is BA. By Corollary 2.2, the function $f: X \to \{0, 1\}$ whose $s$th component is the indicator of $A(s)$ is BA. The result of operation $(A)$ on the system $\{A(s) \mid s \in \mathbb{Z}\}$ is

\[
\bigcup_{\ell(s) \neq 0} \cap_{\delta(s) \neq 0} A(s) = f^{-1}(A),
\]

and this is BA by the remark following Theorem 5. Q. E. D.

If $\rho$ is a BP function from $\{0, 1\}^s$ to $\{0, 1\}^s$ satisfying (4), then $\rho_{ni}$ defined by (5), (6) can fail to be BP [9]. If $\{g, \sigma < \varphi\}$ is a BA approach to $g$, it is not known if $g$ can fail to be BA. It is not known whether the BA $\sigma$-field properly contains the BP $\sigma$-field, nor whether the BA $\sigma$-field is properly contained in the $\sigma$-field of absolutely measurable sets. The relation between the BP sets, the BA sets and the $R$ sets [8] has not been determined. Indeed, the cardinality of the class of BA sets is not known. A particularly intriguing question is whether the product of the BA $\sigma$-fields in $X$ and $Y$ is the BA $\sigma$-field in $X \times Y$.

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References


A stabilization property and its applications in the theory of sections

by

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Abstract. We introduce a stabilization property in descriptive set theory which generalizes the topological and measure theoretical situations. An associated theory of sections for measurable sets in products is developed.

I. Preliminaries. The aim of this section is to make the text more self-contained. We will introduce the various classical notions and properties, which are the starting point of this work. They can also be found in [12].

DEFINITION 1.1. Let $E$ be a set. A paving on $E$ will be a class $\mathcal{E}$ of subsets of $E$ containing the empty set. We will call $(E, \mathcal{E})$ a paved set.

DEFINITION 1.2. If $(E, \mathcal{E})$ is a paved set, we denote by $\mathcal{E}^\circ$: the class of subsets $A$ of $E$ such that $E \setminus A$ belongs to $\mathcal{E}$, $\mathcal{E}^\circ = \mathcal{E} \cap \mathcal{E}^c$.

$\mathcal{E}^\circ$ (resp. $\mathcal{E}^+$, $\mathcal{E}^-$): the stabilization of $\mathcal{E}$ for finite intersection (resp. finite union, finite intersection and finite union, countable intersection and countable union).

$\mathcal{E}(\mathcal{E})$: the $\sigma$-algebra generated by $\mathcal{E}$.

DEFINITION 1.3. Let $(E_i, \mathcal{E}_i)_{i \in I}$ be a family of paved sets. The set $\mathcal{E}$ of subsets of $E = \prod E_i$ of the form $\prod A_i$, where $A_i \in \mathcal{E}_i$ for each $i \in I$, is called the product paving $\mathcal{E}$.

PROPOSITION 1.4. Let $(E_i, \mathcal{E}_i)_{i \in I}$ be paved sets such that $E_i \in \mathcal{E}_i$, for each $i \in I$. Then $\mathcal{E}(\prod \mathcal{E}_i)$ contains the product $\sigma$-algebra $\bigotimes_i \mathcal{E}(\mathcal{E}_i)$. If moreover $I$ is countable, then $\mathcal{E}(\prod \mathcal{E}_i) = \bigotimes_i \mathcal{E}(\mathcal{E}_i)$.

In fact, only finite and countable products will be involved here.

Let $(E, \mathcal{E})$ be a paved set and let $(K_i)_{i \in I}$ be a family of elements of $\mathcal{E}$. We will say that $(K_i)_{i \in I}$ has the finite intersection property provided $\bigcap J K_i \neq \emptyset$ whenever $J$ is a finite subset of $I$. 

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DEFINITION 1.5. A paving \( \mathcal{E} \) on a set \( E \) is said to be compact (resp. semi-compact) if every family (resp. every countable family) of elements of \( \mathcal{E} \), possessing the finite intersection property, has nonempty intersection.

By a simple ultra-filter argument, we obtain

PROPOSITION 1.6. If \( \mathcal{E} \) is a compact (resp. semi-compact) paving on \( E \), then also \( \mathcal{E}^* \) is compact (resp. semi-compact).

The following proposition is immediate

PROPOSITION 1.7. Let \( (E_i, \mathcal{E}_i)_{i \in I} \) be a family of paved sets. If each \( E_i \) is compact (resp. semi-compact), then \( \prod_{i \in I} \mathcal{E}_i \) is compact (resp. semi-compact).

We now pass to a proposition which will often be used later (especially in product situations).

