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The sets of fixed points of families of affine continuous mappings

by

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Abstract. In this paper, fixed point theorems are proved for families of continuous mappings on compact spaces. First, we prove that under some conditions the set of common fixed points of a finite family of mappings on a compact set coinsides with the set of fixed points of a mapping on the set. Furthermore, we obtain a generalization of the Day fixed point theorem on a compact groupoid.

1. Introduction. In 1961, Day [2] obtained the following fixed point theorem: Let X be a compact convex subset of a locally convex topological vector space and Σ be a left amenable semigroup of continuous affine mappings on X, then there exists an element x in X such that $Tx = x$ for all $T \in \Sigma$. This is an extension of the Markov–Kakutani fixed point theorem for the case of which Σ is commutative. Recently, Anzai–Ishikawa [1] gave another extension of the Markov–Kakutani fixed point theorem; see Theorem 1 in this paper. On the other hand, Roberts [6] considered a compact groupoid $(X, \cdot)$, i.e., X is compact Hausdorff and $\cdot\colon X \times X \to X$ is continuous. It is obvious that if X is a compact convex subset of a locally convex topological vector space with $\cdot$ as the midpoint function, then $(X, \cdot)$ is a compact groupoid.

In this paper, we first give a simple proof of Anzai–Ishikawa's theorem by using the Krein–Milman theorem. Furthermore, we obtain an extension of their result. Finally, we prove a fixed point theorem for a family of continuous mappings on a compact groupoid. This is an extension of the Day fixed point theorem.

2. Sets of fixed points. Let T be a mapping of a set X into itself. Then we denote by $F(T)$ the set of fixed points of T. Let D be a subset of a topological vector space. We denote by $\operatorname{co} D$ the convex hull of D, $\overline{\operatorname{co}}\, D$ the closure of $\operatorname{co} D$, and $\operatorname{ex} D$ the set of extreme points of D. Recently, Anzai–Ishikawa proved the following theorem in [1]. We simplify the argument here.

THEOREM 1. *Let X be a compact convex subset of a locally convex topological vector space and $\{T_i\}_{i=1}^{n}$ be a finite commutative family of continuous affine mappings* *of X into itself. Then we have*

$$F\Big(\sum_{i=1}^{n} a_i T_i\Big) = \bigcap_{i=1}^{n} F(T_i)$$

for any positive numbers a_i with $\sum_{i=1}^{n} a_i = 1$ and hence $\bigcap_{i=1}^{n} F(T_i)$ is nonempty.

Proof. Let $P = \frac{1}{2}T_1 + \frac{1}{2}T_2$, then it is sufficient to show that $F(P) = F(T_1) \cap F(T_2)$. Let $x \in \operatorname{ex} F(P)$. Then, since $T_1 x = T_1 P x = P T_1 x$, we have $T_1 x \in F(P)$. Similarly, $T_2 x \in F(P)$. So, $x \in \operatorname{ex} F(P)$ and $x = Px = \frac{1}{2}T_1 x + \frac{1}{2}T_2 x$ imply $T_1 x = T_2 x = x$. By the Krein–Milman theorem, we have

$$F(P) = \overline{\mathrm{co}}\, \operatorname{ex} F(P) \subset F(T_1) \cap F(T_2).$$

Since the inverse inclusion is trivial, we have $F(P) = F(T_1) \cap F(T_2)$. Since $F(P)$ is nonempty by the Tychonoff fixed point theorem, $\bigcap_{i=1}^{n} F(T_i)$ is nonempty.

By Theorem 1, we can prove the Markov–Kakutani fixed point theorem: A commutative family of continuous affine mappings of X into itself has a common fixed point in X. We obtain the following theorem for the case of which $\{T_i\}_{i=1}^{n}$ in Theorem 1 is noncommutative.

THEOREM 2. *Let X be a compact convex subset of a locally convex topological vector space E and $\{T_i\}_{i=1}^{n}$ be a finite family of affine mappings of X into itself. Suppose that the semigroup Σ generated by $\{T_i\}_{i=1}^{n}$ is equicontinuous and $\bigcap_{i=1}^{n} F(T_i) \cap D \neq \emptyset$ for each $\{T_i\}_{i=1}^{n}$-invariant compact convex subset D of X. Then, we have that*

$$F\Big(\sum_{i=1}^{n} a_i T_i\Big) = \bigcap_{i=1}^{n} F(T_i)$$

for any positive numbers a_i with $\sum_{i=1}^{n} a_i = 1$.

