that $j^{m+2} \boldsymbol{g}_m = j^{m+2,m} f_m$ and $\boldsymbol{g}_m|(A, x)^m = j^{m+2,m} f_m|(A, x)^m$.

Consider the map of inverse systems $\varrho^{m+1,m}\colon (B, y)^m \to (B, y)^{m+1}$ defined by $\varrho^{m+1,m} = (\sigma_m, \varrho^{m+1,m}_{n,n^*})$. One can show that $\varrho^{m+1,m} \simeq j^{m+1,m}\colon (B, y)^m \to (B, y)^{m+1}$ so that $(B, y)^* = [(B, y)^m, [\varrho^{m+1,m}]]$ and the sequence $(g_m)\colon (X, x)^m \to (B, y)^{m+2}$ will define a map of direct systems, if

$$\boldsymbol{g}_{m+1} i^{m+1,m} = \varrho^{m+3,m+2} \boldsymbol{g}_m.$$

Let $n^* = \sigma_{m+2}(n)$, $n^{**} = \sigma_{m+1}(n^*)$, $n^{***} = \sigma_m(n^{**})$, $g_m(n) = n'$, $g_{m+1}(n) = n''$, and $n\$ = f_{m+1}(n^{***})$. Then it suffices to show that for all $n \in N$, there is a $k \in N$, $k \geqslant g_m(n^*) = n^{*\prime}$, $g_{m+1}(n) = n''$ such that

$$\text{(A)} \qquad g^{m+1}_n i^{m+1,m}_{n''} p^m_{n'',k} \simeq \varrho^{m+3,m+2}_{n,n^*} g^m_{n^*} p^m_{n^{*\prime},k}.$$

Note that $n^{***} \geqslant n^{**}$, so $n\$ = f_{m+1}(n^{***}) \geqslant f_{m+1}(n^{**}) = n''$. Choose $k \in N$, so that $k \geqslant f_m(n^{***}) = n^{*\prime}$, $f_{m+1}(n^{***}) = n\$$ and

$$f^{m+1}_{n^{***}} i^{m+1,m}_{n\$} p^m_{n\$,k} \simeq j^{m+1,m}_{n^{***}} f^m_{n^{***}} p^m_{n^{*\prime},k}.$$

Then

$$\text{(B)} \qquad \varrho^{m+1,m}_{n^{**},n^{***}} f^m_{n^{***}} p^m_{n^{*\prime},k} \simeq f^{m+1}_{n^{**}} p^{m+1}_{n'',n\$} i^{m+1,m}_{n\$} p^m_{n\$,k}.$$

Consider the left side of (A),

$$g^{m+1}_n i^{m+1,m}_{n''} p^m_{n'',k} = \varrho^{m+3,m+2}_{n,n^*} \psi^{m+2,m+1}_n i^{m+1,m}_{n''} p^m_{n'',k}.$$

By (7.4) it suffices to show that

$$j^{m+2}_{n^*} g^m_{n^*} p^m_{n^{*\prime},k} \simeq j^{m+2}_{n^*} \psi^{m+2,m+1}_n i^{m+1,m}_{n''} p^m_{n'',k}.$$

This may be shown to follow from (B) using straight forward manipulations.

Both $JG = F$ and $G|(A, x)^* = F|(A, x)^*$ are consequences of (7.5).

