Null sequence cellular decompositions of S^3

by

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Abstract. In this paper a technique is presented which allows one to produce many examples of null sequence cellular decompositions of S^3 whose decomposition space is not S^3. Specifically, it is shown that if $\{X_i\}_{i\in\omega}$ is a countable collection of non-degenerate continua each of which admits a cellular embedding in S^3, then there are embeddings $h_i: X_i \to S^3$ so that $\{h_i(X_i)\}_{i\in\omega}$ is a null sequence of disjoint cellular continua and $S^3/\{h_i(X_i)\}$ is not homeomorphic to S^3.

1. Introduction. A technique for constructing null sequence decompositions of S^3 is developed which proves the following theorem.

MAIN THEOREM. *Let $\{X_i\}_{i\in\omega}$ be a countable collection of non-degenerate continua each of which admits a cellular embedding in S^3. Then there are embeddings $h_i: X_i \to S^3$ so that $\{h_i(X_i)\}_{i\in\omega}$ is a null sequence of disjoint cellular continua and $S^3/\{h_i(X_i)\}$ is not homeomorphic to S^3.*

In [1, § 2–5], Bing produced a null sequence of disjoint cellular continua $\{G_i\}$ so that $S^3/\{G_i\}$ is not homeomorphic to S^3. In that example, each G_i is an indecomposable continuum. In [2, Theorem 3] Bing showed that if G is an upper semicontinuous decomposition of S^3 with only a countable number of nondegenerate elements each of which is a tame arc, then S^3/G is homeomorphic to S^3. The papers [3, 5] establish the existence of a null sequence $\{G_i\}$ of disjoint cellular arcs so that $S^3/\{G_i\}$ is not homeomorphic to S^3.

Recent results have given conditions under which a countable decomposition of S^3 yields S^3. One such result which follows from [4; 6, Theorem 4.1; 7, Theorem 1], is the following theorem.

THEOREM. *Let G be an upper semicontinuous decomposition of S^3 into points and countably many cellular continua $\{P_i\}$ each of which has a mapping cylinder neighborhood. Then S^3/G is homeomorphic to S^3.*

The Main Theorem here shows that some hypothesis on the embedding of continua $\{G_i\}$ in S^3 beyond cellularity is needed in order to conclude that $S^3/\{G_i\}$ is homeomorphic to S^3.

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The construction needed to prove the Main Theorem is found in § 4 in the proof of Lemma 4.1. Perhaps the most efficient method for the reader to obtain an understanding of the construction would be for him to read the eight steps in the proof of Lemma 4.1, referring back to notation and previous lemmas as required. Sections 2 and 3 contain statements of Lemmas used in § 4.

2. Convergence lemmas. In this section three lemmas are stated which give conditions under which a sequence of maps of a continuum X into S^3 will converge to a cellular embedding of X. The first lemma presents a condition under which a sequence of continuous functions will converge to a continuous function or map. Lemma 2.2 adds a hypothesis to Lemma 2.1 to enable one to conclude that the limit map is an embedding. Lemma 2.3 adds another hypothesis which implies that the limit map is a cellular embedding. Since the proofs of these lemmas are as easy to construct as the statements are to understand, no proofs are given.

LEMMA 2.1. *Let X be a topological space, $g_i: X \to S^3$ ($i \in \omega$) be a sequence of maps and for each i in ω, let $\mathcal{V}_i = \{V_{i,j}\}_{j=1}^{n_i}$ be a collection of open sets so that $g_i(X) \subset \bigcup_{j=1}^{n_i} V_{i,j}$ and $\operatorname{diam} V_{i,j} < 1/i$ for each $j = 1, \ldots, n_i$. Suppose that for each i in ω, $\mathcal{V}_{i+1}$ is a closed star refinement of $\mathcal{V}_i$, that is, for each $V_{(i+1),j}$ in $\mathcal{V}_{i+1}$,*

$$\operatorname{Cl}(\operatorname{St}(V_{(i+1),j}, \mathcal{V}_{i+1})) \subset V_{i,k}$$

for some k. Suppose that for each $i \in \omega$ and $x \in X$, $g_{i+1}(x) \in \operatorname{St}(g_i(x), \mathcal{V}_{i+1})$.

