Locally starlike decompositions of separable metric spaces

by

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Abstract. Among the main results of this paper are the following.

THEOREM. *Let G be a locally compact, locally null, locally starlike decomposition of a separable metric space (X, d). Then X/G is homeomorphic to X and G is a shrinkable decomposition of X.*

COROLLARY. *Let G be a locally null, locally starlike decomposition of an n-manifold M (not necessarily connected). Then M/G is homeomorphic to M and G is a shrinkable decomposition of M.*

To illustrate these results, let $\{C_k^n\}$ be a sequence of disjoint, open n-cells and let M_n be the free union of $\{C_k^n\}$. If G is a decomposition of M_n such that each $G(C_k^n)$ is a locally null, starlike-equivalent decomposition, then M_n/G is homeomorphic to M_n and G is shrinkable. A somewhat different corollary implies that if $n = 2$ and G is a monotone, pointlike, usc, 0-dimensional decomposition of M_2, then M_2/G is homeomorphic to M_2 and G is shrinkable.

§ 1. **Introduction.** In [2], R. J. Bean showed that null, starlike equivalent decompositions of E^3 yield E^3; in [9], T. M. Price showed that decompositions of E^n satisfying a special 0-dimensional condition yield E^n. These results were strengthened and were generalized to include decompositions of locally compact, SC-WR-CE metric spaces in [10] by this author. In this paper, we introduce the notions of locally starlike decompositions, locally star-0-dimensional decompositions, and locally shrinkable decompositions. These notions help us to extend the results of [2], [9], and [10] to separable metric spaces in a natural way. In particular, these results are extended to n-manifolds (as defined in Section 2).

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COROLLARY. *Let G be a locally null, locally starlike decomposition of an n-manifold M. Then M/G is homeomorphic to M and G is a shrinkable decomposition of M.*

To illustrate these results, let $\{C_k^n\}$ be a sequence of disjoint, open n-cells and let M_n be the free union of $\{C_k^n\}$. If G is a decomposition of M_n such that each $G(C_k^n)$ is a locally null, starlike-equivalent decomposition, then M_n/G is homeomorphic to M_n and G is shrinkable. A somewhat different corollary implies that if $n = 2$ and G is a monotone, pointlike, usc, 0-dimensional decomposition of M_2, then M_2/G is homeomorphic to M_2 and G is shrinkable.

We note that two examples due to R. H. Bing ([3] and [4]) show that each of the conditions of locally null and locally starlike is non-superfluous in the above theorem. Because the spaces of this paper are not assumed to be locally compact or connected, we can not show G is a shrinkable decomposition of a space X and then conclude that X/G is homeomorphic to X (see Theorem 4 of [6]). We must construct the homeomorphism between X and X/G; however, the construction of this homeomorphism allows us to conclude that G is shrinkable.

In Section 2 we give definitions, in Section 3 we obtain lemmas concerning coverings of the non-degenerate elements of a decomposition, and in Section 4 we prove the main results of this paper. *Throughout this paper, unless stated otherwise, we are always in a separable metric space* (X, d).

§ 2. Definitions and notation. For the definitions of SC (*strongly convex*) and WR (*without ramifications*), see [11]. In this paper an SC metric space is *not* assumed to be complete. Let (X, d) possess a unique segment $[ab]$ joining each $a, b \in X$ (which is the case if (X, d) is a locally compact, SC metric space (Proposition 2.1 of [10])). We say (X, d) has *closed edges* (CE) if for each $p \in X$, $\{y \in X\colon [py]$ is maximal$\} \cup \{p\}$ is closed. The set A is *properly starlike w.r.t. p* if for each $x \in A-\{p\}$, $[px]$ is not maximal and is contained in A.

