More Lusin properties in the product space S^n

by

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Abstract. The work of an earlier paper on the subject is extended to include a generalization of the concept of concentrated spaces that applies to the Cartesian product, S^n. Particular attention is paid to the properties of being hereditary, being preserved by continuous functions, and being preserved by unions.

1. Introduction. In an earlier paper [C] on this subject, generalizations of properties L and ν as found in Section 40 of [K] were introduced, and basic relationships about them were proved and several related examples were given. The purpose of the present paper is to go deeper into the subject by generalizing property P (i.e., concentrated about a countable dense subset of itself) in a similar fashion. We keep the notation of [C], and the definitions of properties C'' and C as found in Section 40 of [K] are referred to. We call spaces of universal measure 0 [L] β spaces. We assume that our spaces are hereditarily separable, regular, Hausdorff spaces. By n, we always mean a positive integer.

DEFINITION. S is P^n means that there exists a countable dense subset B of S such that S^n is concentrated $S^n-(S-B)^n$.

DEFINITION. S is P^∞ means that S is P^n for every n.

Note. We let $H\nu^n$, $H\nu^\infty$, HP^n, HP^∞ denote *hereditarily* ν^n, ν^∞, P^n, P^∞.

2. Basic theorems.

THEOREM 1. *Each of the following is true.*

(1) *If S is countable, S is $H\nu^\infty$.*

(2) *If S is ν^{n+1}, S is ν^n.*

(3) *If S is P^{n+1}, S is P^n.*

(4) *If S is ν^n, S is P^n.*

Proof. (1) and (4) are obvious. (2) and (3) have essentially the same argument, and that argument is given in Theorem 1 of [C]. ■

THEOREM 2. *If S is P^n (or HP^n) and f maps S continuously onto T, then T is P^n (or HP^n).*

Proof. We prove the theorem for SP^n. If S is HP^n, then if $T' \subseteq T$, let $S' = f^{-1}(T')$ and $f|S'$ maps S' continuously onto T' and S' is P^n.

Given $f\colon S \to T$, a continuous surjection, let $F\colon S^n \to T^n$ be defined by $F(s_1, \ldots, s_n) = (f(s_1), \ldots, f(s_n))$. F, likewise is a continuous surjection. Now letting B be countable dense in S such that S^n is concentrated about the B-grid, we show that T^n is concentrated about the $f(B)$-grid. As $f(B)$ is countable and dense in T, this will prove the theorem.

Suppose that O is open in T^n containing the $f(B)$-grid. $F^{-1}(O)$ is open in S^n and contains the B-grid. As $S^n - F^{-1}(O)$ is countable, $T^n - O$ is too. ■

Next, we investigate the analogue to the following propositions:

(1) If each S_i is dense in $T = \bigcup_{i=1}^{\infty} S_i$ and each S_i enjoys property ν, then so does T.

(2) If each S_i enjoys property P, then so does $T = \bigcup_{i=1}^{\infty} S_i$.

The reason for the density requirement in (1) is seen by the fact that under CH, uncountable ν subspaces of a Cantor set in R exist, yet when we union it with the rationals, which is ν, the result contains an uncountable, nowhere dense subset, and is not ν.

We start with a lemma.

LEMMA A. *Let T be second countable, and suppose that $S \subseteq T$ is $H\nu^n$ and $C \subseteq T$ is countable. Suppose further that if B is dense in $S \cup C$ and u is open* (rel S), *then there is $b \in B$ that is a point or limit point of u. Then $S \cup C$ is ν^n.*

Proof. By induction. When $n = 1$, $B \cap \mathrm{cl}(S)$ is dense in $\mathrm{cl}(S)$, hence S is concentrated about $B \cap \mathrm{cl}(S)$, so $S \cup C$ is concentrated about B.