PROPOSITION 1.8. Let \( (E, \mathcal{E}) \) be a paved set and \( f \) an application of \( E \) into a set \( F \). We assume that, for each \( x \in E \), the paving consisting of the sets \( f^{-1}(x) \cap A, A \in \mathcal{E} \), is semi-compact. If \( (A_\alpha) \) is a decreasing sequence in \( \mathcal{E} \), then \( \bigcap_{\alpha} A_\alpha = f(\bigcap_{\alpha} A_\alpha) \).

Proof. It is clear that if \( x \in \bigcap_{\alpha} A_\alpha \), then the family \( f^{-1}(x) \cap A_\alpha \) has the finite intersection property. By hypothesis, the set \( f^{-1}(x) \cap A_\alpha \) contains some point \( y \in E \). Hence \( x = f(y) \in f(\bigcap_{\alpha} A_\alpha) \), completing the proof.

\( N \) will denote the set of all positive integers \( 1, 2, \ldots \). Let \( \mathcal{A} = \bigcup_{k=1}^\infty N^k \), consisting of the finite complexes of integers. Take \( \mathcal{K}^* = \mathcal{A} \cup \{ \emptyset \} \). If \( c \in \mathcal{K}^* \), let \( |c| \) be the length of \( c \). If \( c, d \in \mathcal{K}^* \), we write \( c < d \) if \( c \) is an initial section of \( d \). Let \( \mathcal{N}^* = N^\omega \).

DEFINITION 1.9. Let \( (E, \mathcal{E}) \) be a paved set. A Souslin scheme \( (A_\alpha)_{\alpha \in \mathcal{K}^*} \) on \( \mathcal{E} \) will be a mapping of \( \mathcal{K}^* \) into \( \mathcal{E} \). The scheme \( (A_\alpha)_{\alpha \in \mathcal{K}^*} \) is said to be regular if \( A_\alpha \supseteq A_\beta \) whenever \( c < d \). The result of the scheme \( (A_\alpha)_{\alpha \in \mathcal{K}^*} \) is the set \( \bigcup_{\alpha \in \mathcal{K}^*} A_\alpha = \bigcup_{g \in \mathcal{N}^*} A_g \), where \( g \) runs over \( \mathcal{N}^* \).

Let \( (E, \mathcal{E}) \) be a paved set and \( (A_\alpha)_{\alpha \in \mathcal{K}^*} \) a scheme on \( \mathcal{E} \). For each complex \( c \in \mathcal{K}^* \), we introduce the following sets:

\[ A_{c_0} = \bigcup_{\alpha \in c_0} A_{\alpha_0}, \]
\[ A(c) = \bigcup_{\alpha \in c} A_{\alpha}, \]

where \( \alpha \) runs over \( \mathcal{N}^* = \{ v \in \mathcal{N}^*; c < v \} \).

Obviously, the following properties hold

PROPOSITION 1.10. If \( c \in \mathcal{K}^* \), then

\[ A_{c_0} \subseteq \mathcal{E}, \]
\[ A(c) \subseteq A_{c_0}, \]
\[ A[c] = \bigcup_{\alpha \in c} A[c, \alpha], \]

\[ \emptyset \cup \{ A[c, \alpha]; c \in \mathcal{K}^* \} \text{ is a paving on } \mathcal{N}, \text{ which we denote by } \mathcal{F}. \]

The reader will easily verify

PROPOSITION 1.11. \( \mathcal{F} \) is a compact paving on \( \mathcal{N} \).

The following result is basic in the theory of analytic sets.

PROPOSITION 1.12. Let \( (E, \mathcal{E}) \) be a paved set and \( (A_\alpha)_{\alpha \in \mathcal{K}^*} \) a regular scheme on \( \mathcal{E} \), with result \( A \). If \( \forall \alpha \in \mathcal{N} \), then \( \bigcap_{\alpha \in \mathcal{N}} A_{[\alpha]} \subseteq A \).

Proof. Suppose \( x \in \bigcap_{\alpha} A_{[\alpha]} \). For each \( k \in \mathcal{N} \), we introduce the set

\[ K_k = \{ \mu \in \mathcal{N}; \mu_k \leq \nu_1, \ldots, \mu_k \leq \nu_k \text{ and } x \in A_{\mu_1, \ldots, \nu_k} \}, \]

which is clearly a nonempty member of \( \mathcal{F}^* \). By the regularity of the scheme, the sequence \( (K_k) \) is decreasing.

