and so there exists a natural number p_0 such that the set

$$D = \bigcap_{M=1}^{\infty} \bigcup_{m=M+1}^{\infty} \bigcup_{n=M+1}^{\infty} \{x: |f_{m,n}(x)| > 1/p_0\}$$

is of the second category. D is obviously a set having the Baire property, and so $D = G \Delta P$, where G is open and non-empty and P is of the first category. Let $K(x, r)$ be an open ball with the centre x and the radius $r>0$ included in G.

Put $H_{m,n} = \{x: |f_{m,n}(x)| > 1/p_0\}$ for $m, n \in N$. These sets have the Baire property, and so there exist open sets $\{G_{m,n}\}_{m,n\in N}$ and the sets of the first category $\{P_{m,n}\}_{m,n\in N}$ such that $H_{m,n} = G_{m,n} \Delta P_{m,n}$ for every $m, n \in N$.

It is not difficult to prove that for every natural M the set $\bigcup_{m=M+1}^{\infty} \bigcup_{n=M+1}^{\infty} G_{m,n}$ is dense in $K(x, r)$.

Let $\{G_k: k \in N\}$ be a basis for a subspace topology in $K(x, r)$ such that $\{G_k: k = j, j+1, \ldots\}$ is also a basis for every $j \in N$.

The set $\bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty} G_{m,n}$ is dense in $K(x, r)$, and so there exist natural numbers m_1 and n_1 such that $G_1 \cap G_{m_1,n_1} \neq \emptyset$. The set $\bigcup_{m=\max(m_1,n_1)+1}^{\infty} \bigcup_{n=\max(m_1,n_1)+1}^{\infty} G_{m,n}$ is also dense in $K(x, r)$, and so there exist natural numbers $m_2 > m_1$ and $n_2 > n_1$ such that $G_2 \cap G_{m_2,n_2} \neq \emptyset$. Proceeding in this way, we obtain two increasing sequences $\{m_k\}_{k\in N}$ and $\{n_k\}_{k\in N}$ of natural numbers such that $G_k \cap G_{m_k,n_k} \neq \emptyset$ for every $k \in N$. Hence the set $\bigcup_{k=1}^{\infty} G_{m_k,n_k}$ is dense in $K(x, r)$ and, moreover, for every $j \in N$, the set $\bigcup_{k=j}^{\infty} G_{m_k,n_k}$ is also dense in $K(x, r)$. So, for every $j \in N$, the set $\bigcup_{k=j}^{\infty} G_{m_k,n_k}$ is residual in $K(x, r)$ and from the fact that $\bigcup_{k=j}^{\infty} H_{m_k,n_k} \supset \bigcup_{k=j}^{\infty} G_{m_k,n_k} - \bigcup_{k=j}^{\infty} P_{m_k,n_k}$ it follows that $\limsup_k H_{m_k,n_k}$ is residual in $K(x, r)$. Hence $\limsup_k H_{m_k,n_k} \notin \mathcal{I}$. But if $x \in \limsup_k H_{m_k,n_k}$, then $\lim_{k\to\infty} f_{m_k,n_k}(x)$ is not equal to zero — a contradiction. This ends the proof.

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Circularity of graphs and continua: topology

by

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Abstract. A chain in a space X is a finite collection $\{K_1, \ldots, K_n\}$ of distinct closed and connected sets such that $K_i \cap K_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A circular chain in X is a collection $\mathcal{K}$ such that for any $K \in \mathcal{K}$, $\mathcal{K} - \{K\}$ is a chain. For any locally connected, connected space X, $m(X)$, the *circularity* of X is defined by

$$m(X) = \sup\{n: X \text{ can be represented as a union of a circular chain with exactly } n \text{ elements}\}.$$

The circularity, $\sigma(G)$, of a finite connected graph G is defined by

$$\sigma(G) = \sup\{n: G \text{ can be represented as the union of a circular chain } \mathcal{K} \text{ in } G \text{ such that every member of } \mathcal{K} \text{ contains at least one vertex of } G\}.$$

The principal results in this paper are: (1) if G is a (planar) graph, then G can be embedded in a (planar) Peano continuum X with $\sigma(G) = m(X)$. (2) If X is a planar Peano continuum, then $m(X)$ is infinite or even. (3) If G contains a cycle, then $\sigma(G) \geqslant 6$ and if G is planar, then $\sigma(G)$ is even. (4) The G is one of the Kuratowski non-planar graphs, then $\sigma(G) = 6$.