Proof. Let $P = T_1/2 + T_2/2$, then it is sufficient to show that $F(P) \subset F(T_1) \cap F(T_2)$. Since

$$\frac{1}{n}\sum_{k=0}^{n-1} P^k \subset \prod_{x \in X} X_x \quad (X_x = X)$$

and $\prod_{x\in X} X_x$ is compact, there exists a subnet $\{(1/n_\alpha)\sum_{k=0}^{n_\alpha - 1} P^k\}_{\alpha \in A}$ of $\{(1/n)\sum_{k=0}^{n-1} P^k\}$ which converges to an element Q in $\prod_{x\in X} X_x$. Then, by equicontinuity of Σ, Q is affine and

continuous. Furthermore for each $x \in X$, we have

$$Qx - PQx = \lim_{\alpha}\left(\frac{1}{n_\alpha}\sum_{k=0}^{n_\alpha - 1} P^k x - P\left(\frac{1}{n_\alpha}\sum_{k=0}^{n_\alpha - 1} P^k x\right)\right)$$
$$= \lim_{\alpha} \frac{1}{n_\alpha}(x - P^{n_\alpha} x) = 0,$$

and hence Q mapps X onto $F(P)$. Now, assume that $y \in \operatorname{ex} F(P)$. Let $O(y) = \overline{\mathrm{co}}\{Ty\colon T \in \Sigma\}$ and

$$O_0(y) = \{y\},\ O_1(y) = \mathrm{co}\{T_1 y, T_2 y\},\quad O_2(y) = \mathrm{co}\{T_1^2 y, T_1T_2 y, T_2T_1 y, T_2^2 y\}, \ldots,$$

then $O(y) = \overline{\bigcup_{k\geq 0} O_k(y)}$. Since for each $k \geq 0$, $y = P^k y = (\frac{1}{2}T_1 + \frac{1}{2}T_2)^k y$, it is easily seen that for $x \in O_k(y)$ $(x \neq y)$, there exists $z \in O_k(y)$ such that $y = \lambda x + (1-\lambda) z$ for some λ $(0<\lambda<1)$. Let $x \in \bigcup_{k\geq 0} O_k(y)$ $(x \neq y)$, then there exists an integer $k \geq 0$ such that $x \in O_k(y)$. By the above, we have $z \in O_k(y)$ and λ $(0<\lambda<1)$ such that $y = \lambda x + (1-\lambda) z$. Then $Qy = \lambda Qx + (1-\lambda) Qz$. Since $y \in \operatorname{ex} F(P)$ and $Qx, Qz \in F(P)$, it follows that $y = Qy = Qx = Qz$. Consequently $Q\big(\bigcup_{k\geq 0} O_k(y)\big) = \{y\}$. By continuity of Q, $Q\big(O(y)\big) = \{y\}$. On the other hand, by hypothesis, there exists $x_0 \in O(y) \cap F(T_1) \cap F(T_2)$. Since $x_0 = Qx_0 = y$, we have $y \in F(T_1) \cap F(T_2)$. Therefore $\operatorname{ex} F(P) \subset F(T_1) \cap F(T_2)$.

Remark. In Theorem 2, we do not know whether "equicontinuous" can be replaced by "continuous".

3. Day fixed point theorem. Let X be a compact Hausdorff space. Then we denote by $C(X)$ the continuous real valued functions on X and by $M(X)$ all probability measures on X. Since to each norm one linear functional l on $C(X)$ such that $l(1) = 1$, there corresponds a unique probability measure $\mu \in M(X)$ such that $l(f) = \int f\, d\mu$ for each $f \in C(X)$, $M(X)$ is weak* compact. Roberts [6] considered a compact groupoid $(X, \cdot)$, i.e., X is compact Hausdorff and a continuous mapping $\cdot$ of $X \times X$ into X is defined. On this compact groupoid $(X, \cdot)$ we define a real valued function f to be convex if for every $x, y \in X$,

$$f(x\cdot y) \leq \tfrac{1}{2} f(x) + \tfrac{1}{2} f(y).$$

Let C be the family of all continuous convex functions on X and $\mathrm{core}(X)$ be the set of all elements of X such that $x \cdot x = x$. If $\mu, \nu \in M(X)$, then