Since, by 1.6, also \( \mathcal{F}^* \) is compact, we obtain some \( \mu \in \bigcap \mathcal{K}_k \). It follows that

\[ x \in \bigcap_{\alpha \in \mathcal{N}} A_{[\alpha]} \subseteq A, \text{ completing the proof.} \]

DEFINITION 1.13. Let \( (E, \mathcal{E}) \) be a paved set. A subset \( A \) of \( E \) is said to be \( \mathcal{E} \)-analytic if it is the result of a Souslin scheme on \( \mathcal{E} \). Let \( \mathcal{A}(\mathcal{E}) \) denote the class of all \( \mathcal{E} \)-analytic subsets of \( E \). The members of \( \mathcal{A}(\mathcal{E}) \) (resp. \( b \mathcal{A}(\mathcal{E}) \)) are called \( \mathcal{E} \)-coanalytic (resp. \( \mathcal{E} \)-bi analytic).

The main property of \( \mathcal{A}(\mathcal{E}) \) is the following:

PROPOSITION 1.14. \( b \mathcal{A}(\mathcal{E}) = \mathcal{A}(\mathcal{E}) \).

In fact the proof of this property consists in the reduction of a scheme of schemes to a single scheme. Although the idea is quite simple, its working-out is rather complicated. For the details, we refer the reader to [15] for instance.

The class of the analytic sets is stable under projection in the following sense:

PROPOSITION 1.15. Let \( (E, \mathcal{E}) \) and \( (F, \mathcal{F}) \) be paved sets, such that the paving \( \mathcal{F} \) is semi-compact. If \( A \subseteq E \times F \) belongs to \( \mathcal{A}(\mathcal{E} \times \mathcal{F}) \), then \( \pi(A) \) is a member of \( \mathcal{A}(\mathcal{E}) \), if \( \pi: E \times F \to E \) is the projection.

Proof. Let \( A \) be the result of the scheme \( (E, \mathcal{E}) \) on \( \mathcal{F} \), and \( F_\alpha \in \mathcal{F} \) for each \( \alpha \in \mathcal{K} \). We define a scheme \( (B_\alpha)_{\alpha \in \mathcal{K}} \) on \( \mathcal{E} \) by taking \( B_\alpha = E_\alpha \).
if \( \bigcap_{k=1}^{n} F_{ik} \neq \emptyset \) and \( R_k = \emptyset \) otherwise. Since for each \( v \in \mathcal{A} \) we obtain that
\[
\prod_{k \in \mathcal{A}} R_k = \bigcap_{k \in \mathcal{A}} R_k,
\]
the result of the scheme \((R_k)_{k \in \mathcal{A}}\) is precisely \(\pi(A)\).

To each subset \( R \) of \( \mathcal{A}^* \) we associate a transfinite system \((R_k)_{k \in \mathcal{A}}\), which
we define inductively as following
\[
R_0 = R,
R_{k+1} = \{c \in R \mid \text{there exists } d \in R_k \text{ with } c < d \text{ and } c \neq d\}.
\]
If \( y \) is a limit ordinal, take \( R_y = \bigcap_{k < y} R_k \). It is easily verified that the sequence \((R_k)_{k \in \mathcal{A}}\)
is decreasing. Because \( R \) is at most countable, the sequence stabilizes. Let
\[
i(R) = \inf\{c \in \mathcal{A} \mid R_k = R_{k+1} \}
\]
which is called the ordinal of \( R \).

We are now able to introduce the Lusin–Sierpiński index, which is of fundamental importance in the study of Souslin schemes.

**Definition 1.16.** Let \((E, \mathcal{S})\) be a paved set and \((A_k)_{k \in \mathcal{A}}\) a regular scheme on \( \mathcal{S} \).

Suppose \( x \in E \) and consider \( R(x) = \{a \mid a \in \mathcal{S} \wedge x \in A_a\} \). Let \( \eta \in i(R(x)) \).
If \( R(\eta) = \emptyset \), let \( i(\eta) = \eta \). If \( R(\eta) \neq \emptyset \), let \( i(\eta) = a_0 \). The ordinal \( i(\eta) \) is called the Lusin–Sierpiński index of the scheme \((A_k)_{k \in \mathcal{A}}\) in the point \( x \).

Remark that a predecessor of a member of \( R(\eta) \) is also in \( R(\eta) \) and in particular \( R(\eta) \neq \emptyset \) if and only if \( \emptyset \notin R(\eta) \).

**Proposition 1.17.** If \( A \) is the result of the regular scheme \((A_k)_{k \in \mathcal{A}}\), then \( i(\eta) = a_0 \) if and only if \( \emptyset \notin A \).

**Proof.** 1. If \( x \in A \), then \( x \in \bigcap_{k < a_0} A_k \), for some \( v \in \mathcal{A} \). It is easily verified, using induction, that for each \( x < a_0 \), the set \( R(x) \) contains every initial section of \( v \).