Let G be a decomposition of (X, d). We let X/G denote the decomposition space determined by G, P denote the projection of X onto X/G, $H(G)$ denote the non-degenerate elements of G, and $G(\varepsilon)$ denote $\{g \in H(G)|\ \operatorname{diam} g \geqslant \varepsilon\}$ where $\varepsilon>0$. If $B \subset X$, let $G(B)$ be the decomposition of X such that $H(G(B)) = \{g \in H(G)|\ g \subset B\}$ and let BG denote $\bigcup \{g \in G|\ g \subset B\}$. For the definitions of *usc* (*upper semicontinuous*), *monotone*, *0-dimensional*, *null*, *locally null*, *open covering of* $H(G)$, *shrinkable*, and *shrinkable at* g (where $g \in H(G)$) see [1], [6], [8], [12]. Note that a monotone decomposition is usc by definition (as in [6]). We say G is *properly starlike-equivalent* if each $g \in H(G)$ is equivalent under a space homeomorphism to a compact, properly starlike set. If N is a neighborhood (open) of p, then the *edge of N w.r.t. p*, or $\mathrm{Ed}_p(N)$, is $\{y \in \mathrm{Cl}(N)\colon [py]$ is maximal$\}$. We say G is *star-0-dimensional* if for each $g \in H(G)$, there is a neighborhood base $\{U_n\}$ for g such that for each n, $\mathrm{Bd}(U_n) \cap (\bigcup H(G)) = \emptyset$, $U_n \supset \mathrm{Cl}(U_{n+1})$, and $\mathrm{Cl}(U_n)$ is compact and homeomorphic to the closure of an open, starlike w.r.t. p_n set with empty edge w.r.t. p_n.

We say G is *locally compact* if each point of $\mathrm{Cl}(\bigcup H(G))$ possesses a neighborhood (in X) with compact closure. We say G is *locally starlike* (*locally star-0-dimensional*) if for each $g \in H(G)$, there is an open set V_g containing g and a metric d_g on V_g such that

(1) (V_g, d) is homeomorphic to (V_g, d_g);

(2) (V_g, d_g) is a locally compact, SC-WR-CE metric space; and

(3) g is properly starlike w.r.t. p as a subset of (V_g, d_g) ($G(V_g)$ is a star-0-dimensional, usc decomposition of (V_g, d_g)).

We say G is *locally shrinkable* if each $g \in H(G)$ possesses a neighborhood V such that V is locally compact and connected and $G(V)$ is a shrinkable, usc decomposition of (V, d). We say M is an *n-manifold* if M is a separable metric space and each point of M possesses a neighborhood homeomorphic to either E^n or E^n_+.

Remark. It is easily seen that condition (3) of locally starlike is strictly weaker than the following statement: $G(V_g)$ is a properly starlike-equivalent decomposition of (V_g, d_g). Condition (3) of locally starlike and locally star-0-dimensional guarantees that such decompositions are monotone.

§ 3. Coverings of $H(G)$. The following results are needed in Section 4. Unless stated otherwise, (X, d) is a separable metric space.

PROPOSITION 3.1. *Let G be a monotone decomposition of X. If G is locally compact, then the following are equivalent*:

(1) *G is locally null*;

(2) *for each $\delta>0$, every subcollection of $G(\delta)$ has a closed point-set union*; *and*

(3) *every subcollection of G is* usc.

If one of the above holds, $H(G)$ is countable; *hence G is 0-dimensional.*

LEMMA 3.1. *Let G be a locally null, locally compact, monotone decomposition of X, let $\varepsilon>0$, and let $\mathscr{K} = \{O_\alpha\}$ be an open covering of $G(\varepsilon)$. Then there is an open refinement $\mathscr{K}' = \{O_n'\}$ of $\mathscr{K}$ such that $\mathscr{K}'$ is a countable collection, $\mathrm{Cl}(O_n') \cap \mathrm{Cl}(O_m') = \emptyset$ for $m \neq n$, and for each $g_n \in G(\varepsilon)$, $g_n \in O_m'$ if and only if $m = n$.*