Now suppose the result is true for $n-1$, but not for n. Let M be an uncountable closed subset of $(S \cup C)^n$ that misses the B-grid in $(S \cup C)^n$. Let $M' = M \cap (S \cup B)^n$, which is closed in $(S \cup B)^n$ and still uncountable, for the only points removed are in $\pi_i^{-1}(C-B)$ and M can only intersect $\pi_i^{-1}(c)$ (which is homeomorphic to $(S \cup C)^{n-1}$) in a countable set.

Let $\{U_0, U_1, U_2, \ldots\}$ be a basis for S. Pick $d_0 \in S \cap U_0$ such that $\{d_0\}^n$ misses M. If this were not possible, then $b \in B \cap \mathrm{cl}(U_0)$ is such that $(b, \ldots, b) \in M$. Next pick $d_1 \in S \cap U_1$ such that $\{d_0, d_1\}^n$ misses M. As before, such a point d_1 must exist. In general, pick $d_i \in S \cap U_i$ such that $\{d_0, \ldots, d_i\}^n$ misses M.

Let $D = \{d_0, d_1, d_2, \ldots\}$ which lies in S and is dense in S. For each $i \geqslant 0$, let

$$E_i = \bigcup_{j=1}^{n} \Big(\bigcup_{k=1}^{n} \pi_k(\pi_j^{-1}(d_i) \cap M') \Big) \quad \text{and} \quad E = \bigcup_{i=0}^{\infty} E_i - D.$$

E is countable since the sets, $\pi_j^{-1}(d_i) \cap M'$ are.

Let $S' = S - E$ and we show that S' is not ν^n, contradicting that S is $H\nu^n$. $D \subseteq S'$, $M' \cap S'^n$ is still uncountable and misses the D-grid. To see this, suppose $m \in M'$ and m is in the D-grid. Some coordinate of m is therefore in D, and in fact, every coordinate is in D, else such a coordinate would be in E. Thus $m \in D^n$, but $\{d_0, \ldots, d_i\}^n$ misses M for each i. ■

Now, for relatively little additional work, we can prove our theorem.

THEOREM 3. *Let T be second countable, and suppose that S is a dense subspace of T and that S is $H\nu^n$ and $C \subseteq T$ is countable. Then $S \cup C$ is $H\nu^n$.*

Proof. An arbitrary subspace of $S \cup C$ may be written as $S' \cup C'$ with $S' \subseteq S$ and $C' \subseteq C$. As S' has property ν, we assume that S' is somewhere dense in T, else S' is countable. Let O be the maximum open set in T such that $S' \cap O$ is dense in O. Thus $S' - O$ is nowhere dense in T and therefore countable.

Write $S' \cup C'$ as $S'' \cup C''$ where $S'' = S' \cap O$ and $C'' = C' \cup (S' - O)$. Lemma A is applicable and $S'' \cup C''$ is ν^n. Thus, $S \cup C$ is $H\nu^n$. ■

The proof for the analogous statement concerning P^n spaces is much easier.

THEOREM 4. *If $S \subseteq T$ is P^n (or HP^n) and $C \subseteq T$ is countable, then $S \cup C$ is P^n (HP^n).*

Proof. If S^n is concentrated about a B-grid in S^n, $(S \cup C)^n$ is concentrated about the $(B \cup C)$-grid in $(S \cup C)^n$. This also applies to $S' \cup C' \subseteq S \cup C$. ■

Remark. As weak as Theorems 3 and 4 are, we shall see (Example 7) that they are the best possible. With regard to Theorem 3, the requirement that S be $H\nu^n$ as opposed to ν^n is necessary, even to conclude that $S \cup C$ is ν^n, as shown by Theorem 7 of [C].

We end this section by giving two theorems that provide a foundation for later examples.

THEOREM 5. *If S is P^n, then S^n is C''.*

Proof. See [T, Corollary 2.5]. ■

THEOREM 6. *If $\bigoplus_{i=1}^{n} S = R$, then S^n does not have property C.*

Proof. This is more or less a generalization of [S]. It is proved using properties of the inner product space, R^n. Let L be the line in R^n containing $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1)$. Given a point $x \in R^n$, its projection, $P(x)$, onto L is the point with each coordinate $\Sigma x_i/n$. From some elementary inequalities, if x and y are in R^n, $\|P(x) - P(y)\| \leqslant \|x - y\|$.