2. If \( i(\eta) = a_0 \), then \( R(\eta) = R(\eta) \). Suppose \( k \in R(\eta) \), and therefore every element of \( R(x) \), has a strict successor in \( R(x) \). Assume \( R(\eta) \neq \emptyset \). Then we find some \( v \in \mathcal{A} \) so that \( v \mid k \in R(\eta) \) for each \( k \in \mathcal{N} \). Hence also \( v \mid k \in R(\eta) \) for each \( k \in \mathcal{N} \), implying \( x \in A \).

**Proposition 1.18.** If \( i(\eta) < a_0 \), then \( i(\eta) \) is never a limit ordinal.

**Proof.** If \( i(\eta) \) would be a limit ordinal, we would obtain that
\[
R(\eta) = \bigcap_{k \in \mathcal{A}} R(\eta),
\]
for each \( x < \eta \) we have that \( R(x) = R(\eta) \), and hence \( R(\eta) = \emptyset \).

It follows that \( \emptyset \in R(\eta) \), which is a contradiction.

**Definition 1.19.** Let \((E, \mathcal{S})\) be a paved set and \((A_k)_{k \in \mathcal{A}}\) a regular scheme on \( \mathcal{S} \).

If \( x \in E \), then we define for each \( c \in \mathcal{A}^* \) a subset \( R(c, x) \) of \( \mathcal{A}^* \) and an ordinal \( i(c, x) \) by taking
\[
R(\emptyset, x) = \emptyset,
R(c, x) = \{d \in \mathcal{A}^* \mid x \in A_d\} \text{ if } c \neq \emptyset.
\]
If \( \eta = i(R(c, x)) \), let \( i(c, x) = \eta \) if \( R(c, x) = \emptyset \) and \( i(c, x) = a_0 \) if \( R(c, x) = \emptyset \).

Of course \( i(\emptyset, x) = i(\emptyset) \). If \( c \neq \emptyset \), then \( i(c, x) \) is the Lusin–Sierpiński index of the scheme \((A_k)_{k \in \mathcal{A}}\) if \( x \in A_k \). In virtue of 1.17 and 1.18, we obtain that
\[
i(c, x) = a_0 \text{ if and only if } x \in \bigcap_{k \in \mathcal{A}} A_k \text{ and otherwise } i(c, x) \text{ is never a limit ordinal}.
\]

**Proposition 1.20.** If \( a < a_0 \), and \( c, d \in \mathcal{A}^* \), then \( d \in R(c, x) \) if and only if \( (c, d) \in R(\cdot, x) \).

**Proof.** If \( c = \emptyset \), there is nothing to prove. If \( c \neq \emptyset \), we proceed again by induction on \( \emptyset \).

**Proposition 1.21.** If \( c \in \mathcal{A}^* \), then \( i(c, x) = \inf(a_0, \sup(i(c, x), x) + 1) \).

**Proof.** If \( i(c, x) = a_0 \), then \( R(c, x) \) contains every initial section of some \( v \in \mathcal{A} \). Therefore \( R((c, x), x) \) contains every section of the sequence \( u \) defined by \( \mu_k = \gamma_{x+1} \). It follows that \( i((c, x), x) = a_0 \).

Assume now \( i(c, x) < a_0 \). Then also \( i((c, x), x) < a_0 \) for each \( n \in \mathcal{N} \).

1. If \( n \in \mathcal{N} \) and \( a < \inf((c, n), x) \), then \( R((c, n), x) = \emptyset \) and thus contains \( \emptyset \).

It follows that \( n \in R(c, x) \), and thus \( \emptyset \in R(c, x) \). Therefore \( R(c, x) \neq \emptyset \). Since \( i((c, n), x) \) is not a limit ordinal, it follows that \( i((c, x), x) > \inf((c, n), x) \). Therefore \( i((c, x), x) \neq \emptyset \). This completes the proof.

Proceeding by induction, we deduce easily from 1.21.

**Proposition 1.22.** If \((A_k)_{k \in \mathcal{A}}\) is a regular scheme on \( \mathcal{A} \), then \( \{x \in E \mid (i(c, x) > 0) \} \) is a member of \( \mathcal{A}^* \) whenever \( c \in \mathcal{A}^* \) and \( c < a_0 \).

**II. A stabilization property.** The topic of this section is to define a stabilization property, which we will call \( (S) \). It will provide us a generalization of various situations, especially the topological and measure-theoretical case.