Proof. Since G satisfies Proposition 3.1 (2), $H(G)$ is countable and G is 0-dimensional. Let $G(\varepsilon) = \{g_1, g_2, \ldots\}$. We may assume $\mathscr{K}$ is countable, $\mathscr{K} = \{O_n\}$, and that for each n, $g_n \subset O_n$. Therefore one may obtain a countable covering $\{W_n\}$ of $G(\varepsilon)$ such that $g_n \subset W_n$ for each n and $\mathrm{Cl}(W_n) \cap \mathrm{Cl}(W_m) = \emptyset$ if $m \neq n$. Let $U_n = O_n \cap W_n$ for each n and let G_1 denote the decomposition of X such that $H(G_1) = G(\varepsilon)$. Applying [Lemma 6, [6]] to G_1 and $\{U_n\}$, we obtain a refinement $\{O_n'\}$ of $\{U_n\}$ such that $\{O_n'\}$ is a countable collection of disjoint, open sets covering $G(\varepsilon)$. We may reorder $\{O_n'\}$ such that $g_n \in O_n'$ for each $g_n \in G(\varepsilon)$. Thus $O_n' \subset U_n$ for each n, and it follows if $m \neq n$, then $\mathrm{Cl}(O_n') \cap \mathrm{Cl}(O_m') = \emptyset$. Thus for each $g_n \in G(\varepsilon)$ $g_n \in O_m'$ if and only if $m = n$.

LEMMA 3.2. *Let G be a monotone,* usc *decomposition of X. Let $\mathscr{K}$ be a locally null collection of subsets of X such that each point of $X-\mathrm{Cl}(\bigcup H(G))$ possesses a neighborhood which intersects only finitely many members of $\mathscr{K}$. If G is a locally compact decomposition, then $P(\mathscr{K}) = \{P(K)|\ K \in \mathscr{K}\}$ is a locally null collection of subsets of X/G.*

Proof. Let $x \in X/G$; we must find a neighborhood A of x such that $\{P(K)|\ P(K) \cap A \neq \emptyset\}$ is a null collection of subsets of X/G. First assume $x \in X/G-\mathrm{Cl}\big(P(\bigcup H(G))\big) = X/G-P\big(\mathrm{Cl}(\bigcup H(G))\big)$. We have a neighborhood B of x such that $B \cap P\big(\mathrm{Cl}(\bigcup H(G))\big) = \emptyset$. Let $y = p^{-1}(x)$ and let C be a neighborhood of y in X such that $\{K|\ K \cap C \neq \emptyset\}$ is a finite collection and $C \subset P^{-1}(B)$. Let $A = P(CG)$; then A is a neighborhood of x and it follows that

$$\{P(K)|\ P(K) \cap A \neq \emptyset\}$$

is a finite and hence null collection. Now let $x\in \mathrm{Cl}(P(\bigcup H(G)))$. Then $y=P^{-1}(x)$ is a point in $\mathrm{Cl}(\bigcup H(G))$ or is an element of $H(G)$. In either case y possesses a neighborhood C in X such that $\mathrm{Cl}(C)$ is compact. Hence CG is an open set containing y such that $\mathrm{Cl}(CG)$ is compact. It follows that $P(CG)$ is a neighborhood of x having compact closure. The proof now follows exactly as in the proof of [Theorem 1, [6]].

PROPOSITION 3.2. *Let G be a decomposition of a topological space X and let $\mathscr{K}$ be a disjoint, open covering of $H(G)$. Then $\mathrm{Bd}(O_\alpha)\cap(\bigcup H(G))=\emptyset$ for each $O_\alpha\in\mathscr{K}$.*

LEMMA 3.3. *Let G be a locally null, monotone decomposition of X. Let $\mathscr{K}$ be an open covering of $H(G)$. Then there exists a refinement $\mathscr{K}'$ of $\mathscr{K}$ such that*

(1) *$\mathscr{K}'$ and $P(\mathscr{K}')=\{P(K')|\ K'\in\mathscr{K}'\}$ are countable, disjoint collections of subsets of X and X/G, respectively;*