Then $g = \lim_{i\to\infty} g_i$ exists, is continuous, and for every x in X and i in ω, $g(x) \in \operatorname{St}(g_i(x), \mathcal{V}_i)$.

LEMMA 2.2. *Suppose the hypotheses of Lemma 2.1 are given. In addition suppose X is a metric continuum with metric ϱ and for each i in ω, and points x, y in X, if $\varrho(x, y) > 1/i+1$,*

$$\operatorname{St}(\operatorname{St}(g_i(x), \mathcal{V}_{i+1}), \mathcal{V}_{i+1}) \cap \operatorname{St}(\operatorname{St}(g_i(y), \mathcal{V}_{i+1}), \mathcal{V}_{i+1}) = \emptyset .$$

Then $g = \lim_{i\to\infty} g_i$ is an embedding.

LEMMA 2.3. *Suppose the hypotheses of Lemmas 2.1 and 2.2 are given and in addition for each element $V_{i,j}$ of $\mathcal{V}_i$, $V_{i,j} \cap g_i(X) \neq \emptyset$ and for each i, there is a 3-cell B_i so that $\bigcup \mathcal{V}_{i+1} \subset B_i \subset \bigcup \mathcal{V}_i$.*

Then $g(X)$ is a cellular set, so g is a cellular embedding of X, and

$$g(X) = \bigcap_{i\in\omega} (\bigcup \mathcal{V}_i) = \bigcap_{i\in\omega} \operatorname{Cl}(\bigcup \mathcal{V}_i) .$$

3. The two disk property. In order to prove the Main Theorem, it is necessary to define a property which distinguishes a decomposition space from S^3. The property used here, as well as in [1, 3] and elsewhere, is the two disk property defined below.

DEFINITION. A finite collection M of disjoint closed subsets in the interior of a solid torus T has the *two disk property* if and only if for every pair of disjoint

meridional disks D_0, D_1 of T, there is an element of M which intersects both D_0 and D_1.

DEFINITION. A decomposition G of S^3 is *defined by sequence* $\{M_i\}_{i\in\omega}$ if and only if for each i, M_i is a finite collection of disjoint closed sets in S^3 so that $\bigcup M_{i+1} \subset \bigcup M_i$ and each element of G is either a component of $\bigcap_{i\in\omega} (\bigcup M_i)$ or a point of $S^3 - \bigcap_{i\in\omega} (\bigcup M_i)$.

THEOREM 3.1. *Let T be a solid torus and G be a cellular decomposition of S^3 defined by the sequence $\{M_i\}_{i\in\omega}$ where each M_i has the two disk property in T. Then S^3/G is not homeomorphic to S^3.*

The properties mentioned in Theorem 3.1 are a favorite way of showing that a decomposition space of S^3 is topologically different from S^3. We do not repeat the proof of Theorem 3.1.

The Main Theorem here will be proved by constructing an appropriate defining sequence $\{M_i\}$ for which each M_i has the two disk property. The following lemmas state that if a given collection M has the two disk property then certain modifications of M will yield a new collection with the two disk property. These lemmas are then used to prove inductively that each M_i in the defining sequence $\{M_i\}$ has the two disk property. The following lemmas were proved in [3]. The proofs are not repeated here.