**Definition 2.1.** Let \( E \) be a set and \( \mathcal{E}, \mathcal{R} \) pavings on \( E \). We agree to say that \((E, \mathcal{E}, \mathcal{R})\) is basic, if:
1. \( \mathcal{E} \) is stable under finite intersection.
2. If \( A \in \mathcal{R} \) and \( B \in \mathcal{A} \), then also \( B \in \mathcal{R} \).

**Definition 2.2.** Let \((E, \mathcal{E}, \mathcal{R})\) be basic. We say that \((E, \mathcal{E}, \mathcal{R}) \) has property \( (S) \) if moreover the following is true:

Let \((A_k)_{k \in \mathcal{A}}\) be a regular scheme on \( \mathcal{A} \) with index \( i \). Then either the result of the scheme is nonempty or \( \{x \in E \mid (i(c, x) > 0) \} \) for some \( a < a_0 \), and hence for the preceding countable ordinals. It is clear that \( (S) \) is preserved if \( \mathcal{E} \) decreases and \( \mathcal{R} \) increases. The following proposition will provide us a more explicit formulation of property \( (S) \).

**Proposition 2.3.** Let \((E, \mathcal{E}, \mathcal{R})\) be basic. Then the following properties are equivalent:
1. Let for each $c \in \mathcal{A}^*$ a transfinite system $(A^s_n)_{n<\omega}$ of sets in $\mathcal{A}^*$ be given, verifying:

1. $(A^s_n)_{n<\omega}$ is a regular scheme on $\mathcal{A}^*$.
2. $A^s_0 \supseteq A^s_1$ if $s < t$.
3. $A^s_{n+1} \supseteq \bigcup A^s_n$.

Then either $(A^s_n)_{n<\omega}$ has a nonempty result or $A^s_n \in \mathcal{R}$ for some $n<\omega$.

II. $(E, \mathcal{A}^*, \mathcal{R})$ has property $(S)$.

III. The same as $(I)$, but where $\mathcal{A}^*$ is replaced by $2^\mathcal{A}^*$.

Proof. I $\Rightarrow$ II. Assume $(A^s_n)_{n<\omega}$ a regular scheme on $\mathcal{A}^*$ and define $A^*_s = \{ x \in E; i(c, x) > s \}$, which belongs to $\mathcal{A}^*$ by (1). Applying 1.21, we see that the conditions of (I) are satisfied. Therefore either $(A^s_n)_{n<\omega}$ has nonempty result or $A^s_n = \{ x \in E; i(x) > s \}$ for some $n<\omega$.

II $\Rightarrow$ III. Let for each $c \in \mathcal{A}^*$ a transfinite system $(A^s_n)_{n<\omega}$ of subsets of $E$ be given, satisfying (1), (2), (3). We consider the scheme $(A^s_n)_{n<\omega}$ on $\mathcal{A}^*$, in which the reader will easily verify by induction on $n<\omega$, $A^s_n = \{ x \in E; i(c, x) > n \}$.

If $(A^s_n)_{n<\omega}$ has an empty result, then $(x \in E; i(x) > n)$ and hence $A^s_n \in \mathcal{R}$ for some $n<\omega$.

III $\Rightarrow$ I. This is obvious.

It is clear that if $(E, \mathcal{A}^*, \mathcal{R})$ has $(S)$, then also $(E, \mathcal{A}^*, \mathcal{R}_1)$ has $(S)$, where $\mathcal{R}_1 = \{ A \in \mathcal{A}; \mathcal{A} \in \mathcal{A}_1, A \in \mathcal{R} \cap \mathcal{A}^* \}$. Some examples are in order. The first example requires the notion of a capacity.

Definition 2.4. Let $(E, \mathcal{A}^*)$ be a paved set such that $\mathcal{A}^*$ is stable under finite union and finite intersection. An $\mathcal{A}^*$-capacity on $E$ will be a real valued function $I$ defined on $\mathcal{A}^*$, verifying the following conditions:

1. $I$ is increasing: $A \subseteq B \Rightarrow I(A) \leq I(B)$.
2. If $A^*_n$ is an increasing sequence of subsets of $E$, then $\bigcup A^*_n = \sup I(A^*_n)$.
3. If $A^*_n$ is a decreasing sequence in $\mathcal{A}^*$, then $\bigcap A^*_n = \inf I(A^*_n)$.

Example I. Let $(E, \mathcal{A}^*)$ be a paved set such that $\mathcal{A}^*$ is stable under finite union and finite intersection. Let $I$ be an $\mathcal{A}^*$-capacity on $E$ by $I(\emptyset) = 0$. If we take $\mathcal{R} = \{ A \in \mathcal{A}; I(A) = 0 \}$, then $(E, \mathcal{A}^*, \mathcal{R})$ has property $(S)$.