(2) *$\mathscr{K}'$ is a locally null open covering of $H(G)$ such that each point of $X-\mathrm{Cl}(\bigcup H(G))$ possesses a neighborhood which intersects only finitely many members of $\mathscr{K}'$, and $P(\mathscr{K}')$ is a collection of open sets of X/G; and*

(3) *$\mathrm{Bd}(K)\cap(\bigcup H(G))=\emptyset$ for each $K\in\mathscr{K}'$.*

Proof. By Proposition 3.1, $H(G)$ is countable and G is usc and 0-dimensional. Let $H(G)=\{g_1,g_2,\ldots\}$. By (Lemma 6, [6]) we have that $\mathscr{K}$ is refined by $\mathscr{K}''$, where $\mathscr{K}''$ is a countable, disjoint, open covering of $H(G)$. For each n let $K_n''\in\mathscr{K}''$ such that $g_n\subset K_n''$, let $\delta_n>0$ such that $\delta_n<1/n$ and $N(g_n,\delta_n)\subset K_n''$, and let $K_n'=N(g_n,\delta_n)G$. We choose $\mathscr{K}'$ be to $\{K_n'\}$. Clearly $\mathscr{K}'$ is a countable, disjoint, open covering of $H(G)$. By Proposition 3.2, (3) of the conclusion is satisfied. By the definition of $\mathscr{K}'$, the definition of P, and the monotonicity and upper semicontinuity of G, $P(\mathscr{K}')$ is a countable, disjoint, open collection of subsets of X/G. We now show $\mathscr{K}'$ is locally null. Let $x\in X$. We must find an open set A containing x such that $\{K'|\ K'\cap A\neq\emptyset\}$ is a null family. We distinguish three cases. First assume $x\notin\mathrm{Cl}(\bigcup H(G))$. Then there is $\delta>0$ such that $N(x,\delta)\cap\mathrm{Cl}(\bigcup H(G))=\emptyset$. It follows that $d(g_n,N(x,\delta/2))\geqslant\delta/2$ for each $g_n\in H(G)$. Choose a positive integer N such that $1/N<\delta/2$. Then $n\geqslant N$ implies $N(g_n,1/n)\cap N(x,\delta/2)=\emptyset$. Thus $N(x,\delta/2)$ intersects only finitely many members of $\mathscr{K}'$, i.e. $\{K'|\ K'\cap N(x,\delta/2)\neq\emptyset\}$ is a null family. Now assume $x\in\mathrm{Cl}(\bigcup H(G))-\bigcup H(G)$. Let U be an open set containing x such that $\{g_n|\ g_n\cap U\neq\emptyset\}$ is a null family. Since $\{P(x)\}\cup P(H(G))$ is a countable and hence 0-dimensional subset of X/G, by (Lemma 5, [6]) there is a neighborhood V of x such that $V\subset U$ and $\mathrm{Bd}(V)\cap(\bigcup H(G))=\emptyset$. Then $\{g_n|\ g_n\cap V\neq\emptyset\}$ is a null family and equals $\{g_n|\ g_n\subset V\}$ by the monotonicity of G. Denote this collection by $\{g_m\}$. Now $\operatorname{diam}N(g_m,1/m)\leqslant\operatorname{diam}g_m+2/m$. Let $\varepsilon>0$ and choose a positive integer N such that $m\geqslant N$ implies $\operatorname{diam}g_m\leqslant\varepsilon/2$ and $2/m\leqslant\varepsilon/2$. If $m\geqslant N$, then $\operatorname{diam}N(g_m,1/m)\leqslant\varepsilon$. Hence only finitely many members of $\{K_n'|\ K_n'\cap V\neq\emptyset\}$ have diameter greater than ε, i.e. $\{K_n'|\ K_n'\cap V\neq\emptyset\}$ is a null family. Now assume $x\in g$ where $g\in H(G)$. Let G_1 be the decomposition of X such that $H(G_1)=H(G)-\{g\}$. Then G_1 is a locally null, monotone decomposition of X and hence is usc and 0-dimensional. This case now follows as in the second case. Therefore $\mathscr{K}'$ is a locally null family. That $\mathscr{K}'$ has the property that each point of $X-\mathrm{Cl}(\bigcup H(G))$ possesses a neighborhood which intersects only finitely many members of $\mathscr{K}'$ has been shown in the first case. Thus $\mathscr{K}'$ and $P(\mathscr{K}')$ satisfy all conditions of the conclusion.