In another paper, Circularity of Graphs and continua: Combinatorics, the authors develope combinatorial techniques for the evaluation of the circularity of graphs and show that for any integer $k \geqslant 6$, there exists a non-planar graph G_k with $\sigma(G_k) = k$.

1. Introduction. Throughout this paper X will denote a locally connected, connected normal space. For $A \subset X$, $b_0(A)$ denotes the number of components of A less one (or ∞ if this number is infinite). The *degree of multicoherence*, $r(X)$, of X is defined by

$$(*) \quad r(X) = \sup\{b_0(H \cap K): X = H \cup K \text{ and } H \text{ and } K \text{ are closed and connected subsets of } X\}.$$

If $r(X) = 0$, X is said to be *unicoherent* and we say that X is *multicoherent* otherwise. X is said to be *finitely multicoherent* if $0 < r(X) < \infty$. If this value is never attained, i.e. $b_0(H \cap K) < \infty$ for representations $X = H \cup K$ as in $(*)$, X is said to be *weakly-finitely multicoherent*. A. H. Stone has studied multicoherent spaces extensively [6, 7, 8, 9] and many authors have studied unicoherent spaces. Stone has raised several interesting questions concerning multicoherent spaces.

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By a *chain* $\mathscr{K}$ in X we mean a finite collection $\{K_1, \ldots, K_n\}$ of distinct closed and connected subsets of X such that $K_i \cap K_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A *circular chain* in X is a collection of closed and connected sets $\mathscr{K}$ such that no three members of $\mathscr{K}$ have a point in common and if $K \in \mathscr{K}$, then $\mathscr{K} - \{K\}$ is a chain in X. Let $n > 2$ be an integer and let $S(n)$ denote the following statement:

$S(n)$: X is multicoherent if and only if X can be represented as the union of a circular chain containing exactly n elements.

In a private communication A. H. Stone conjectured that $S(n)$ is true for all $n > 2$ and he stated that he had established $S(n)$ for all $n > 2$ when X was finitely multicoherent and he announced $S(3)$ for locally connected, connected normal spaces in [6]. In [4], Dickman proved that $S(6)$ is true for compact spaces and he showed that $S(n)$ is true for $n > 2$ whenever X is weakly-finitely multicoherent. In [6], it was shown that $S(4)$ always obtains for a large class of spaces. Finally, Bell and Dickman gave an example of a plane multicoherent, one-dimensional Peano continuum C for which $S(7)$ fails, i.e. C could not be represented as the union of a circular chain of seven subcontinua [3]. It was known from [5], that C could be represented as the union of a circular chain with six members. The results led naturally to the following definition:

For any locally connected, connected space X, let $m(X)$, the *circularity* of X, be defined by:

(**) $\quad m(X) = \sup\{n\colon X \text{ has a circular chain representation with } n \text{ links}\}$.

In [3], the authors indicated that they were unable to construct an example of a plane Peano continuum where $m(X)$ is an odd integer, however, for any integer $k \geqslant 3$, they gave an example of a plane Peano continuum C_k for which $m(C_k) = 2k$. In the example C of [3], for which $m(C) = 6$, the continuum C was obtained as the closure of the union of a nested sequence of planar graphs $G_1, G_2, \ldots, G_n, \ldots$ where G_1 is a triangle and G_{i+1} is obtained by subdividing G_i and adding certain new edges. This observation in turn, led to the definition of a new topological invariant for graphs.

For completeness we include the following standard definitions:

A *graph* G is a finite, nonempty set V together with a set E (disjoint from V) of two-element subsets of (distinct) elements of V. Each element of V is called a *vertex* and V itself is called the *vertex set* of G; the members of the *edge set* E are called *edges*. In the present discussion, the topological realization of an abstract graph G will also be denoted by G. Note that if u and v distinct vertices of G, then u and v are joined by one edge or no edges, i.e. there are no parallel edges. This restriction is for convenience only. It can be shown that the addition of a parallel edge to a graph does not change the invariant defined herein.