Proof. Let for each $c \in \mathcal{A}^*$ a transfinite system $(A^s_n)_{n<\omega}$ of subsets of $E$ be given, such that (1), (2), (3) of Proposition 2.3 are satisfied.

If $c \in \mathcal{A}^*$ with $|c| = k$ and $n<\omega$, let

$$A^s_n = \bigcup_{n<\omega} A^s_n.$$  

Assume $A^s_n \notin \mathcal{R}$ for each $n<\omega$. Then there is some $k>0$ with $I(A^s_n) > \varepsilon$ for each $n<\omega$. By induction on $k$, we construct a sequence $(n_k)\subseteq\omega$ of integers satisfying

$I(A^s_{n_k}) > \varepsilon$ for each $n<\omega$ and $k \in \mathbb{N}$.

For each $x < \omega$, we have that $I(A^s_{n_k}) > \varepsilon$ and $A^s_{n_k+1} \subseteq \bigcup A^s_{n_k}$. Therefore there must be some $n_k \in \mathbb{N}$ so that $I(A^s_{n_k}) > \varepsilon$ for each $x < \omega$. Suppose $n_1, \ldots, n_k$ obtained verifying $I(A^s_{n_k}) > \varepsilon$ for each $x < \omega$. Then for each $x < \omega$, we have that $A^s_{n_k+1, \ldots, n_k} \subseteq \bigcup A^s_{n_k, \ldots, n_k}$. Therefore there must be some $n_{k+1} < \omega$ so that $I(A^s_{n_k, \ldots, n_k}) > \varepsilon$ for each $x < \omega$.

So the construction is complete.

Since in particular $(A^s_{n_k, \ldots, n_k})$ is a decreasing sequence in $\mathcal{A}^*$ and $I(A^s_{n_k, \ldots, n_k}) > \varepsilon$ for each $x < \omega$, we find that $\bigcap A^s_{n_k, \ldots, n_k} \notin \mathcal{R}$. But by 1.12, this set is contained in the result of the scheme $(A^s_n)_{n<\omega}$, which is therefore also nonempty.

Example II. Let $(E, \mathcal{A}^*)$ be a paved set such that $\mathcal{A}^*$ is semi-compact and stable under finite union and finite intersection. If $\mathcal{R} = \{ \emptyset \}$, then $(E, \mathcal{A}^*, \mathcal{R})$ has property $(S)$.

Proof. Define $I$ on $2^E$ by taking $I(\emptyset) = 0$ and $I(A) = 1$ if $A \neq \emptyset$. Clearly $I$ is an $\mathcal{A}^*$-capacity. We obtain a special case of Example I.

The following example is of different nature.

Example III. Let $(E, \mathcal{A}^*)$ be a paved set such that $\mathcal{A}^*$ is stable under countable union and countable intersection. Let $\mathcal{R}$ be a class of subsets of $E$, such that:

1. $\mathcal{R}$ is $\mathcal{A}^*$-ideal.
2. If $(A^s_n)_{n<\omega}$ is decreasing in $\mathcal{A}^*$, then there is some $\eta < \omega_1$ such that $A^s_\eta \notin \mathcal{R}$ whenever $n > \eta$.

Then $(E, \mathcal{A}^*, \mathcal{R})$ has property $(S)$.

Proof. Let for each $c \in \mathcal{A}^*$ a transfinite system $(A^s_n)_{n<\omega}$ of subsets of $E$ be given, such that (1), (2), (3) of Proposition 2.3 are satisfied. There exists $x < \omega_1$ so that $A^s_{x+1} \notin \mathcal{R}$ for each $x \in \mathcal{A}^*$ and $x \neq \eta$. Remark that $\bigcup (A^s_{x+1}) \in \mathcal{R}$. If $A^s_\eta \notin \mathcal{R}$, then there is $x \in A^s_\eta$ not belonging to $\bigcup (A^s_{x+1}) \in \mathcal{R}$. By induction on $k$ we construct a sequence $(n_k)\subseteq\omega$ of integers satisfying $x \in A^s_{n_k}$ for each $k \in \mathbb{N}$.

Since $x \in A^s_n$ and $x \notin A^s_{n_k+1}$, we obtain that $x \in A^s_n \subseteq \bigcup A^s_{n_k}$. Thus there is $n_k \in \mathbb{N}$ with $x \in A^s_{n_k}$. Suppose $n_1, \ldots, n_k$ obtained such that $x \in A^s_{n_k}$. Since $x \notin A^s_{n_k+1}$, we obtain $x \in A^s_{n_k+1} \subseteq \bigcup A^s_{n_k}$. Thus there is $n_{k+1} \in \mathbb{N}$ with $x \in A^s_{n_{k+1}}$, completing the construction.