LEMMA 3.4. *If we assume in Lemma 3.3 that G is a locally compact decomposition, then $P(\mathscr{K}')$ will have the additional property of being a locally null collection of subsets of X/G.*

Proof. This follows from Lemmas 3.2 and 3.3.

§ 4. Main results.

LEMMA 4.1. *Let G be a monotone,* usc *decomposition of X and let A be an open set in X.*

(1) *Let G_1 be the decomposition of X such that $H(G_1)=H(G(A))$ and let $P\colon X\to X/G$ and $P_1\colon X\to X/G_1$ be the projections. If $\mathrm{Bd}(A)\cap(\bigcup H(G))=\emptyset$, then $P_1(\mathrm{Cl}(A))$ is homeomorphic to $P(\mathrm{Cl}(A))$.*

(2) *Let G_1 be the decomposition of A such that $H(G_1)=H(G(A))$ and let $P\colon X\to X/G$ and $P_1\colon A\to A/G_1$ be the projections. If $\bigcup H(G)\subset A$, then $P_1(A)$ is homeomorphic to $P(A)$ and hence $P_1(B)$ is homeomorphic to $P(B)$ for each $B\subset A$.*

LEMMA 4.2. *Let G be a locally null, locally starlike decomposition of X. If X is locally compact and connected, then G is shrinkable and X/G is homeomorphic to X.*

Proof. We emphasize that X is metrized by d. Let $\varepsilon>0$ and let U be an open set containing $\bigcup H(G)$. Since G is locally null and monotone, we may let $G(\varepsilon)=\{g_1,g_2,\ldots\}$. For each $g_n\in G(\varepsilon)$, let (V_n',d_n) be the open subspace of X such that (V_n',d_n) is a SC-WR-CE metric space and g_n is a properly starlike w.r.t. p_n subset of (V_n',d_n), where $p_n\in g_n$. Now $\{(V_n',d_n)\}$ is an open cover of $G(\varepsilon)$. By Lemma 3.1 and (Lemma 9, [6]) we may refine $\{(V_n',d_n)\}$ by a collection $\{(V_n,d_n)\}$ of open sets covering $H(G)$ such that

(1) each $\mathrm{Cl}(V_n)$ is compact;

(2) $g_n\subset V_n\subset\mathrm{Cl}(V_n)\subset V_n'\cap U$ for each n;

(3) $\mathrm{Cl}(V_n)\cap\mathrm{Cl}(V_m)=\emptyset$ if $n\neq m$;

(4) $\mathrm{Bd}(V_n)\cap(\bigcup H(G))=\emptyset$ for each n; and

(5) $\{V_n\}$ is a locally null collection of subsets of X with respect to the metric d.

For each n let i_n be the identity map from (V_n,d) onto (V_n,d_n). Then i_n is a homeomorphism and each of i_n and i_n^{-1} is uniformly continuous on its domain For each n, $G(V_n)$ is a locally null, monotone decomposition of X and may be viewed as such of (V_n,d) and hence of (V_n,d_n). Let $\varepsilon_n>0$ such that $d_n(i_n(x),i_n(y))<\varepsilon_n$ implies $d(x,y)<\varepsilon/2$, where x and $y\in V_n$. By (Lemma 5.1, [10]) there are open sets U_n and M_n and there is a homeomorphism h_n' from (V_n,d_n) onto (V_n,d_n) satisfying

(1) $g_n\subset M_n\subset\mathrm{Cl}(M_n)\subset U_n\subset V_n$;

(2) $\mathrm{Bd}(U_n)\cap(\bigcup H(G))=\emptyset$;

(3) $h_n'|(V_n-M_n)$ is the identity; and

(4) $\operatorname{diam} h_n'(g) < \varepsilon_n$ for each $g \in H(G(U_n))$ in the d_n metric.