For any graph G, define the *circularity* of G, $\sigma(G)$, by

(***) $\quad \sigma(G) = \sup\{n\colon G \text{ can be covered by a circular chain of } n \text{ elements each of which contains at least one vertex of } G\}$.

Note that each $A \in \mathscr{A}$ is a closed and connected subset of the topological realization of G and thus is not necessarily a union of whole edges and vertices of G. Frequently we will call the circular chain representation $\mathscr{A} = \{A_1, A_2, \ldots, A_n\}$ of a continuum (or a graph) X, a circular covering if $X = \bigcup_{i=1}^{n} A_i$, $A_i \cap A_{i+1} \neq \emptyset$ for $i = 1, 2, \ldots, n$ (where we take $n+1 = 1$), and no three of the A_i's have a point in common (and each A_i contains at least one vertex of the graph).

Since for a large class of continua, the circularity is infinite, (i.e. for any weakly-finitely multicoherent space X, $m(X) = \infty$) the study of the circularity of graphs is inherently more interesting than the study of the circularity of continua. However the two concepts are intimately related. For example, in this paper we prove that if G is any graph, there is a Peano continuum X containing G such that $m(X) = \sigma(G)$.

The principal result of this paper is that the circularity of any planar Peano continuum (respectively, planar graph) is always even or infinite (respectively, always even). Furthermore, we establish some basic results concerning the circularity of graphs and calculate the circularity of several classes of graphs.

In another paper, *Circularity of graphs and continua: combinatorics*, we continue our investigations and show that for any $n \geqslant 3$, there exist a planar graph with circularity $2n$ and a non-planar graph with circularity $2n+1$. The techniques of the second paper are combinatorial in nature, whereas the techniques of this paper are principally topological. Definitions not given herein may be found in [1] or [11].

II. Basic results-circularity of graphs. Hereafter we shall only consider (finite) connected graphs with at least two vertices.

LEMMA (2.1). *For any graph G, $\sigma(G) \geqslant 2$.*

Proof. Let $A_1 = G = A_2$. Then $\mathscr{A} = \{A_1, A_2\}$ is the desired circular covering.

THEOREM (2.2). *Let G be a graph that contains a cycle. Then $\sigma(G) \geqslant 6$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices of a cycle in G, $n \geqslant 3$, with v_1 and v_2 adjacent. Let $\{L_1, L_2, L_3\}$ be a maximal collection of mutually disjoint continua such that $v_1 \in L_1$, $v_2 \in L_2$ and $\{v_3, \ldots, v_n\} \subset L_3$ and such that each L_i, $i = 1, 2, 3$, is the union of edges and vertices of G. (Here maximal means that if $\{L_1', L_2', L_3'\}$ is any other such collection and $L_1 \subset L_1'$, $L_2 \subset L_2'$ and $L_3 \subset L_3'$, then $L_i = L_i'$, $i = 1, 2, 3$.) Note that every edge in $G/\{L_1, L_2, L_3\}$ has its vertices in exactly two of the L_i's. Thus for each edge $e = uv$ of G let $m(e)$ be a point of e distinct from u and v and for $i, j = 1, 2, 3$, $i \neq j$, let

$$K_{ij} = \bigcup \{[u, m(e)]\colon e = uv \in E(G), u \in L_i \text{ and } v \in L_j \cup L_i\}.$$

Then $\{K_{ij}\}$ forms a circular chain of continua with 6 members and $\sigma(G) \geqslant 6$.

COROLLARY (2.3). *If G is a graph, then $\sigma(G) = 2$ if and only if G is acyclic, i.e. G is a tree.*

Proof. The result follows immediately from (2.1), (2.2) and the observation that trees are unicoherent.

DEFINITIONS. Let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively, of a graph G. Let u, v be members of $V(G)$ and define the *distance* $d_G(u, v)$ relative to G as the length of the shortest path in G from u to v, i.e. the minimum number of edges contained in a path in G from u to v. Note $d_G(u, v) = 1$ if and only if u and v are adjacent in G. Let $J = \{v_0, v_1, \ldots, v_n\}$ be a cycle in G. We say that J is a *proper cycle* if $d_G(V_i, V_j) = d_J(V_i, V_j)$ for all pairs i, j (here we are considering J to be a subgraph of G.)