In particular $x \in A^s_{n_k}$ for each $k \in \mathbb{N}$. Hence $x$ belongs to the result of the scheme $(A^s_n)_{n<\omega}$.

The following example reduces as well to (I) as to (III):

Example IV. Let $(E, \mathcal{A}^*, \mu)$ be a probability space and take

$$\mathcal{R} = \{ A \subset E; \mu^*(A) = 0 \}.$$
Then $(E, \delta, \mathcal{K})$ has property (S). Also the following example, which is an application of (III), is worth to be mentioned.

**Example V.** Let $E$ be a separable metric space, $\delta$ the Baire $\sigma$-algebra and $\mathcal{K}$ the class of first category sets. Then $(E, \delta, \mathcal{K})$ has property (S).

**Proposition 2.5.** Assume $(E, \delta, \mathcal{K})$ with property (S) and let $(K, \mathcal{K})$ be a paved set such that $\mathcal{K}$ is semi-compact and stable under finite intersection. Let $\pi: E \times K \rightarrow E$ be the projection and consider $\pi^{-1}(\mathcal{K}) = \{A \subseteq E \times K : \pi(A) \in \mathcal{K}\}$. Then $(E \times K, \delta \times \mathcal{K}, \pi^{-1}(\mathcal{K}))$ has property (S).

**Proof.** First, remark that $(E \times K, \delta \times \mathcal{K}, \pi^{-1}(\mathcal{K}))$ is basic. For each $c \in \mathcal{K}$, let $(A_k)_{k \in \mathcal{K}}$ be a transfinite system of subsets of $E \times K$ satisfying (1), (2), (3) of 2.3. Then the subsets $\pi(A_k^c)$ of $E$ also satisfy (1), (2), (3) of 2.3, with respect to the paving $\mathcal{A}$. Suppose there is $c \in \mathcal{A}$ so that $\bigcap_{c \in \mathcal{A}} \pi(A_k^c) \neq \emptyset$. Since $\bigcap_{c \in \mathcal{A}} \pi(A_k^c) = \pi(\bigcap_{c \in \mathcal{A}} A_k^c)$, by 1.8, we see that also $(A_k^c)_{k \in \mathcal{K}}$ has a nonempty result. Otherwise $A_k^c \in \pi^{-1}(\mathcal{K})$ for some $c \in \mathcal{K}$.

The next result requires the following lemma, which is more technical than basically difficult.

**Proposition 2.6.** Assume $(E, \delta, \mathcal{K})$ with property (S). Let for each $k \in \mathbb{N}$ and $(c_1, ..., c_k) \in (\mathcal{K}^k)$ a set $W_{c_1, ..., c_k}$ in $\mathcal{A}$ and a transfinite system $(V_{c_1, ..., c_k})_{c_1, ..., c_k}$ of subsets of $E$ be given, so that following properties are satisfied:

1. $W_{c_1, ..., c_k} \subseteq V_{c_1, ..., c_k}$ if $c_1 < c_2, ..., c_k < c_1$.
2. $W_{c_1, ..., c_k} \subseteq V_{c_1, ..., c_k}$.
3. $V_{c_1, ..., c_k} \subseteq W_{c_1, ..., c_k}$.
4. $V_{c_1, ..., c_k} \subseteq V_{c_1, ..., c_k}$ if $c_1 < \beta$.
5. $V_{c_1, ..., c_k} = \bigcup_{c_1, ..., c_k} W_{c_1, ..., c_k}$.
6. $V_{c_1, ..., c_k}^{\alpha} \subseteq V_{c_1, ..., c_k}$.

Then one of the following alternatives must occur:

1. $\exists k \in \mathbb{N}$ for some $\alpha < c_1$.
2. There is a sequence $(\delta_k)_{k \in \mathbb{N}}$ such that $\bigcap_{k \in \mathbb{N}} W_{\delta_k, ..., \delta_k} = \emptyset$.

**Proof.** The Cantor enumeration of $\mathbb{N} \times \mathbb{N}$ induces a map

$\mathcal{A} \rightarrow \bigcup_{k \in \mathbb{N}} \mathcal{A}^k : c \mapsto (d_1^c, ..., d_k^c)$,

where the number $k_0(c)$ of complexes is of course only dependent on $|c|$. This map is extended to $\mathcal{K}^k$ by taking $k_0 = 1$ and $d_{k+1} = \emptyset$.

For each $c \in \mathcal{K}^k$, we define

$A^c = W_{d_1^c, ..., d_k^c}$ and $A^c = V_{d_1^c, ..., d_k^c}$ if $\alpha > 0$.