Let h_n be defined from $\mathrm{Cl}(V_n)$ onto $\mathrm{Cl}(V_n)$ by

$$h_n(x) = i_n^{-1}\big(h_n'(i_n(x))\big).$$

Now h_n is the identity on $\mathrm{Cl}(V_n) - U_n$. If $g \in H(G(V_n))$, then either $g \subset V_n - U_n$, $h_n(g) = g$, and hence $\operatorname{diam} h_n(g) = \operatorname{diam} g < \varepsilon$ in the metric d, or $g \subset U_n$ and hence $\operatorname{diam} h_n(g) \leqslant \varepsilon/2 < \varepsilon$ in the metric d. Let h be defined from X onto X such that

$$h(x) = \begin{cases} h_n(x), & x \in \mathrm{Cl}(V_n), \\ x, & x \in X - \bigcup V_n. \end{cases}$$

Then it follows that h is a homeomorphism of X onto X (by applying (Theorem 2, [6]) to each of h and h^{-1}), $\operatorname{diam} h(g) < \varepsilon$ for each $g \in H(G)$ in the metric d, and $h|\ (X-U)$ is the identity. Thus G is a shrinkable decomposition of X and by (Theorem 4, [6]) X/G is homeomorphic to X.

LEMMA 4.3. *Let G be a monotone, 0-dimensional* usc *decomposition of X. If G is locally shrinkable and X is locally compact and connected, then G is shrinkable and X/G is homeomorphic to X.*

Proof. The lemma follows from (Theorems 7, 10, and 4 of [6]).

THEOREM 4.1. *Let G be a locally compact, locally null, locally starlike decomposition of a separable metric space (X, d). Then for each open set U containing $\bigcup H(G)$, there is a homeomorphism h from X onto X/G such that $h|(X-U) = P|(X-U)$. Also G is a shrinkable decomposition of X.*

Proof. Let U be an open set containing $\bigcup H(G)$. Since G is locally null and monotone, we have $H(G)$ is countable and G is 0-dimensional. Let

$$H(G) = \{g_1, g_2, \ldots\}$$

and let $\{(V_n, d_n)\}$ be the collection of locally compact, SC-WR-CE subspaces open in (X, d) such that $g_n \subset V_n$ and g_n is a properly starlike w.r.t. p_n subset of (V_n, d_n). We now identify a collection of open generalized continua which covers $H(G)$ and is refined by $\{V_n\}$. Let $\mathscr{V}_1^1 = \{V_1\}$ and $\mathscr{V}_k^1 = \{V_n|\ V_n \cap (\bigcup \mathscr{V}_{k-1}^1) \neq \emptyset\}$, let $\mathscr{V}_1^2 = \{V_2\}$ and $\mathscr{V}_k^2 = \{V_n|\ V_n \cap (\bigcup \mathscr{V}_{k-1}^2) \neq \emptyset\}$, and continue this process so that for each positive integer m, the collection $\{\mathscr{V}_k^m|\ K = 1, 2, \ldots\}$ of subcollections of $\{V_n\}$ is defined. Now for each positive integer m let $C_m = \bigcup_{k=1}^{\infty} (\bigcup \mathscr{V}_k^m)$. It follows that C_m is the component of $\bigcup \{V_n\}$ containing V_m. We note that each (C_m, d) is a locally compact, connected metric space. Let G_m be the decomposition of C_m such that $H(G_m) = H(G(C_m))$. From either the fact that $\mathrm{Bd}(C_m) \cap (\bigcup H(G)) = \emptyset$ or the fact that G_m is locally null we conclude G_m is a locally compact, locally starlike, usc decomposition of C_m. Applying Lemma 4.2 to each G_m and C_m, we have G_m is a shrinkable decomposition of C_m. Now the collection $\{U \cap C_m\}$ is an open covering of $H(G)$ in X and by Lemma 3.4 is refined by $\{B_k^m\}$ such that