LEMMA 3.2 [3, Lemma 3.5]. *Let T be a solid torus and M be a collection of subsets of T with the two disk property one element of which is a cube with n handles H. Then*

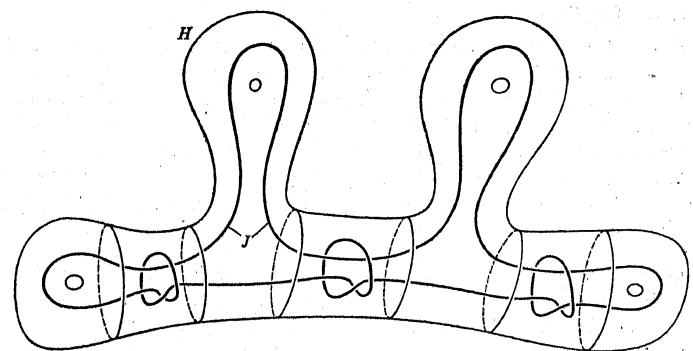

Fig. 3.1

the collection M', obtained from M by replacing H by a simple closed curve J which is embedded in H as illustrated in Figure 3.1, also has the two disk property.

Figure 3.2 represents a solid torus W in which two disjoint sets Y and Z are embedded. The set Y is one pair of small eyeglasses while $Z = \bigcup Z_i$ is the union of several eyeglasses. The chain of eyeglasses $Y \cup Z$ goes around W twice.

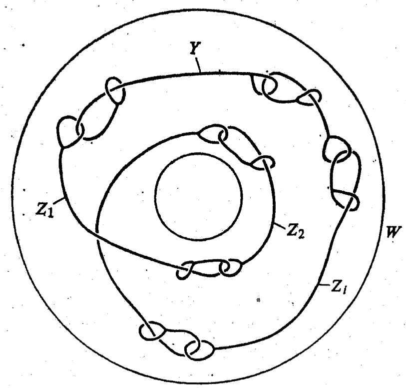

Fig. 3.2

LEMMA 3.3 [3, Lemma 3.3]. *Let M be a collection of subsets of a torus T which has the two disk property. Suppose the solid torus W is an element of M. Then the collection M', obtained from M by replacing W by the two sets Y and Z as described above, still has the two disk property.*

4. Proof of the Main Theorem. Let $\{X_P | P$ is an n-tuple of 0's and 1's starting and ending with $0\}$ be a countable collection of non-degenerate continua each of which admits a cellular embedding in S^3. Our objective is to produce, for each such n-tuple P, an embedding $h_P\colon X_P \to S^3$ so that $\{h_P(X_P)\}$ is a null sequence of disjoint cellular continua and $S^3/\{h_P(X_P)\}$ is not homeomorphic to S^3.

We will produce a defining sequence $\{M_i\}_{i\in\omega}$ for a decomposition of S^3 so that for each i, M_i has the two disk property and so that the non-degenerate elements of $\bigcap_{i\in\omega}(\bigcup M_i)$ form a collection $\{h_P(X_P)\}$ as desired.

Notation. Let $\tilde{Q}_n = \{Q | Q$ is an n-tuple of 0's and 1's which starts with $0\}$. For each Q in $\tilde{Q}_n$, let Q' be the i-tuple ($i \leqslant n$) obtained from Q by truncating Q immediately after the last 0 in Q.

In order to describe the defining sequence $\{M_n\}$, we produce several things for each n-tuple Q in $\tilde{Q}_n$: namely, K_Q, a 1-complex; g_Q, an embedding of $X_{Q'}$; and $\mathscr{V}_Q$, a collection of open sets which cover $g_Q(X_{Q'})$. These items will be produced inductively on n, the length of the tuples Q, in such a way that $M_n = \{\mathrm{Cl}(\bigcup \mathscr{V}_Q)\}_{Q\in\tilde{Q}_n}$ has the two disk property for each n. Also, for each n-tuple P in $\tilde{Q}_n$ which ends in 0, $\lim\{g_Q | Q' = P\}$ will be a cellular embedding h_P of X_P so that $\{h_P(X_P)\}$ is a null sequence and is equal to the set of non-degenerate elements of $\bigcap_{n\in\omega}(\bigcup M_n)$.

The exact requirements for each K_Q, g_Q and $\mathscr{V}_Q$ are contained in the following lemma.

Let ϱ be a metric for the X_P's.