Now suppose $\bigoplus_{i=1}^{n} S = R$. Let, for each i $d_i = 2^{-i}$. Given any sequence, $\langle x_1, x_2, \ldots \rangle$ of points from S^n, let $p_i = P(x_i)$ for each i. As some point, $(p, p, \ldots, p) \in L$ is not covered by the collection of neighbourhoods,

$$\{N(p_1, d_1), N(p_2, d_2), \ldots\},$$

and as some $x \in S^n$ projects onto $(p, p, \ldots, p)$ (namely, x such that $\Sigma x_i = np$), then because the neighbourhoods do not shrink through projection, x is not covered by $\{N(x_1, d_1), N(x_2, d_2), \ldots\}$. Therefore, S^n does not have property C. ■

3. Examples. The examples given here are similar to those found in Theorems 6 and 7 of [C]. In fact, Example 8 improves Theorem 6. We first define a useful condition and prove two lemmas. The reader should consult [C] for definitions of α, β, and γ. Although they are defined there for I^n, there is no problem in extending their meaning to an arbitrary S^n. The only possible confusion might be with α, where the change involves changing the condition that "x belongs to the segment (a, b)" to "x belongs to the open set u in S".

DEFINITION. S has property H_n means that if O is a dense open set in S^n and $\beta(O)$ contains no open (rel diag (S^n)) subset of diag (S^n), then $\gamma(O)$ is countable.

LEMMA B. *If S is H_n, S is ν^n.*

Proof. Let B be dense in S and O be open in S^n containing the B-grid. Then $\beta(O)$ contains no open subset of diag (S^n). Also, $S^n - O = \beta(O)$. $\alpha(O)$ is empty, for if $p \in \alpha(O)$, then one coordinate of p is allowed to move and the result remains in $\beta(O)$; this would put a point of the B-grid in $\beta(O)$, which cannot happen. Thus $\beta(O) = \gamma(O)$, so S is ν^n. ∎

Before proceeding, we direct the reader's attention to two useful properties regarding β, α and γ.

(1) If $S \subseteq T$ and S is dense in T and O' is open in S^n and O is open in T^n with $O \cap S^n = O'$, then $\beta(O') = \beta(O) \cap S^n$, $\alpha(O') = \alpha(O) \cap S^n$, and $\gamma(O') = \gamma(O) \cap S^n$. We use ambiguous notation. $\beta(O')$ refers to the boundary of O' *in* S^n and $\beta(O)$ to the boundary of O *in* T^n. Similar interpretations apply for α and γ.

(2) If $S \subseteq T$, O' is a dense open set in S^n and O is open in T^n such that $O \cap S^n = O'$ and also such that O contains $T^n - \mathrm{cl}(S^n)$, then $\beta(O') = \beta(O) \cap S^n$, and $\alpha(O) \cap S^n \subseteq \alpha(O')$, so $\gamma(O') \subseteq \gamma(O) \cap S^n$.

Both of these are easily verified.

LEMMA C. *If T is H_n and $S \subseteq T$, then S is H_n.*

Proof. Suppose O' is a dense open set in S^n for which $\beta(O')$ contains no open subset of diag (S^n). Let O be open in T^n such that $O \cap S^n = O'$ and also such that O contains $T^n - \mathrm{cl}(S^n)$. $\beta(O)$ fails to contain an open subset of diag (T^n), for supposing that v is open in T and if $x \in v$, $(x, x, \ldots, x) \in \beta(O)$, then $v \cap S$ is dense in v since O contains $T^n - \mathrm{cl}(S^n)$. So $v' = v \cap S$ is open relative to S and if $x \in v'$, $(x, x, \ldots, x) \in \beta(O')$.