(1) $\{B_k^m\}$ and $P(\{B_k^m\}) = \{P(B_k^m)\}$ are countable, disjoint collections of subsets of X and X/G, respectively;

(2) $\{B_k^m\}$ is a locally null, open covering of $H(G)$ in X and $P(\{B_k^m\})$ is a locally null, open collection in X/G;

(3) $\mathrm{Bd}(B_k^m) \cap (\bigcup H(G)) = \emptyset$ for each B_k^m; and

(4) $\mathrm{Cl}(B_k^m) \subset C_m \cap U$ for each B_k^m.

For each k and m let G_k^m be the decomposition such that $H(G_k^m) = H(G_m(B_k^m))$ and viewing G_k^m as a decomposition of C_m, let $p_k^m\colon C_m \to C_m/G_k^m$ be the projection, and viewing G_k^m as a decomposition of X, let ${}_XP_k^m\colon X \to X/G_k^m$ be the projection. By (Theorem 4, [6]) there exists for each m a homeomorphism H_m^k of C_m onto C_m/G_k^m such that $H_m^k|(C_m - B_k^m) = P_k^m(C_m - B_k^m)$. Applying Lemma 4.1 (1) to X, B_k^m, G, and G_k^m, we have that $P(\mathrm{Cl}(B_k^m))$ is homeomorphic to ${}_XP_k^m(\mathrm{Cl}(B_k^m))$, applying Lemma 4.1 (2) to X, G_k^m, C_m, and $\mathrm{Cl}(B_k^m)$, we have that ${}_XP_k^m(\mathrm{Cl}(B_k^m))$ is homeomorphic to $P_k^m(\mathrm{Cl}(B_k^m))$, i.e. $P(\mathrm{Cl}(B_k^m))$ is homeomorphic to $P_k^m(\mathrm{Cl}(B_k^m))$. Letting $i_k^m\colon P_k^m(\mathrm{Cl}(B_k^m)) \to P(\mathrm{Cl}(B_k^m))$ denote the identity, we have i_k^m is a homeomorphism. Now let $h_k^m\colon \mathrm{Cl}(B_k^m) \to P_m(\mathrm{Cl}(B_k^m))$ be defined by $h_k^m = H_m^k|\mathrm{Cl}(B_k^m)$. We have $h_k^m|\mathrm{Bd}(B_k^m) = P_k^m|\mathrm{Bd}(B_k^m) = P|\mathrm{Bd}(B_k^m)$. We now define h from X onto X/G by

$$h(x) = \begin{cases} i_k^m(h_k^m(x)), & x \in B_k^m, \\ P(x), & x \in X - \bigcup B_k^m \end{cases}$$

and claim h is the required homemorphism. It is clear that h is well-defined, onto, and one-to-one. Since P^{-1} is a continuous map of $X/G - \bigcup P(B_k^m)$ into X and $\{B_k^m\}$ and $P(\{B_k^m\})$ are locally null collections of disjoint, open sets in X and X/G, respectively, it follows by a double application of (Theorem 2, [6]) that each of h and h^{-1} are continuous, i.e. h is a homeomorphism. From the construction of h it is clear that $h|(X-U) = P|(X-U)$. As for the shrinkability of G, we note that the only time the local compactness of X is needed in the proof of (Theorem 5, [6]) is when appeal is made to (Lemma 9, [6]). To conclude that G is shrinkable, we use Lemma 3.4 in place of (Lemma 9, [6]) in the proof of (Theorem 5, [6]). The proof of the theorem is complete.

THEOREM 4.2. *Let G be a locally shrinkable, monotone, 0-dimensional,* usc *decomposition of a locally compact, separable metric space (X, d). Then for each open set U containing $\bigcup H(G)$, there is a homeomorphism h from X onto X/G such that $h|(X-U) = P|(X-U)$. Thus G is a shrinkable decomposition of X.*

Proof. The theorem follows from Lemma 4.3 just as Theorem 4.1 follows from Lemma 4.2 except that to refine coverings of $H(G)$ we are allowed to use (Lemma 9, [6]) instead of Lemma 3.4. The shrinkability of G follows from the first assertion of this theorem and (Theorem 5, [6]).