For any graph G, let $p(G)$ the *proper girth* of G, be defined by:

$$p(G) = \max\{n \in N\colon\ G \text{ contains a proper cycle with } n \text{ vertices}\}.$$

THEOREM (2.4). *For any connected graph G, $\sigma(G) \leqslant 2p(G)$.*

Proof. Let $\sigma(G) = n$ and let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be a circular covering of G. Let $\varphi\colon G \to S^1 = \{z \in \mathbb{C}\colon |z| = 1\}$ be any continuous map such that for each $i = 1, 2, \ldots, n$, $\varphi(A_i) \subseteq \{e^{i\theta}\colon 2\pi(i-1)/n \leqslant \theta \leqslant 2\pi i/n\}$ and $\varphi^{-1}(e^{2\pi i/n}) = A_i \cap A_{i+1}$ (where we take $n+1 = 1$).

Let $J = \{v_0, v_1 \ldots, v_q\}$ be a cycle in G of minimal length such that the topological index $\mu(f\varphi J, 0) \neq 0$. It is cleary that J is a proper cycle. Since $\mu(f\varphi J, 0) \neq 0$ it follows that $f(J) = S^1$. It follows now that J must intersect each A_i. Since each vertex of J_0 lies in at most two A_i's, $\sigma(G) \leqslant 2q \leqslant 2p(G)$.

COROLLARY (2.5). *If $G = \{v_0, \ldots, v_n\}$ is a cycle, then $\sigma(G) = 2n$.*

Proof. Clearly G can be represented as the union of $2n$-half-edges each containing a vertex of G and so $\sigma(G) \geqslant 2n$. Then by (2.4), $\sigma(G) = 2n$.

COROLLARY (2.6). *If K_n is a complete graph on n vertices, $n \geqslant 3$, then $\sigma(K_n) = 6$.*

Proof. Since K_n is complete, $p(G) = 3$ and so by (2.2) and (2.4), $\sigma(G) = 6$.

III. Embedding graphs in continua.

LEMMA (3.1) (Lemma θ of [2]). *Let X be a locally connected, connected normal space and let a, b be distinct points of X such that $X \setminus \{a, b\}$ contains (at least) three distinct non-empty components, R, P and Q with $\operatorname{cl} R \cap \operatorname{cl} P \cap \operatorname{cl} Q = \{a, b\}$. Then if $\mathscr{C}$ is a circular cover of X and some $C \in \mathscr{C}$ lies entirely in R, then R contains all but at most four members of $\mathscr{C}$.*

THEOREM (3.2) *Let G be any (planar) graph. Then G can be embedded in a (planar) Peano continuum X with $\sigma(G) = m(X)$.*

Proof. If $\sigma(G) < 6$, then by (2.3), G is a tree and $\sigma(G) = 2$. In this case, let $X = G$. Thus we assume that $\sigma(G) \geqslant 6$. In [1] a planar Peano continuum C was constructed with $\sigma(C) = 6$. The continuum C contains a closed interval $[a, b]$ where $C \setminus \{a, b\}$ has exactly three components P, Q and R and $\operatorname{cl} P \cap \operatorname{cl} Q \cap \operatorname{cl} R = \{a, b\}$. Let X be the (planar) continuum obtained by replacing each edge e in G with a copy, C_e, of C in the natural way, so that each edge e of G is identified with a copy, $[a_e, b_e]$, of $[a, b]$.

Clearly every circular covering of G can be extended, link by link, to a circular covering of X. Thus $\sigma(G) \leqslant m(X)$.

Let $\{K_1, \ldots, K_n\}$ be a circular covering of X. If some K_j contains no vertex of G, then a component of some $C_e \setminus \{a_e, b_e\}$ contains K_j. Then by the proof of Theorem 1 of [1], $m(X) \leqslant 6 \leqslant \sigma(G)$.

If on the other hand if every K_j contains a vertex of G, we write each $e = u_e v_e \in E(G)$ as the union of two non-degenerate arcs, $[u_e, p_e]$ and $[p_e, v_e]$ with $[u_e, p_e] \cap [p_e, v_e] = \{p_e\}$ and let

$$K_i' = \bigcup \{[u_e, p_e]\colon\ e = u_e v_e \in E(G) \text{ and } u_e \in K_i \text{ and } v_e \in (K_{i-1} \cup K_i \cup K_{i+1})\},$$

$i = 1, 2, \ldots, n$ (where we take $n+1 = 1$). Then $\{K_1', K_2', \ldots, K_n'\}$ is a circular covering of G and $\sigma(G) \geqslant m(X)$. This completes the proof.