8. Whitehead theorem for maps. Let $f\colon (X, x_0) \to (Y, y_0)$ be a map between two σ-compact metric spaces. Let (Z, z_0) denote the (pointed) mapping cylinder of f and let $i\colon (X, x_0) \to (Z, z_0)$ and $j\colon (Y, y_0) \to (Z, z_0)$ denote the usual embeddings. Then,

(8.1) (i) *(Z, z_0) is a σ-compact Hausdorff space,*

(ii) $\dim Z \leqslant \max\{1+\dim X, \dim Y\}$,

(iii) *f is a CG-shape equivalence iff i is a CG-shape equivalence, and*

(iv) *f induces a monomorphism (epimorphism) of homotopy bi-groups iff i induces one.*

We can now state and prove our main result:

(8.2) *Let (X, x_0) and (Y, y_0) be (pointed) σ-compact metric spaces, connected and finite-dimensional and let $f\colon (X, x_0) \to (Y, y_0)$ be a map which induces bimorphisms $F_k\colon \pi_k(X, x_0)^* \to \pi_k(Y, y_0)^*$ of homotopy bi-groups for*

$$0 \leqslant k \leqslant K = \max\{1+\dim X, \dim Y\}$$

and an epimorphism for $k = K+1$. Then f is a CG-shape equivalence, i.e. there is a CG-shape map $G\colon (Y, y_0) \to (X, x_0)$ such that $FG = I$ and $GF = I$. Here F denotes the CG-shape map induced by f and I denotes the identity shape maps.

Proof. Let (Z, z_0) be the mapping cylindar of f. Z is σ-compact, connected and $\dim Z \leqslant K < \infty$. The inclusion $i\colon (X, x_0) \to (Z, z_0)$ induces a bimorphism I_k of homotopy bi-groups for $0 \leqslant k \leqslant K$ and an epimorphism for $k = K+1$. By (5.1) there is an exact sequence of homotopy bi-groups belonging to the pair (Z, X, z_0). By (2.6), $\pi_k(Z, X, x_0)^* = 0$ for $1 \leqslant k \leqslant K+1$. It follows by (7.2) that the inclusion $i\colon (X, x_0) \to (Z, z_0)$ is a CG-shape equivalence, which implies that f is a CG-shape equivalence.

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DEPARTMENT OF MATHEMATICS
U. S. NAVAL ACADEMY
Annapolis, MD

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Topological games and products I

by

Yukinobu Yajima (Yokohama)

Abstract. Our main purpose in this paper is to show the following result: If a paracompact space X has a σ-closure-preserving cover by compact sets and Y is a paracompact space, then the inequality $\dim X\times Y\leqslant \dim X+\dim Y$ holds. As a matter of fact, we shall prove it in the more generalized form. The main tool of its proof is the topological game (in the sense of R. Telgársky).

§ 1. Introduction. R. Telgársky [12] introduced and studied the concept of topological game $G(K, X)$. Moreover, making use of it, he showed that the topological product of paracompact spaces one of which has a σ-closure-preserving cover by compact sets is paracompact (cf. [12, Theorem 14.7]). In fact, he obtained this result by proving the form which is generalized in terms of topological game (cf. [12, Theorem 14.6]). In § 2, we prove our main theorem. It is another generalization of the above result. Besides, the product inequality of covering dimension simultaneously holds in it, which is given by proving that a locally finite open cover of product space has a locally finite refinement by cozero rectangles. Here, the product space with this topological property is named to be strongly rectangular. In § 3, we apply the technique used in the proof of the above theorem to the product of Hurewicz spaces. In § 4, furthermore, we investigate what kind of a topological product is strongly rectangular. In § 5, we state several questions unanswered.

Throughout this paper, each space is assumed to be a Hausdorff space. However, for a topological product $X\times Y$, we shall mainly discuss in the case $X\times Y$ is normal and we assume either X or Y is non-empty. Non-negative integers are denoted by the letters i, j, k, m, n etc, and μ denotes an infinite cardinal number.

The descriptions and the details of the topological game $G(K, X)$ are found in [12]. Let us note that a sequence $(E_n\colon n\geqslant 0)$ of closed subsets of X is a play for $G(K, X)$ if and only if each finite subsequence $(E_0, \ldots, E_n)$ of it is admissible for $G(K, X)$. In particular, we consider as K, in this paper, the following two classes of spaces:

DC — the class of all spaces which can be decomposed into a discrete collection by compact sets.