LEMMA 4.1. *Let T be a standardly embedded torus in S^3 with diameter less than 1 and $\{X_P | P \in \tilde{Q}_n$ for some n and P ends in $0\}$ be a countable collection of non-degenerate continua each of which admits a cellular embedding in S^3.*

Then for each $Q \in \tilde{Q}_n$ and $(Q, 0)$ and $(Q, 1) \in \tilde{Q}_{n+1}$, there are K_Q's, $\mathscr{V}_Q$'s and g_Q's which satisfy the following conditions:

(i) *$\mathscr{V}_Q$ is a finite collection of open sets each of diameter less than $1/n$;*

(ii) *$\{\mathrm{Cl}(\bigcup \mathscr{V}_Q) | Q \in \tilde{Q}_n\}$ is a collection of 2^{n-1} disjoint subsets of $\mathrm{Int}\,T$;*

(iii) *K_Q is a connected, PL 1-complex contained in $\bigcup \mathscr{V}_Q$;*

(iv) *$\{K_Q | Q \in \tilde{Q}_n\}$ has the two disk property;*

(v) *$\mathrm{Cl}(\bigcup \mathscr{V}_{(Q,\varepsilon)})$ is contained in a 3-cell in $\bigcup \mathscr{V}_Q$ for $\varepsilon = 0$ or 1;*

(vi) *$g_Q\colon X_{Q'} \to \bigcup \mathscr{V}_Q$ is a cellular embedding;*

(vii) *for each element V of $\mathscr{V}_Q$, there is a point $x \in X_{Q'}$ so that $g_Q(x) \in V$;*

(viii) *for each $x \in K_Q$, there is a point $y \in X_{Q'}$ and a ball B in $\bigcup \mathscr{V}_Q$ so that x and $g_Q(y)$ belong to B and $\mathrm{diam}\,B < \frac{1}{3}\delta_Q$ where δ_Q is a Lebesgue number associated with the cover $\mathscr{V}_Q$ of $g_Q(X_{Q'})$;*

(ix) *if $x, y \in X_{Q'}$ and $\varrho(x, y) > 1/n+1$, then*

$$\mathrm{St}\big(\mathrm{St}(g_Q(x), \mathscr{V}_{(Q,1)}), \mathscr{V}_{(Q,1)}\big) \cap \mathrm{St}\big(\mathrm{St}(g_Q(y), \mathscr{V}_{(Q,1)}), \mathscr{V}_{(Q,1)}\big) = \emptyset;$$

(x) *$\mathscr{V}_{(Q,1)}$ is a closed star refinement of $\mathscr{V}_Q$, that is, for each V in $\mathscr{V}_{(Q,1)}$, there is a U in $\mathscr{V}_Q$ so that $\mathrm{Cl}(\mathrm{St}(V, \mathscr{V}_{(Q,1)})) \subset U$;*

(xi) *$\mathscr{V}_{(Q,0)}$ is a single open set in $\bigcup \mathscr{V}_Q$;*

(xii) *there is an isotopy $h_t\colon S^3 \to S^3$ fixed outside a finite number of disjoint sets in $\mathscr{V}_{(Q,1)}$ so that $h_0 = \mathrm{id}$ and $g_{(Q,1)} = h_1 \circ g_Q\colon X_{Q'} \to S^3$.*

Proof of Lemma 4.1. The proof proceeds by induction on n. It will be proved that if K_Q, $\mathscr{V}_Q$, and g_Q have been defined for each $Q \in \tilde{Q}_n$ so that they satisfy conclusions (i) → (viii) of Lemma 4.1, then $K_{(Q,\varepsilon)}$, $\mathscr{V}_{(Q,\varepsilon)}$, $g_{(Q,\varepsilon)}$ ($\varepsilon = 0, 1$) can be constructed in such a manner that they satisfy conclusions (ix)–(xii) for the nth Lemma 4.1 and conclusions (i)–(viii) of the $(n+1)$st Lemma 4.1.