Since T has H_n, $\gamma(O)$ is countable and by (2) above, $\gamma(O')$ is countable. ∎

Remark. The main reason for (1) is that we shall construct dense subspaces S, of R, looking at the open sets in R^n. We want to assume that by forcing countability conditions on $\gamma(O)$, for these open sets O, we have in fact forced countability conditions on $\gamma(O \cap S^n)$. As a matter of fact, Theorem 6 of [C] does adhere to this situation.

EXAMPLE 7 (CH). There exists two $H\nu^\infty$ subspaces, X and Y, of R, each uncountably dense in R, but such that $X \cup Y$ is not P^2.

Proof. Let $\{B_\theta\}$, $\theta < \omega_1$, list the countable dense subsets of R in such a way that each countable dense subset of R appears ω_1 times. For each n, list the dense open sets in R^n whose boundaries contain no open subset of diag (R^n), $\{O_\theta^n\}$, $\theta < \omega_1$.

Let $\langle u_0, u_1, u_2, \ldots\rangle$ be a sequence such that both $\{u_0, u_2, u_4, \ldots\}$ and $\{u_1, u_3, u_5, \ldots\}$ are bases for R. The even sequence will pertain to X, the odd one to Y.

Let h_0 be a homeomorphism from R onto R that maps B_0 into its complement. Let $F_0 = G_0$ be a first category in R set such that if $x \in R - F_0$, $\{x\}^n \cap \gamma(O_0^n) = \emptyset$ for each n. This is possible since none of the open sets have boundaries containing open subsets of diagonals. Now let $Q_0 = \big(B_0 \cup h_0(B_0) \cup h_0^{-1}(B_0) \cup h_0^{-1}(h_0^{-1}(B_0))\big)$ which is first category in R, and pick $x_0 \in u_0 - (Q_0 \cup F_0 \cup h_0^{-1}(G_0))$ and let $y_0 = h_0(x_0)$. The transfinite induction is now started.

For $\tau < \omega_1$, if $B_\tau = B_\eta$ for some $\eta < \tau$, let h_τ be the homeomorphism defined at ordinal η. Otherwise, let h_τ be a homeomorphism defined from R onto R that maps $\bigcup_{\sigma<\tau} \{x_\sigma, y_\sigma\} \cup B_\tau$ into its complement. Let F_τ be a first category in R set such that if $x \in R - F_\tau$, $\theta \leqslant \tau$, and n is a positive integer, $[(\bigcup_{\sigma<\tau} \{x_\sigma\} \cup \{x\})^n - (\bigcup_{\sigma<\theta} \{x_\sigma\})^n] \cap \gamma(O_\theta^n) = \emptyset$. (The argument for the existence of F_τ is very similar to one found in Lemma C of [C]. It is a combinatorial consideration of different first category sets, and thus omitted here.) Let G_τ be a similar set for the y_σ's. Let

$$Q_\tau = \bigcup_{\sigma\leqslant\tau} [B_\sigma \cup h_\sigma(B_\sigma) \cup h_\sigma^{-1}(B_\sigma) \cup h_\sigma^{-1}(h_\sigma^{-1}(B_\sigma))] \cup \\ \cup \bigcup_{\sigma<\tau} [h_\tau^{-1}(h_\sigma(B_\sigma)) \cup h_\tau^{-1}(h_\sigma^{-1}(B_\sigma))].$$

If $[\tau]$ is even, x_τ is picked from $u_{[\tau]}$. If $[\tau]$ is odd, x_τ is picked from $h_\tau^{-1}(u_{[\tau]})$. So pick x_τ in accordance with that rule and also such that $x_\tau \notin (Q_\tau \cup F_\tau \cup h_\tau^{-1}(G_\tau))$. Furthermore, pick x_τ outside $\bigcup_{\sigma<\tau} \{x_\sigma\}$ and such that $h_\tau(x_\tau)$ is not in $\bigcup_{\sigma<\tau} \{y_\sigma\}$ (i.e., x_τ is not equal to $h_\tau^{-1}(y_\sigma)$ for a previous σ). Let $y_\tau = h_\tau(x_\tau)$.