THEOREM (3.3). *Let X be a multicoherent Peano continuum. Then*

$$m(X) = \sup\{m(P)\colon\ P \text{ is a cyclic element of } X\}.$$

Proof. Let P be any cyclic element of X and let $\{A_1, A_2, \ldots, A_n\}$ be a circular covering of P where $m(P) = n$. By Proposition (2.1) of [11, p. 66], if B_i is the union of A_i together with the components of $X \setminus P$ with limit points in A_i, then $\{B_1, B_2, \ldots, B_n\}$ is a circular covering of X. Hence $m(P) \leqslant m(X)$.

Suppose $\{C_1, \ldots, C_m\}$ is a circular covering of X. Now X contains a simple closed curve J that meets each A_i. Let Q be the cyclic element of X that contains J. By Propositions (3.5) and (3.1) of [11, p. 66], each of the sets $C_i \cap Q$ is connected. Thus $Q = \bigcup_{i=1}^{n} (C_i \cap Q)$ is a circular covering of Q. It then follows that $m(X) = \sup\{m(P)\colon P \text{ is a cyclic element of } X\}$.

DEFINITION (3.4). A graph G is *separable* if it is the union of two non-degenerate subgraphs H_1 and H_2 such that H_1 and H_2 have exactly one vertex in common. We say that G is *non-separable* if it can not be so represented.

THEOREM (3.5) [10, Thm 12]. *A graph G may be decomposed into its non-separable components in a unique manner.*

THEOREM (3.6). *Let G be any graph with $\sigma(G) \geqslant 6$. If $G_1, G_2, \ldots, G_n$ are the non-separable components of G, then $\sigma(G) = \max\{\sigma(G_1), \sigma(G_2), \ldots, \sigma(G_n)\}$.*

Proof. The proposition follows from (3.2) and (3.3).

IV. Planar graphs and continua.

LEMMA (4.1). *Let P be a plane continuum, let $\mathscr{D}$ be the set of bounded complementary domains of P and let $T(P) = P \cup (\bigcup \{D\colon D \in \mathscr{D}\})$. If $\varphi\colon P \to S^1$ is continuous, then φ extends continuously to a map $\varphi\colon T(P) \to S^1$ if and only if φ extends continuously to a map $\varphi_D\colon P \cup D \to S^1$ for each $D \in \mathscr{D}$. Thus, if $\varphi\colon P \to S^1$ is not null-homotopic, $\varphi(\operatorname{Fr} D) = S^1$ for some $D \in \mathscr{D}$.*

Proof. By the Tietze Extension Theorem, φ extends continuously to a map $\varphi_1\colon T(P) \to \{z \in \mathbb{C}\colon |z| \leqslant 1\}$. Let $\mathscr{D}_1 = \{D \in \mathscr{D}\colon \varphi_1^{-1}(0) \cap D \neq \varnothing\}$. For each $D \in \mathscr{D}_1$, let φ_D be a continuous extension of φ to a map $\varphi_D\colon P\cup D\to S^1$. Then, since $\mathscr{D}_1$ is a finite set, the function $\varphi\colon T(P)\to S^1$ defined by

$$\varphi(x)=\begin{cases}\varphi_D(\lambda) & \text{if } x\in D\in\mathscr{D}_1,\\ \varphi_1(x)/\|\varphi_1(x)\| & \text{otherwise},\end{cases}$$

is continuous.