For $n = 0$, $\tilde{Q}_n$ has one element, (0). Let $K_{(0)}$ be a centerline of T. Let $\mathscr{V}_{(0)}$ contain one open set V, the interior of a regular neighborhood of $K_{(0)}$. Let $g_{(0)}\colon X_{(0)} \to V$ be a cellular embedding.

Suppose K_Q, $\mathscr{V}_Q$, and g_Q have been defined for each $Q \in \tilde{Q}_n$ so that they satisfy conclusions (i)–(viii) of Lemma 4.1. We show how to construct $K_{(Q,\varepsilon)}$, $\mathscr{V}_{(Q,\varepsilon)}$ and $g_{(Q,\varepsilon)}$ ($\varepsilon = 0, 1$) in several steps.

Step 1. Use the fact that $g_Q(X_{Q'})$ is cellular to push K_Q off $g_Q(X_{Q'})$ inside $\bigcup \mathscr{V}_Q$.

Let K'_Q be the moved K_Q. Maintain the property (viii) that for each point $x \in K'_Q$ there is a point $y \in X_{Q'}$ and a ball B in $\bigcup \mathscr{V}_Q$ so that x and $g_Q(y)$ belong to B and $\operatorname{diam} B < \frac{1}{3}\delta_Q$.

Step 2. Take a small regular neighborhood of K'_Q and use Lemma 3.2 to replace K'_Q by a solid torus W_Q so that $\{W_Q \mid Q \in \tilde{Q}_n\}$ has the two disk property, and each point of W_Q lies in a $\frac{1}{3}\delta_Q$ ball in $\bigcup \mathscr{V}_Q$ which contains a point of $g_Q(X_{Q'})$.

Step 3. Use Lemma 3.3 to replace each W_Q by two objects, an eyeglass Y and a union of eyeglasses $\{Z_i\}_{i=1}^m$ so that Y and each Z_i is contained in a ball B in $\bigcup \mathscr{V}_Q$ of diameter less than $\frac{1}{3}\delta_Q$.

Step 4. Join each Z_i to one point of $g_Q(X_{Q'})$ by a small (i.e. diameter less than $\frac{1}{3}\delta_Q$) arc A_i in such a way that $g_Q(X_{Q'}) \cup (\bigcup_{i=1}^m A_i) \cup (\bigcup_{i=1}^m Z_i)$ is contained in a cell in $\bigcup \mathscr{V}_Q$. This step can be accomplished by having the arc A_i meet the eyeglass Z_i as shown in Figure 4.1. Notice how Z_1 then Z_2, etc., can be isotoped down to the arcs A_1, A_2, etc., and thence into as close a neighborhood of $g_Q(X_{Q'})$ as desired.

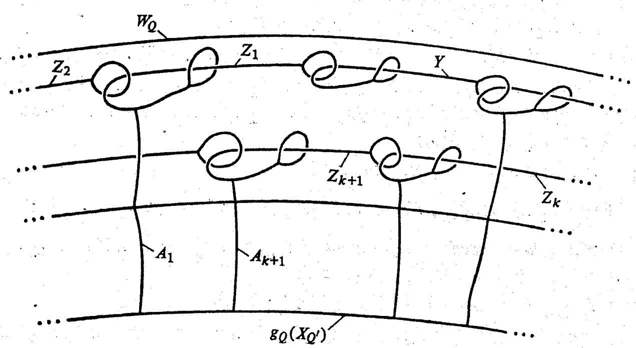

Fig. 4.1

Step 5. Cover $g_Q(X_{Q'}) \cup (\bigcup A_i) \cup (\bigcup Z_i)$ by a finite open cover $\mathscr{V}_{(Q,1)}$ so that

(a) if $x, y \in X_{Q'}$ and $\varrho(x, y) > 1/n+1$, then $\operatorname{St}(\operatorname{St}(g_Q(x), \mathscr{V}_{(Q,1)}), \mathscr{V}_{(Q,1)}) \cap \cap \operatorname{St}(\operatorname{St}(g_Q(y), \mathscr{V}_{(Q,1)}), \mathscr{V}_{(Q,1)}) = \emptyset$,