Let $X = \bigcup_{\tau<\omega_1} \{x_\tau\}$ and $Y = \bigcup_{\tau<\omega_1} \{y_\tau\}$, both of which are uncountably dense in R. Both spaces have property H_n for every n (recall Proposition (1) between Lemmas B and C). Thus X and Y are $H\nu^\infty$.

Now we show that $X \cup Y$ is not P^2. Suppose that B is countable and dense in $X \cup Y$. There is a first ordinal δ such that $B = B_\delta$. We are able to find an uncountable closed set in $(X \cup Y)^2$ that misses the B-grid, namely the graph of h_δ when intersected with $(X \cup Y)^2$. The graph is clearly closed, and has uncountable intersection with $X \times Y \subseteq (X \cup Y)^2$, because for each $\tau > \delta$ with $B_\tau = B_\delta$, (x_τ, y_τ) is a new point of the graph.

This uncountable closed set surely misses B^2. Now consider what happens if some $(b, z) \in (X \cup Y)^2$ is on this graph. Then $z = h_\delta(b)$, yet $h_\delta(b)$ is not in $X \cup Y$, for it could not be picked prior to the δ-level since h_δ maps B_δ away from $\bigcup_{\sigma<\delta} \{x_\sigma, y_\sigma\}$. It is not picked at the δ-level, for $h_\delta(b)$ would be either x_δ or y_δ, which means x_δ is in Q_δ. It is not picked after the δ-level, for $h_\delta(b) = x_\tau$ $(\tau > \delta)$ means $x_\tau \in h_\delta(B_\delta)$ and for $h_\delta(b) = y_\tau$ means $x_\tau \in h_\tau^{-1}(h_\delta(B_\delta))$, both of which were disallowed. Similar reasoning takes care of $(z, b) \in (X \cup Y)^2$. Thus, $X \cup Y$ is not P^2. ■

Remark. As X and Y are both dense in R, $X \cup Y$ is ν as well as P.

Remark. There does not appear to be the result that property C'' does not imply property P in the literature. X above has the property that X^2 is not P (for if B is countable in X^2, B misses $\pi^{-1}(x)$ for some $x \in X$ which is closed and uncountable), yet by Theorem 5, X^2 is C''.

The next example improves Theorem 6 of [C]. First we state two lemmas, very similar to Lemmas A and C of [C]. For this reason, the proofs given are lacking in details.

Lemma D. *If $t \in R$, then the subset P of R^{n+1}, consisting of all x such that $x_1 + \ldots + x_{n+1} = t$, has the property that if k is an integer, $1 \leqslant k \leqslant n$, and s is an increasing finite subsequence of $(1, 2, \ldots, n+1)$ with k terms and F is first category in $\Pi_{j \in s} R_j$, then $\pi_s^{-1}(F) \cap P$ is first category in P.*

Proof. It suffices to show that if F is closed and nowhere dense in $\prod_{j \in s} R_j$, then $\pi_s^{-1}(F) \cap P$ is closed and nowhere dense in P. This, in turn, follows from the fact that if U is an open rectangle in R^{n+1} and U intersects P, then $\pi_s(U \cap P)$ is open in $\prod_{j \in s} R_j$. To see that the latter is true, suppose $(p_{i_1}, \ldots, p_{i_k}) \in \pi_s(U \cap P)$ yet is a limit point of non-members of $\pi_s(U \cap P)$. Find a sequence, $\langle (q_{i_1}^y, \ldots, q_{i_k}^y) \rangle$ that converges to $(p_{i_1}, \ldots, p_{i_k})$ with y, but such that no point of this sequence is in $\pi_s(U \cap P)$. Let $(p_1, \ldots, p_{n+1}) \in P \cap U$ project to $(p_{i_1}, \ldots, p_{i_k})$. Let z be an integer, $1 \leqslant z \leqslant n+1$ such that $z \neq i_j$ for any j. Let, for $1 \leqslant m \leqslant n+1$, $m \neq z$,

$$r_m^y = \begin{cases} q_m^y & \text{if } m \text{ is term of } s, \\ p_m & \text{if } m \text{ is not}. \end{cases}$$