$$l(f) = \int f(x\cdot y)\, d\mu(x) \times \nu(y)$$

defines a norm one linear functional l on $C(X)$ such that $l(1) = 1$. Hence $l(f) = \int f\, d\psi$ for some $\psi \in M(X)$. We shall denote the measure ψ by $\mu * \nu$. Define a map $S\colon M(X) \to M(X)$ by $S\mu = \mu * \mu$ for every $\mu \in M(X)$, then S is weak* continuous. Let $\mu \in M(X)$. Then the Bair sets in $X \times X$ are $\mu \times \mu$-measurable and the continuous functions on $X \times X$ are Bair measurable. Thus there exists a measure ν in $M(X \times X)$ such that for every $f \in C(X \times X)$,

$$\int f\,d\nu = \int f\,d\mu \times \mu$$

and

$$(\operatorname{supp} \nu) \supset (\operatorname{supp} \mu) \times (\operatorname{supp} \mu) .$$

THEOREM 3. *Let $(X, \cdot)$ be a compact groupoid whose C separates points in X. Let Σ be a family of continuous mappings on X satisfying the following conditions:*

(1) *There exists a Σ-invariant probability measure on X;*

(2) *$T(x \cdot y) = Tx \cdot Ty$ or $T(x \cdot y) = Tx \cdot y$ for $x, y \in X$ and $T \in \Sigma$.*

Then, we have an element $x \in \operatorname{core}(X)$ such that $Tx = x$ for every $T \in \Sigma$.

Proof. For $T \in \Sigma$, we define a Markov operator $\hat{T}$ on $C(X)$ by $\hat{T}f(x) = f(Tx)$. If $M_0(X) = \{\mu \in M(X)\colon \hat{T}^*\mu = \mu, T \in \Sigma\}$, then M_0 is weak* compact and convex. Since for $\mu \in M_0$ and $T \in \Sigma$,

$$\begin{aligned} \hat{T}^* S\mu(f) &= \int f\big(T(x \cdot y)\big)\,d\mu(x) \times \mu(y) \\ &= \int f(Tx \cdot Ty)\,d\mu(x) \times \mu(y) \\ &= \int f(x \cdot y)\,d\mu(x) \times \mu(y) = S\mu(f) , \end{aligned}$$

S is a weak* continuous mapping of M_0 into itself. Using the Tychonoff fixed point theorem [7], we obtain an element $\mu \in M_0$ such that $S\mu = \mu$. Suppose $a, b \in (\operatorname{supp} \mu)$ and $a \neq b$, then there exists $f \in C$ such that $f(a \cdot b) < \frac{1}{2}f(a) + \frac{1}{2}f(b)$. But then

$$\begin{aligned} \mu(f) = S\mu(f) &= \int f(x \cdot y)\,d\mu(x) \times \mu(y) \\ &< \int \big(\tfrac{1}{2}f(x) + \tfrac{1}{2}f(y)\big)\,d\mu(x) \times \mu(y) \\ &= \int f\,d\mu = \mu(f) . \end{aligned}$$

This is a contradiction. Therefore $(\operatorname{supp} \mu)$ must consist of a single point x. Since μ is the point measure δ_x, we obtain

$$\delta_x = S\delta_x = \delta_x * \delta_x = \delta_{x \cdot x}$$

and hence $x \in \operatorname{core}(X)$. Similarly, we can also prove the theorem for the case of which $T(x \cdot y) = Tx \cdot y$ for every $x, y \in X$ and $T \in \Sigma$.

COROLLARY (Day). *Let X be a compact convex subset of a locally convex topological vector space and Σ be a left amenable semigroup of continuous affine mappings of X into itself, then there exists an element $x \in X$ such that $Tx = x$ for all $T \in \Sigma$.*

Proof. Putting $x \cdot y = \frac{1}{2}x + \frac{1}{2}y$ for every $x, y \in X$, it follows that $T(x \cdot y) = Tx \cdot Ty$ for all $T \in \Sigma$. Furthermore, since Σ is amenable, there exists a Σ-invariant probability measure on X without using the Day fixed point theorem. In fact, define a functional μ on $C(X)$ by $\mu(f) = m_T(f(Tx))$, where m is a left invariant mean on Σ and $x \in X$. Then μ is a Σ-invariant probability measure on X. By Theorem 3, the family Σ has a common fixed point in $X = \operatorname{core}(X)$.

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