(b) for each i, $A_i \cup Z_i$ is contained in a single element of $\mathscr{V}_{(Q,1)}$,

(c) if $i \neq j$, $(A_i \cup Z_i) \subset V \in \mathscr{V}_{(Q,1)}$, and $(A_j \cup Z_j) \subset W \in \mathscr{V}_{(Q,1)}$, then $V \cap W = \emptyset$,

(d) $\bigcup \mathscr{V}_{(Q,1)}$ is contained in a cell in $\bigcup \mathscr{V}_Q$, and

(e) $\operatorname{Cl}(\bigcup \mathscr{V}_{(Q,1)}) \cap Y = \emptyset$.

Note that these conclusions are easily obtained with (b) possible since for each i, $(A_i \cup Z_i) \cap g_Q(X_{Q'})$ is a single point and $A_i \cup Z_i$ is small.

Step 6. The embedding $g_{(Q,1)}: X_{Q'} \to \bigcup \mathscr{V}_{(Q,1)}$ can now be defined. For each $A_i \cup Z_i$ $(i = 1, \ldots, m)$ there is an open set V_i in $\mathscr{V}_{(Q,1)}$ which contains it. Let $h_t\colon S^3 \to S^3$ $(t \in [0, 1])$ be an isotopy so that $h_0 = \mathrm{id}$, for each t in $[0, 1]$, $h_t | S^3 - \bigcup_{i=1}^m V_i = \mathrm{id}$, and for each point $x \in \bigcup_{i=1}^m (A_i \cup Z_i)$, there is a point $y \in X_{Q'}$ and a ball B in $\bigcup \mathscr{V}_{(Q,1)}$ so that $\operatorname{diam} B < \frac{1}{3}\delta_{(Q,1)}$ and x and $h_1 \circ g_Q(y)$ belong to B. Then $g_{(Q,1)} = h_1 \circ g_Q$.

Step 7. The connected 1-complex $K_{(Q,1)}$ is obtained by joining the free ends of the $A_i \cup Z_i$'s together by arcs each point of which can be joined to $g_{(Q,1)}(X_{Q'})$ by an arc in $\bigcup \mathscr{V}_{(Q,1)}$ of size less than $\frac{1}{3}\delta_{(Q,1)}$.

Step 8. The set $K_{(Q,0)}$ is Y, the eyeglass. A small neighborhood of $K_{(Q,0)}$ is the single open set in $\mathscr{V}_{(Q,0)}$. The embedding $g_{(Q,0)}: X_{(Q,0)} \to \bigcup \mathscr{V}_{(Q,0)}$ is a cellular embedding guaranteed by the hypothesis on $X_{(Q,0)}$.

These steps complete the proof of Lemma 4.1.

The Main Theorem is now proved as follows. Let $M_n = \{\operatorname{Cl}(\bigcup \mathscr{V}_Q) \mid Q \in \tilde{Q}_n\}$. Then $\{M_n\}_{n \in \omega}$ is a defining sequence for a decomposition of S^3. The non-degenerate components of $\bigcap_{n \in \omega} (\bigcup M_n)$ form a null sequence. Each element in the null sequence is a cellular embedding of an X_P. In fact, for each n-tuple P in $\tilde{Q}_n$ which ends in 0, $\lim\{g_Q \mid Q' = P\}$ is a cellular embedding of X_P and is equal to $\bigcap_{Q'=P} \operatorname{Cl}(\bigcup \mathscr{V}_Q)$ by Lemma 2.3. Since each M_n has the two disk property, the decomposition space S^3/G determined by the M_n's is not homeomorphic to S^3 and the Main Theorem is proved. Note that the statement of the Main Theorem could be strengthened slightly to reflect the fact that the final embedding of each X_i is obtained from a given cellular embedding by means of a pseudo isotopy of space.

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