LEMMA (4.2). *Let $\{A_1, A_2, \dots, A_n\}$, $n>2$, be a circular covering of the plane multicoherent continuum X. Then there exists an embedding of X in the plane such that the boundary of some bounded complementary domain of X and the boundary of the unbounded complementary domain of X each intersect every A_i, $i=1,2,\dots,n$.*

Proof. For each $i\in\{1,2,\dots,n\}$, let $Z_i=\{z\in C\colon z=e^{2\pi i\theta}\colon (i-1)/n\leqslant\theta\leqslant i/n\}$ and let $\varphi\colon X\to S^1$ be any continuous map of X onto S^1 such that $\varphi(A_i)\subset Z_i$. Clearly φ is not null-homotopic, so by Lemma (4.1), there must be a bounded complementary domain Q so that $\varphi|\operatorname{Fr}Q$ is not null-homotopic. Thus $\varphi(\operatorname{Fr}Q)=S^1$ and $\operatorname{Fr}Q$ meets every A_i. Now let $p\in Q$ and embed X in the plane so that $Q-\{p\}$ is homeomorphic to the unbounded complementary domain of X. Another application of Lemma (4.1) to this embedding assures us that there is a bounded complementary domain C of this embedding such that $\varphi(\operatorname{Fr}C)=S^1$ and this completes the proof.

LEMMA (4.3). *Let Y be the union of $k=(2n-1)$, $n\geqslant 2$, simple arcs $\{M_1,\dots,M_k\}$ with common endpoints a and b and such that $M_i\cap M_j\neq\{a,b\}$ if and only if $\{i,j\}=\{k,k+1\}$ for $k=1,2,\dots,n$ (where we take $n+1=1$). Then Y contains a collection of simple arcs from a to b $S_1,\dots,S_n$ such that $S_i\cap S_j=\{a,b\}$, for $i\neq j$, $1\leqslant i,j\leqslant n$* (1).

Proof. By our hypothesis no collection of fewer than n points of Y separates a from b in Y. The proposition then follows from Whyburn's n-arc Connectedness Theorem [12].

THEOREM (4.4). *If X is any multicoherent planar Peano continuum and $m(X)<\infty$, then $m(X)$ is even.*

Proof. By (3.3) we may assume that X is cyclic. Suppose that $m(X)=k=2n-1$ is odd where $n>3$ (recall that $m(X)\geqslant 6$). Let $\{A_1, A_2, \dots, A_k\}$ be a circular covering of X where each A_i is a Peano continuum. (We may choose the A_i's to be locally connected by Theorem (15.4) of [11, p. 21].

By Lemma (4.2), we may suppose that X is embedded in the plane so that the boundary I of some bounded complementary domain Q of X ($\operatorname{Fr}Q=I$) and the boundary 0 of the unbounded complementary U of X ($\operatorname{Fr}U=0$) each meet every A_i, i.e., $0\cap A_i\neq\emptyset\neq I\cap A_i$, for $i=1,\dots,k$. By (2.5) of [11, p. 107], 0 and I are simple closed curves.

(1) The authors express their appreciation to Howard Cook and F. Burton Jones for pointing out that the n-arc connectedness theorem could be employed in this proof.

Let $p\in U$ and $q\in Q$. By the Schoenflies Theorem (or Theorem (4.4) of [11, p. 113]) we may select a collection $\{M_1,\dots,M_k\}$ of simple arcs from p to q such that $M_i\cap M_j\cap U=\{p\}$, $M_i\cap M_j\cap Q=\{q\}$ for $i\neq j$, and for each i, $M_i\cap X$ is a simple arc (possibly degenerate) in A_i. Then by Lemma (3.4), $\bigcup_{i=1}^{k}M_i$ contains a collection $\{S_1,\dots,S_n\}$ of simple arcs such that $S_i\cap S_j=\{p,q\}$ for $i\neq j$, $1\leqslant i,j\leqslant n$. By re-ordering the S_i's if necessary we may assume that if B_i is the closed 2-cell bounded by $S_i\cup S_{i+1}$, $i=1,n-1$ and B_n is the closure of the unbounded complementary domain of $S_n\cup S_1$, then $\operatorname{int}B_i\cap\operatorname{int}B_j=\emptyset$ if $i\neq j$, and $B_{i-1}\cap B_i=S_i$, for $i=2,\dots,n$ and $B_n\cap B_1=S_1$.

We next argue that $X\cap B_1$, is connected. First of all, by our construction $S_i\cap X=T_i$ simple arc for each $i=1,\dots,n$ and every point of $X\cap B_1$ can be joined to $(S_1\cap X)\cup(S_2\cap X)$ by a simple arc lying in $X\cap B_1$. It remains to show that some component of $X\cap B_1$ meets both $S_1\cap X$ and $S_2\cap X$. Now X separates p and q in the plane and so $X\cap B_1$ separates p and q in B_1. Since B_1 is unicoherent, some component C of $X\cap B_1$ separates p and q in B_1 [3, Thm (4.12)]. Consequently $C\cap S_1\neq\emptyset\neq C\cap S_2$ and thus $X\cap B_1$ is connected. It then follows by a similar proof, that $X\cap B_i$ is connected for $i=1,\dots,n$.