Let r_z^y solve $r_1^y + \ldots + r_z^y + \ldots + r_{n+1}^y = t$. For large y, $(r_1^y, \ldots, r_{n+1}^y) \in P \cap U$ yet projects to $(q_{i_1}^y, \ldots, q_{i_k}^y)$, which is a contradiction. ■

Lemma E. *Let $t \in R$. Let P be defined as above. Suppose that C is a countable subset of R (perhaps empty) and $C' \subseteq C$ and O is open in R^n and $C^n - C'^n$ misses $\gamma(O)$. Then there exists a first category subset F of P such that if $(x_1, \ldots, x_{n+1}) \in P - F$, then $(\{x_1, \ldots, x_{n+1}\} \cup C)^n - C'^n$ misses $\gamma(O)$.* ■

Proof. Almost identical to Lemma C of [C].

Example 8 (CH). Let n be a positive integer. There exists a subspace S of R which is uncountably dense in R such that S is $H\nu^n$ but $\bigoplus_{i=1}^{n+1} S = R$.

Proof. Let $\{u_0, u_1, \ldots\}$ be a basis for R. Arrange the dense open sets in R^n whose boundaries contain no open subset of diag (R^n): $\{O_\theta\}$, $\theta < \omega_1$. Arrange the elements of R: $\{t_\theta\}$, $\theta < \omega_1$. For each θ, let P_θ be the set P in Lemma D with $t = t_\theta$.

Apply Lemma E with $C = C' = \emptyset$ and $O = O_0$. Pick $(x_0^1, \ldots, x_0^{n+1})$ from P_0 such that $\{x_0^1, \ldots, x_0^{n+1}\}^n$ misses $\gamma(O_0)$. Furthermore, do it so that $x_0^1 \in u_0$.

For ordinal $\tau < \omega_1$, let $P = P_\tau$ and $C = \bigcup_{\sigma < \tau} \{x_\sigma^1, \ldots, x_\sigma^{n+1}\}$. For each $\theta \leqslant \tau$, apply Lemma E with $C' = \bigcup_{\sigma < \theta} \{x_\sigma^1, \ldots, x_\sigma^{n+1}\}$ to get $F = F_\theta$. As there are only countably many $\theta \leqslant \tau$, we get a first category subset of P_τ, the complement from which we pick $(x_\tau^1, \ldots, x_\tau^{n+1})$. We also make $x_\tau^1 \in u_{[\tau]}$ and $x_\tau^1 \notin \bigcup_{\sigma < \tau} \{x_\sigma^1\}$.

Let $S = \bigcup_{\tau < \omega_1} \{x_\tau^1, \ldots, x_\tau^{n+1}\}$ which is uncountably dense in R. Furthermore, $\bigoplus_{i=1}^{n+1} S = R$, and S has property H_n. ■

Remark. Coupling this example with Theorems 5 and 6 (and with $C'' \to C$), we have $H\nu^n$ does not imply P^{n+1}. We also refute the converses to (2) and (3) of Theorem 1.

Remark. Although it has appeared in the literature [MS] that $\beta \nrightarrow C$, and in fact, there are set theoretical differences between β and C [L], the above example (for $n = 1$) gives another example. That is, if S is ν, then it is easily verified that S^2 is β, yet if $S \oplus S = R$, S^2 is not C.

The last example refutes the converse to (4) of Theorem 1.

Example 9 (CH). There exists a HP^∞ subspace of R that is not ν.