THEOREM 4.3. *Let G be a locally star-0-dimensional,* usc *decomposition of a locally compact, separable metric space (X, d). Then for each open set U containing $\bigcup H(G)$, there is a homeomorphism h from X onto X/G such that $h|(X-U) = P|(X-U)$. Thus G is a shrinkable decomposition of X.*

Proof. That G is locally shrinkable follows from (Theorem 6.1, [10]) and (Theorems 4 and 5, [6]). The theorem now follows from Theorem 4.2.

Recalling that an n-manifold is a separable metric space (not necessarily connected) having the property that each point possesses a neighborhood homeomorphic to either E^n or E^n_+, we have the following results.

COROLLARY 4.1. *Let G be a decomposition of an n-manifold M such that G satisfies one of the following sets of conditions*:

(1) *locally null and locally starlike*;

(2) *locally shrinkable, monotone,* 0*-dimensional, and* usc; *or*

(3) *locally star-*0*-dimensional and* usc.

Then M/G is homeomorphic to M and G is a shrinkable decomposition of M.

Remark. We gave two examples to illustrate Corollary 4.1 in Section 1.

References

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Toroidal decompositions of S^3 and a family of 3-dimensional ANR's (AR's)

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Abstract. It is shown that there exist an ANR X satisfying (1) $X\times S^1\approx S^3\times S^1$, (2) X does not contain any proper ANR of dimension larger than 1, and (3) the homeomorphism group of X is the trivial group; furthermore, there are uncountably many topologically distinct ANR's with these properties. It follows that the family of 3-dimensional AR's satisfying the properties (2) and (3), as above, is also uncountable. These ANR's are constructed as cell-like images of S^3, and hence, they are generalized manifolds and possess many other desirable properties. There exists a cellular image of S^3 satisfying the assertions, (1)–(3), given above (a suitable result for B^3 also holds). A problem of Bing concerning partitions of Peano continua is answered in the negative. A condition (Δ^a) is given and it is shown that a finite dimensional closed subset of an ANR $X\in(\Delta^a)$ has a locally connected ε-displacement inside X. Several other applications are also given.

1. Introduction and terminology.

(1.1) By an AR (ANR) we mean a compact metrizable absolute (neighborhood) retract in the category of metrizable spaces, see [13] and [21] for more details. An ANR X will be called *strongly irreducible* (Abbreviate: *s-irreducible*) if $X\times S^1\approx S^3\times S^1$ and X does not contain any proper ANR of dimension larger than one. Let E^n, B^n, and S^{n-1}, respectively, denote the n-dimensional Euclidean space, the closed unit ball in E^n, and the unit sphere in E^n. By $X\approx Y$ we mean X is homeomorphic to Y.

A method for constructing s-irreducible ANR's (or AR's) is given in [32] where these ANR's are constructed as decomposition spaces corresponding to certain null cell-like but non-cellular upper semicontinuous decompositions of S^3. We prefer to consider these decompositions for S^3 rather than B^3 to avoid technicalities concerning the boundary. It is routine to construct similar decompositions for B^3 once these decompositions for S^3 are known. By an *s-irreducible* decomposition G of S^3 we mean any cell-like upper semicontinuous decomposition G of S^3 such that S^3/G is an s-irreducible ANR. The purpose of this note is to show (1) there exist cellular s-irreducible decompositions of S^3, and (2) there are uncountably many s-irreducible decompositions of S^3. Other applications will also be given.

(1.2) If A is a subset of a metric space (X, d), the *diameter* $\Delta(A)$ of A is defined by $\Delta(A) = \sup\{d(x, y)\colon x, y\in A\}$. If G is an u.s.c. decomposition ("upper semicon-