Now by Whyburn's Separation Theorem (see Lemma 3 of [4]), $X\cap B_i$ can be represented as the union of two continua D_i^1 and D_i^2 where $(S_i\cap X)\subset(D_i^1\setminus D_i^2)$ and $S_{i+1}\subset(D_i^2\setminus D_i^1)$. Then $\{D_1^1, D_1^2, D_2^1, D_2^2,\dots, D_n^1, D_n^2\}$ is a circular covering of X and so $m(X)\geqslant 2n$. This contradicts our assumption that $m(X)=2n-1$ and this completes the proof.

COROLLARY (4.5). *If G is any planar graph, then $\sigma(G)$ is even.*

Proof. This follows from (3.2) and (4.4).

EXAMPLE (4.6). Here we describe an example of a non-planar graph G with exactly 7 vertices and for which $\sigma(G)=7$. The proof that $\sigma(G)=7$ will appear in *Circularity of graphs and continua: combinatorics.*

Let G_0 be a cycle with seven vertices $v_1v_2,\dots,v_7$ so that v_i and v_{i+1} are adjacent for $i=1,2,\dots,6$ and v_1 and v_7 are adjacent. We add the following edges to G_0: $\{v_1v_3, v_2v_4, v_3, v_5, v_4v_6, v_5v_7, v_6v_1, v_7v_2\}$ so as to obtain G. Now G is non-planar since it contains a homeomorphic copy of $K_{3,3}$ [1].

Since $\sigma(G)=7$, by Theorem (3.2) there exists a non-planar Peano continuum X containing G such that $m(X)=7$.

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Spaces of order arcs in hyperspaces

by

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Abstract. Let X be a metric continuum and let 2^X and $C(X)$ denote respectively the space of closed subsets and the space of subcontinua of X topologized with the Hausdorff metric. An order arc in 2^X ($C(X)$) is an arc α contained in 2^X ($C(X)$) such that if $A, B \in \alpha$, then $A \subseteq B$ or $B \subseteq A$. Let $\Gamma(2^X)$ ($\Gamma(C(X))$) denote the space of order arcs in 2^X ($C(X)$) together with the singletons $\{A\}$, $A \in 2^X$ ($C(X)$), topologized with the Hausdorff metric on 2^{2^X}. In this paper we prove that if X is locally connected, then $\Gamma(2^X)$ is homeomorphic with the Hilbert cube Q and if, in addition, X contains no arc with interior, then $\Gamma(C(X))$ is homeomorphic with Q.

1. Introduction. Let X be a continuum (i.e., a compact connected metric space containing more than one point). The *hyperspaces* of X are the spaces 2^X, consisting of all nonempty closed subsets of X, and $C(X)$, consisting of the connected elements in 2^X, each with the Hausdorff metric H. Basic facts about hyperspaces may be found in [13] and [9].

An *order arc in* 2^X (resp., $C(X)$) is an arc $\alpha \subset 2^X$ (resp. $\alpha \subset C(X)$) such that if $A, B \in \alpha$, then $A \subset B$ or $B \subset A$. Order arcs in hyperspaces were first constructed in [2], as a part of the proof of the following:

1.1. THEOREM. *For any continuum X, 2^X and $C(X)$ are each arcwise connected continua.*

However, the fact that the construction in [2] yielded an arc was not noted until later in [10, Lemma 5]. Since the publication of these two papers, order arcs have been used extensively in studying hyperspaces. However, spaces of order arcs have undergone almost no investigation. In this paper we investigate the spaces

$$\Gamma(2^X) = \{\alpha \subset 2^X: \alpha \text{ is an order arc}\} \cup \{\{A\}: A \in 2^X\}$$

and

$$\Gamma(C(X)) = \{\alpha \subset C(X): \alpha \text{ is an order arc}\} \cup \{\{A\}: A \in C(X)\}$$

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