Proof. We start with the space X of Example 7. We shall map this continuously onto a space that is not ν. In view of Theorem 2, this will give us our conclusion.

Write that space, X, as a countably infinite union of mutually exclusive open sets, $X = \bigcup_{i=1}^{\infty} O_i$. Let $\langle C_1, C_2, \ldots \rangle$ be a sequence of Cantor sets with the property that if u is open in R, then u contains some C_i. Continuously map, for each i, O_i into C_i in a 1-1 fashion. From a well known result on 0-dimensional sets, this is possible. The resulting map is continuous, yet the image contains lots of uncountable nowhere dense sets, and therefore is not ν. ■

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References

[C] G. V. Cox, *Lusin properties in the product space S^n*, Fund. Math. 108 (1980), pp. 199–210.
[K] K. Kuratowski, *Topology*, Vol. 1, New York–London–Warszawa 1966.
[L] R. Laver, *On the consistency of Borel's conjecture*, Acta Math. 137 (1976), pp. 151–169.

[MS] S. Mazurkiewicz and E. Szpilrajn, *Sur la dimension de certains ensembles singuliers*, Fund. Math. 28 (1937), pp. 305–308.

[S] W. Sierpiński, *Sur le produit combinatoire de deux ensembles jouissant de la propriété C*, Fund. Math. 24 (1935), pp. 48–50.

[T] R. Telgársky, *Concerning product of paracompact spaces*, Fund. Math. 74 (1972), pp. 153–159.

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A characterization of expandability of models for ZF to models for KM

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Abstract. In this paper we characterize KM-expandable and KM-non-β-expandable models by means of certain games. Also another characterization is given. It is proved that KM-expandability and KM-non-β-expandability are equivalent in a wide class of models for ZF.

§ 0. **Introduction.** The primary aim of our paper is the characterization, with the aid of a certain closed game, of those KM-expandable models for ZF whose height has a cofinality character equal to ω.

We characterize KM-expandable models in a way similar to that of Bieliński [3] in the case of countable models. We do this by means of approximations for recursive closed game formulas considered by Barwise in [1].

The investigation of properties of KM-expandable models was initiated by Marek and Mostowski in [6]. The authors focus their attention on KM-β-expandable models and give their full characterization. Among other things, they show that KM-expandability is not an elementary property in a 1-st order language. In that paper a characterization of KM-expandable standard models whose height has a cofinality character $>\omega$ is given. In fact, it is shown that any model for KM possessing such a set universe is automatically a β-model, hence KM-expandability can be reduced to KM-β-expandability in that case.

Let K be a language. By $K_{\infty\omega}$ we shall denote the class of all infinitary formulas of the language K. Let L_α: $\alpha \in \mathrm{Ord}$ be the hierarchy of constructible sets. By K_α we shall denote $K_{\infty\omega} \cap L_\alpha$ and by $\mathrm{ZF}^{\mathrm{KM}}_\infty$ the class of all formulas φ from language $(\mathscr{L}_{\mathrm{ZF}})_{\infty\omega}$ such that their relativization φ^V is a theorem in KM. By $\mathrm{ZF}^{\mathrm{KM}}_\alpha$ we shall understand the intersection of $\mathrm{ZF}^{\mathrm{KM}}_\infty$ and L_α. Note that for admissible $\alpha>\omega$,

$$\mathrm{ZF}^{\mathrm{KM}}_\alpha = \{\varphi \in (\mathscr{L}_{\mathrm{ZF}})_\alpha \colon L_\alpha \models (\mathrm{KM} \vdash \varphi^V)\}.$$

K. Bieliński, in [3], shows that although KM-expandability is not an elementary property (in language $(\mathscr{L}_{\mathrm{ZF}})_{\omega\omega}$), nevertheless it can be characterized in a uniform manner in the class of countable models $\underline{M}$ by a theory which is Σ_1 in $\mathrm{HYP}_{\underline{M}}$.

Namely: