and so there exists a natural number $p_0$ such that the set
\[ D = \bigcap_{N=1}^{\infty} \bigcup_{n=m+1}^{\infty} \{ x : |\mathcal{f}_{n}(x)| > 1/p_n \} \]
is of the second category. $D$ is obviously a set having the Baire property, and so $D = G \cap P$, where $G$ is open and non-empty and $P$ is of the first category. Let $K(x, r)$ be an open ball with the centre $x$ and the radius $r > 0$ included in $G$.

Put $H_{m,n} = \{ x : |\mathcal{f}_{n}(x)| > 1/p_n \}$ for $m, n \in N$. These sets have the Baire property, and so there exist open sets $\{G_{m,n}\}_{m,n}$ and the sets of the first category $\{P_{m,n}\}$ such that $H_{m,n} = G_{m,n} \triangle P_{m,n}$ for every $m, n \in N$.

It is not difficult to prove that for every natural $M$ the set $\bigcup \bigcup G_{m,n}$ is dense in $K(x, r)$.

Let $\{G_{k} : k \in N\}$ be a basis for a subspace topology in $K(x, r)$ such that $\{G_{k} : k = j, j+1, \ldots\}$ is also a basis for every $j \in N$.

The set $\bigcup \bigcup G_{m,n}$ is dense in $K(x, r)$, and so there exist natural numbers $m_1$ and $n_1$ such that $G_{m_1} \cap G_{m_1, n_1} \neq \emptyset$. The set $\bigcup_{m=m_1}^{\infty} \bigcup_{n=n_1}^{\infty} G_{m,n}$ is also dense in $K(x, r)$, and so there exist natural numbers $m_2 > m_1$ and $n_2 > n_1$ such that $G_{m_2} \cap G_{m_2, n_2} \neq \emptyset$. Proceeding in this way, we obtain two increasing sequences $\{m_k\}_{k=1}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $G_{m_k} \cap G_{m_k, n_k} \neq \emptyset$ for every $k \in N$.

Hence the set $\bigcup \bigcup G_{m,n}$ is dense in $K(x, r)$ and, moreover, for every $j \in N$, the set $G_{j} \cap \bigcup_{k=j}^{\infty} \bigcup_{n=n_{j}}^{\infty} G_{m,n}$ is also dense in $K(x, r)$. So, for every $j \in N$, the set $G_{j} \cap \bigcup_{k=j}^{\infty} \bigcup_{n=n_{j}}^{\infty} G_{m,n}$ is residual in $K(x, r)$ and from the fact that $H_{m,n} \supseteq \bigcup_{k=j}^{\infty} \bigcup_{n=n_{j}}^{\infty} G_{m,n}$ it follows that $\limsup_{k \to \infty} H_{m,n} = \emptyset$. Hence $\limsup_{k \to \infty} H_{m,n} = \emptyset$. But if $x \in \limsup_{k \to \infty} H_{m,n}$, then for $n \to \infty$, $\mathcal{f}_{n}(x)$ is not equal to zero — a contradiction. This ends the proof.

References


Accepted for publication 18. 12. 1979

Circularity of graphs and continua: topology

by

Harold Bell, Ezra Brown, R.F. Dickman, Jr., and E.L. Green*, (Blacksburg, Va.)

Abstract. A chain in a space $X$ is a finite collection $(K_1, \ldots, K_n)$ of distinct closed and connected subsets such that $K_i \cap K_j \neq \emptyset$ if and only if $|i-j| < 1$. A circular chain in $X$ is a collection $\mathcal{K}$ such that for any $X \in \mathcal{K}$, $X \in \mathcal{K}$ is a chain. For any locally connected, connected space $X$, $\mathcal{C}(X)$, the circularity of $X$ is defined by

$$m(X) = \sup \{ n : X \text{ can be represented as a union of } n \text{ circular chains} \}.$$

The circularity, $\mathcal{C}(G)$, of a finite connected graph $G$ is defined by

$$\mathcal{C}(G) = \sup \{ n : G \text{ can be represented as the union of a circular chain } \mathcal{K} \text{ in } G \}.$$

The principal results in this paper are: (1) If $G$ is a (planar) graph, then $G$ can be embedded in a (planar) Peano continuum $X$ with $\mathcal{C}(G) = m(X)$. (2) If $X$ is a planar Peano continuum, then $m(X)$ is infinite or even. (3) If $G$ contains a cycle, then $\mathcal{C}(G) > 6$ and if $G$ is planar, then $\mathcal{C}(G)$ is even. (4) The $G$ is one of the Kuratowski non-planar graphs, then $\mathcal{C}(G) = 6$. In another paper, Curvature of Graphs and Continua: Combinatorial, the authors develop combinational techniques for the evaluation of the circularity of graphs and show that for any integer $k \geq 6$, there exists a non-planar graph $G$ with $\mathcal{C}(G) = k$.

1. Introduction. Throughout this paper $X$ will denote a locally connected, connected normal space. For $A = X$, $\mathcal{A}(A)$ denotes the number of components of $A$ less one (or 0 if this number is infinite). The degree of multicoherence, $r(X)$, of $X$ is defined by

$$r(X) = \sup \{ \mathcal{A}(H \cap K) : H \cup K \text{ and } H \text{ and } K \text{ are closed and connected subsets of } X \}.$$

If $r(X) = 0$, $X$ is said to be unicoherent and we say that $X$ is multicoherent otherwise. $X$ is said to be finitely multicoherent if $0 < r(X) < \infty$. If this value is never attained, i.e. $\mathcal{A}(H \cap K) < \infty$ for representations $X = H \cup K$ as in $(\ast)$, $X$ is said to be weakly-finitely multicoherent. A. T. Stone has studied multicoherent spaces extensively [6, 7, 8, 9] and many authors have studied multicoherent spaces. Stone has raised several interesting questions concerning multicoherent spaces.

* The last author was partially supported by a grant from the National Science Foundation.
By a chain $\mathcal{X}$ in $X$ we mean a finite collection $\{K_1, ..., K_n\}$ of distinct closed and connected subsets of $X$ such that $K_i \cap K_j = \emptyset$ if and only if $|i-j| \leq 1$. A circular chain in $X$ is a collection of closed and connected sets $\mathcal{X}$ such that no three members of $\mathcal{X}$ have a point in common and if $K \in \mathcal{X}$, then $\mathcal{X} = \{K\}$ is a chain in $X$. Let $n > 2$ be an integer and let $S(n)$ denote the following statement:

$$S(n): \text{X is multicohherent if and only if X can be represented as the union of a circular chain containing exactly n elements.}$$

In a private communication A. H. Stone conjectured that $S(n)$ is true for all $n > 2$ and he stated that he had established $S(n)$ for all $n > 2$ when $X$ was finitely multicohherent and he announced $S(3)$ for locally connected, connected normal spaces in [6]. In [4], Dickman proved that $S(4)$ is true for compact spaces and he showed that $S(n)$ is true for $n > 2$ whenever $X$ is weakly-finitely multicohherent.

In [6], it was shown that $S(4)$ always obtains for a large class of spaces. Finally, Bell and Dickman gave an example of a plane multicohherent, one-dimensional Peano continuum $C$ for which $S(7)$ fails, i.e., $C$ could not be represented as the union of a circular chain of seven continua [3]. It was known from [5], that $C$ could be represented as the union of a circular chain with six members. The results led naturally to the following definition:

For any locally connected, connected space $X$, let $m(X)$, the circularity of $X$, be defined by:

$$m(X) = \sup \{n: X has a circular chain representation with n links\}.$$

In [3], the authors indicated that they were unable to construct an example of a plane Peano continuum where $m(X)$ is an odd integer, however, for any integer $k \geq 3$, they gave an example of a plane Peano continuum $C_k$ for which $m(C_k) = 2k$.

In the example of [3], for which $m(C) = 6$, the continuum $C$ was obtained as the closure of the union of a nested sequence of plane graphs $G_1, G_2, ..., G_{16}$ where $G_1$ is a triangle and $G_{1+i}$ is obtained by subdividing $G_i$ and adding certain new edges. This observation in turn, led to the definition of a new topological invariant for graphs.

For completeness we include the following standard definitions:

- A graph $G$ is a finite, nonempty set $V$ together with a set $E$ (disjoint from $V$) of two-element subsets of (distinct) elements of $V$. Each element of $V$ is called a vertex and $V$ itself is called the vertex set of $G$; the members of the set $E$ are called edges. In the present discussion, the topological realization of an abstract graph $G$ will also be denoted by $G$. Note that if $u$ and $v$ distinct vertices of $G$, then $u$ and $v$ are joined by one edge or no edges, i.e., there are no parallel edges. This restriction is for convenience only. It can be shown that the addition of a parallel edge to a graph does not change the invariant defined herein.

- For any graph $G$, define the circularity of $G$, $\sigma(G)$, by

$$\sigma(G) = \sup \{n: G can be covered by a circular chain of n elements each of which contains at least one vertex of G\}.$$
Definitions. Let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively, of a graph $G$. Let $u, v$ be members of $V(G)$ and define the distance $d_G(u, v)$ relative to $G$ as the length of the shortest path in $G$ from $u$ to $v$, i.e., the minimum number of edges contained in a path in $G$ from $u$ to $v$. Note $d_G(u, v) = 1$ if and only if $u$ and $v$ are adjacent in $G$. Let $J = [u, v_1, v_2, \ldots, v_n]$ be a cycle in $G$. We say that $J$ is a proper cycle if $d_G(V_i, V_j) = d_G(V_j, V_i)$ for all pairs $i, j$ (here we are considering $J$ to be a subgraph of $G$).

For any graph $G$, let $p(G)$ be the proper girth of $G$, be defined by:

$$p(G) = \max\{n \in N : G \text{ contains a proper cycle with } n \text{ vertices} \}.$$  

Theorem (2.4). For any connected graph $G$, $\sigma(G) \leq 2p(G)$.

Proof. Let $G = \{A_1, \ldots, A_n\}$ be a circular covering of $G$. Let $\varphi : G \to S^1 = \{z \in C : |z| = 1\}$ be any continuous mapping such that for each $i = 1, 2, \ldots, n$, $\varphi(A_i) \subseteq [e^{2\pi i}, e^{2\pi i}]$ and $\varphi^{-1}(e^{2\pi in}) = A_i \cap A_{i+1}$ (where we take $n+1 = 1$).

Let $J = [u_0, u_1, \ldots, u_n]$ be a cycle in $G$ of minimal length such that the topological index $\mu(J, 0) \neq 0$. It is clear that $J$ is a proper cycle. Since $\mu(J, 0) \neq 0$, it follows that $f(J) = S^1$. It follows now that $J$ must intersect each $A_i$. Since each vertex of $J_0$ lies in at most two $A_i's$, $\sigma(G) \leq 2p(G)$.

Corollary (2.5). If $G = [u, v_1, \ldots, v_n]$ is a cycle, then $\sigma(G) = 2n$.

Proof. Clearly $G$ can be represented as the union of 2n-halves edges each containing a vertex of $G$ and so $\sigma(G) \geq 2n$ then by (2.4), $\sigma(G) = 2n$.

Corollary (2.6). If $G$ is a complete graph on $n$ vertices, $n > 3$, then $\sigma(G) = 6$.

Proof. Since $G$ is complete, $p(G) = 3$ and so by (2.2) and (2.4), $\sigma(G) = 6$.

III. Embedding graphs in continua.

Lemma (3.1) (Lemma 3 of [2]). Let $X$ be a locally connected, connected normal space and let $a, b$ be different points of $X$ such that $X \setminus \{a, b\}$ contains (at least) three distinct non-empty components, $R, P$ and $Q$ with $d_R \cap d_P \cap d_Q = \{a, b\}$. Then if $\theta$ is a circular cover of $X$ and some $C \subseteq \theta$ lies entirely in $R$, then $C$ contains all but at most four members of $\theta$.

Theorem (3.2). Let $\theta$ be any (planar) graph. Then $\theta$ can be embedded in a (planar) Peano continuum $Y$ with $\sigma(Y) = \{m(X)\}$.

Proof. If $\sigma(Y) = 6$, then by (2.3), $G$ is a tree and $\sigma(G) = 2$. In this case, let $X = G$. Thus we assume that $\sigma(G) > 6$. In [1] a planar Peano continuum $C$ was constructed with $\sigma(C) = 6$. The continuum $C$ contains a closed interval $[a, b]$ where $C \cap (a, b)$ has exactly three components $P, Q$ and $R$ and $d_P \cap d_Q \cap d_R = \{a, b\}$. Let $X$ be the (planar) continuum obtained by replacing each edge $e$ in $G$ with a copy, $C_\epsilon$, of $C$ in the natural way, so that each edge $e$ of $G$ is identified with a copy, $[a, b]_\epsilon$, of $[a, b]$. Clearly every circular covering of $G$ can be extended, link by link, to a circular covering of $X$. Thus $\sigma(Y) = m(X)$.

Let $K, K_1, K_2$ be a circular covering of $X$. If some $K_i$ contains no vertex of $G$, then a component of some $C_i \setminus \{a_i, b_i\}$ contains $K_i$. Then by the proof of Theorem 1 of [1], $m(X) = 6 = \sigma(G)$.

If on the other hand if every $K_i$ contains a vertex of $G$, we write each $e = u, v_a \in E(G)$ as the union of two non-degenerate arcs, $[u_a, v_a]$ and $[v_a, v_b]$ with $[u_a, v_a] \cap [v_a, v_b] = \{v_a\}$ and let

$$K_i = \bigcup \{[u, v_a], [v_a, v_b] \in E(G) \text{ and } u_a \in K_i \text{ and } v_b \in (K_{i-1} \cup K_i \cup K_{i+1}) \},$$

for $i = 1, 2, \ldots, n$ (where we take $n+1 = 1$). Then $\{K_1', K_2', \ldots, K_n'\}$ is a circular covering of $G$ and $\sigma(G) = m(X)$. This completes the proof.

Theorem (3.3). Let $X$ be a multicoherecent Peano continuum. Then

$$m(X) = \sup \{m(P) : P \text{ is a cyclic element of } X \}.$$  

Proof. Let $P$ be any cyclic element of $X$ and let $\{A_1, A_2, \ldots, A_n\}$ be a circular covering of $P$ where $m(P) = n$. By Proposition (2.1) of [1], $P$ is the union of $A_i$, together with the components of $X \setminus P$ with limit points in $A_i$. Then $\{B_1, B_2, \ldots, B_n\}$ is a circular covering of $X$. Hence $m(P) \leq m(X)$.

Suppose $C_1, \ldots, C_n$ is a circular covering of $X$. Now $X$ contains a simple closed curve $J$ that meets each $A_i$. Let $Q$ be the cyclic element of $X$ that contains $J$. By Propositions (3.5) and (3.1) of [1], each of the sets $C_i \cap Q$ is connected. Then $Q = \bigcup \{C_i \cap Q\}$ is a circular covering of $Q$. It then follows that $m(X) = \sup \{m(P) : P \text{ is a cyclic element of } X \}$.

Definition (3.4). A graph $G$ is separable if it is the union of two non-degenerate subgraphs $H_1$ and $H_2$ such that $H_1$ and $H_2$ have exactly one vertex in common. We say that $G$ is non-separable if it can not be so represented.

Theorem (3.5) [10, Thm 12]. A graph $G$ may be decomposed into its non-separable components in a unique manner.

Theorem (3.6). Let $G$ be any graph with $\sigma(G) \geq 6$. If $G_1, G_2, \ldots, G_n$ are the non-separable components of $G$, then $\sigma(G) = \max\{\sigma(G_1), \sigma(G_2), \ldots, \sigma(G_n)\}$.

Proof. The proposition follows from (3.2) and (3.3).

IV. Planar graphs and continua.

Lemma (4.1). Let $P$ be a plane continuum, let $\mathcal{B}$ be the set of bounded complementary domains of $P$ and let $T(P) = P \cup \Big( \bigcup \{B : B \in \mathcal{B} \} \Big)$. If $\varphi : P \to S^1$ is continuous, then $\varphi$ extends continuously to a map $\psi : T(P) \to S^1$ if and only if $\varphi$ extends continuously to a map $\varphi_1 : P \cup \{D \in \mathcal{B} : \varphi(0) \cap D \neq \emptyset \}$. For each $D \in \mathcal{B}$,

Proof. By the Tietze Extension Theorem, $\varphi$ extends continuously to a map $\varphi_1 : T(P) \to \{z \in C : |z| = 1\}$. Let $B_\delta = \{D \in \mathcal{B} : \varphi^{-1}(0) \cap D \neq \emptyset \}$. For each $D \in \mathcal{B}$,
Let $\varphi_0$ be a continuous extension of $\varphi$ to a map $\varphi_0: P \cup D \to S^1$. Then, since $\Delta_1$ is a finite set, the function $\varphi: T(P) \to S^1$ defined by

$$
\varphi(x) = \begin{cases} 
\varphi_0(x) & \text{if } x \in D \in \Delta_1, \\
\varphi_0(x) & \text{if } x \in \Delta_1 \cup (P \setminus \Delta_1)
\end{cases}
$$

is continuous.

**Lemma 4.2.** Let $\{A_1, A_2, \ldots, A_n\}, n \geq 2$, be a circular covering of the plane multi-coherent continuum $X$. Then there exists an embedding of $X$ in the plane such that the boundary of some bounded complementary domain of $X$ and the boundary of the unbounded complementary domain of $X$ intersect each $A_i$, $i = 1, 2, \ldots, n$.

Proof: For each $i \in \{1, 2, \ldots, n\}$, let $Z_i = \{z \in C: z = e^{i\theta}; \theta \in (i-1)\pi/n, i\pi/n\}$ and let $\varphi: X \to S^1$ be any continuous map of $X$ onto $S^1$ such that $\varphi(A_i) = Z_i$. Clearly $\varphi$ is not null-homotopic, so by Lemma 4.1, there must be a bounded complementary domain $Q$ so that $\varphi|Q = S^1$. Thus $\varphi|Q$ is a null-homotopy. Then $\varphi|Q$ meets every $A_i$. Now let $p \in Q$ and embed $X$ in the plane so that $Q - \{p\}$ is homeomorphic to the unbounded complementary domain of $X$. Another application of Lemma 4.1 to this embedding assures us that there is a bounded complementary domain $C$ of this embedding such that $\varphi|C = S^1$ and this completes the proof.

**Lemma 4.3.** Let $Y$ be the union of $k = (2n-1), n \geq 2$, simple arcs $\{A_1, A_2, \ldots, A_n\}$ with common endpoints $a$ and $b$ and such that $A_i \cap A_j \neq \emptyset$ if and only if $i, j \in \{1, 2, \ldots, n\}$ (where we take $n+1 = 1$). Then $Y$ contains a collection of simple arcs from $a$ to $b$ $S_1, S_2, \ldots, S_n$ such that $S_1 \cap S_i = \emptyset$ for $i \neq j, 1 \leq i, j \leq n$.

Proof: By our hypothesis no collection of fewer than $n$ points of $Y$ separates $a$ from $b$ in $Y$. The proposition then follows from Whyburn's $n$-arc Connectedness Theorem [12].

**Theorem 4.4.** If $X$ is any multi-coherent planar Peano continuum and $m(X) < \infty$, then $m(X) = 1$.

Proof: By (3.3) we may assume that $X$ is cyclic. Suppose that $m(X) = k = 2n-1$ is odd where $n \geq 3$ (recall that $m(X) > 0$). Let $\{A_1, A_2, \ldots, A_n\}$ be a circular covering of $X$ where each $A_i$ is a Peano continuum. (We may choose the $A_i$'s to be locally connected by Theorem 15.4 of [11], p. 21).

By Lemma 4.2, we may suppose that $X$ is embedded in the plane so that the boundary $Y$ of some bounded complementary domain $Q$ of $X$ (Fr $Q = \emptyset$) and the boundary $Y$ of the unbounded complementary $U$ of $X$ (Fr $U = \emptyset$) each meet every $A_i$, i.e., $0 \in A_i \neq \emptyset \not\cap A_i$ for $i = 1, \ldots, k$. By (2.3) of [11], p. 107, 0 and $Y$ are simple closed curves.

---

(1) The authors express their appreciation to H. Bell and W. Burton Jones for pointing out that the $n$-arc connectedness theorem could be employed in this proof.

---

Let $p \in U$ and $q \in Q$. By the Schoenflies Theorem (or Theorem 4.4) of [11, p. 113] we may select a collection $\{M_1, \ldots, M_n\}$ of simple arcs from $p$ to $q$ such that $M_i \cap M_j \cup \{p, q\} = \emptyset$ for $i \neq j$ and for each $i$, $M_i \cap X$ is a simple arc (possibly degenerate) in $A_i$. Then by Lemma 4.2, contains a collection of $S_1, S_2, \ldots, S_n$ of simple arcs such that $S_i \cap S_j = \emptyset$ for $i \neq j, 1 \leq i, j \leq n$. By re-ordering the $S_i$'s if necessary we may assume that if $B_j$ is the closed ball bounded by $S_i \cup S_j$, $i = 1, n-1$ and $B_n$ is the closure of the unbounded complementary domain of $S_1$ and $S_n$, then $B_1 \cap B_i = \emptyset$ if $i \neq j$, and $B_{i-1} \cap B_i = S_i$ for $i = 2, 3, \ldots, n$ and $B_0 \cap B_i = S_i$.

We next note that $X \cap B_i$ is connected. First of all, by our construction $S_1 \cap X = T_1$ simple arc for each $i = 1, \ldots, n$ and every point of $X \cap B_i$ can be joined to $S_1 \cap X \cup (S_2 \cap X)$ by a simple arc lying in $X \cap B_i$. It remains to show that some component of $X \cap B_i$ meets both $S_1 \cap X$ and $S_2 \cap X$. Now $X$ separates $p$ and $q$ in the plane and so $X \cap B_i$ separates $p$ and $q$ in $B_i$. Since $B_i$ is unicoherent, some component $C$ of $X \cap B_i$ separates $p$ and $q$ in $B_i$. Consequently $S_1 \cap S_i \cup C \not\cap C$ and thus $X \cap B_i$ is connected. It then follows by a similar proof, that $X \cap B_i$ is connected for $i = 1, \ldots, n$.

Now by Whyburn's Separation Theorem (see Lemma 3 of [4]), $X \cap B_i$ can be represented as the union of two continua $D_1^i$ and $D_2^i$ where $(S_1 \cap X) \cup (S_2 \cap X)$ and $S_1 \cap (D_1^i \cap D_2^i)$. Then $D_1^i, D_2^i, D_1^i \cup D_2^i \cup D_1^j, D_2^j$ is a circular covering of $X$ and so $m(X) \geq 2n$. This contradicts our assumption that $m(X) = 2n-1$ and this completes the proof.

**Corollary 4.5.** If $G$ is any planar graph, then $\sigma(G)$ is even.

Proof: This follows from (3.2) and (4.4).

**Example 4.6.** Here we describe an example of a non-planar graph $G$ with exactly 7 vertices and for which $\sigma(G) = 7$. The proof that $\sigma(G) = 7$ will appear in Circularity of graphs and continua: combinatorics.

Let $G_2$ be a cycle with seven vertices $v_1, v_2, \ldots, v_7$ so that $v_1$ and $v_7$ are adjacent. For $i = 1, 2, \ldots, 6$ and $v_i$ and $v_{i+1}$ are adjacent. We add the following edges to $G_2$: $v_1 v_4, v_2 v_5, v_3 v_6, v_2 v_4, v_5 v_6, v_1 v_3, v_2 v_3$ so as to obtain $G$. Now $G$ is non-planar since it contains a homeomorphic copy of $K_{3,3}$. [1]

Since $\sigma(G) = 7$, by Theorem 3.2 there exists a non-planar Peano continuum $X$ containing $G$ such that $m(X) = 7$.

**References**


Spaces of order arcs in hyperspaces

by

Carl Eberhart, Sam B. Nadler, Jr. * and William O. Nowell, Jr. (Lexington, Ky.)

Abstract. Let $X$ be a metric continuum and let $2^X$ and $C(X)$ denote respectively the space of closed subsets and the space of subcontinua of $X$ topologized with the Hausdorff metric. An order arc in $2^X (C(X))$ is an arc $a$ contained in $2^X (C(X))$ such that if $A, B \in a$, then $A \subseteq B$ or $B \subseteq A$. Let $\Gamma(2^X)$ ($\Gamma(C(X))$) denote the space of order arcs in $2^X (C(X))$ together with the singletons $\{A\}$, $A \in 2^X (C(X))$, topologized with the Hausdorff metric on $2^X$. In this paper we prove that if $X$ is locally connected, then $\Gamma(2^X)$ is homeomorphic with the Hilbert cube $Q$ and if, in addition, $X$ contains no arc with interior, then $\Gamma(C(X))$ is homeomorphic with $Q$.

1. Introduction. Let $X$ be a continuum (i.e., a compact connected metric space containing more than one point). The hyperspace of $X$ are the spaces $2^X$ consisting of all nonempty closed subsets of $X$, and $C(X)$, consisting of the connected elements in $2^X$, each with the Hausdorff metric $H$. Basic facts about hyperspaces may be found in [13] and [9].

An order arc in $2^X$ (resp., $C(X)$) is an arc $a \subseteq 2^X$ (resp. $a \subseteq C(X)$) such that if $A, B \in a$, then $A \subseteq B$ or $B \subseteq A$. Order arcs in hyperspaces were first constructed in [2], as a part of the proof of the following:

1.1. Theorem. For any continuum $X$, $2^X$ and $C(X)$ are each arcwise connected continua.

However, the fact that the construction in [2] yielded an arc was not noted until later in [10, Lemma 5]. Since the publication of these two papers, order arcs have been used extensively in studying hyperspaces. However, spaces of order arcs have undergone almost no investigation. In this paper we investigate the spaces

\[ \Gamma(2^X) = \{ a \subseteq 2^X : a \text{ is an order arc} \} \cup \{ \{A\} : A \in 2^X \} \]

and

\[ \Gamma(C(X)) = \{ a \subseteq C(X) : a \text{ is an order arc} \} \cup \{ \{A\} : A \in C(X) \} \]

* The second author was partially supported by National Research Council (Canada) grant no. A5616.