We show that the conditions (1), (2), (3) of 2.3 are verified.

1. To see that the scheme $(A^c)_{c \in \mathcal{K}}$ on $\delta$ is regular, take $c', c'' \in \mathcal{A}^k$ with $c' < c''$.

Then $k_0(c') \leq k_0(c'')$ and $d_0^c < d_0^r, ..., d_{k-1}^c < d_{k-1}^r$. We only have to apply properties 1 and 2.

(2) This follows immediately from properties 3 and 4.

(3) Assume $c \in \mathcal{K}$ and $|c| = r$. We distinguish 2 cases.

**Case I.** $k_r = k_{r+1}$. There is some $k = 1, ..., k_r$ so that $d_k^c = d_k^r$ if $i \neq k$ and $d_k^c = d_k^r$ whenever $n \in \mathbb{N}$. We find

$A^c = A^{(c^r)} = \bigcup_{c \in \mathcal{K}} V_{d_1^c, ..., d_{k-1}^c, d_k^r, ..., d_{k-1}^c, d_k^r} = \bigcup_{c \in \mathcal{K}} V_{d_1^c, ..., d_{k-1}^c, d_k^r} = \bigcup_{c \in \mathcal{K}} A^c$.

**Case II.** $k_r = k_{r+1}$. Then $d_n^c = d_n^r$ if $1 \leq n \leq k$, and $d_n^c = n$ whenever $n \in \mathbb{N}$. We obtain

$A^c = V_{d_1^c, ..., d_{k-1}^c, d_k^c, ..., d_{k-1}^c} = \bigcup_{c \in \mathcal{K}} V_{d_1^c, ..., d_{k-1}^c, d_k^c} = \bigcup_{c \in \mathcal{K}} A^c$.

Since $(E, \delta, \mathcal{A})$ possesses (S), either $(d_r)_{c \in \mathcal{K}}$ for some $\alpha < c_1$ or there is $c \in \mathcal{A}$ with $\bigcap_{c \in \mathcal{A}} A^c \neq \emptyset$. Remark that $A^c = V^c_{d_r}$. If $c \in \mathcal{A}$, then there is a sequence $(\delta_k)_{k \in \mathbb{N}}$ such that $d_k^c < k$ whenever $r \in \mathbb{N}$ and $k < k$. If $k \in \mathbb{N}$ is fixed, then there exists $r \in \mathbb{N}$ with $k < k$ and $\delta_r^c < d_k^c$ for each $l = 1, ..., k$. Then $A^c = W_{d_1^c, ..., d_k^c} = \emptyset$.

This completes the proof.

**Theorem 2.7.** Assume $(E, \delta, \mathcal{K})$ with property (S) and $(K, \mathcal{K})$ a paved set such that $\mathcal{K}$ is semi-compact and stable under finite intersection. We consider the projections $\pi_k: E \times K^k \rightarrow E \times K^k$ and $p_k: E \times K^k \rightarrow E$ for each $k \in \mathbb{N}$, let $(X_k)_{k \in \mathbb{N}}$ be a transfinite system of subsets of $E \times K^k$ such that following properties are satisfied:

1. $X_k^c = (\delta \times \mathcal{K}^k)$-analytic in $E \times K^k$.
2. $X_k^c = X_k^c$ if $c < \beta$.
3. $X_k^c = p_k(X_k^{c+1})$.

Assume $p_k(X_k) \notin \mathcal{K}$ for each $k < c_1$. Then there exist $x \in E$ and $(y_k)_{k \in \mathbb{N}}$ in $K^k$ such that $(x, y_1, ..., y_k) \in X_k^c$ for each $k \in \mathbb{N}$.

**Proof.** Let $X_k^c$ be the result of a regular scheme $(Y^c, c \in \mathcal{K})$. For each $k \in \mathbb{N}$ and $(c_1, ..., c_k) \in (\mathcal{K}^k)$, define

$W_{c_1, ..., c_k} = p_k(\bigcap_{c \in \mathcal{K}} (Y_{c, c_1, ..., c_k}^c \times K^{k-1}))$

and

$V_{c_1, ..., c_k} = p_k(\bigcap_{c \in \mathcal{K}} (Y_{c, c_1, ..., c_k}^c \times K^{k-1}) \times X_1)$.

The reader will easily make out that (1) → (6) of Proposition 2.6 are verified. Hence there are 2 possibilities:

1. There is $\alpha < c_1$ such that $V_{c_1, ..., c_k} = p_k(X_k^c) \in \mathcal{K}$.
2. There is a sequence $(\delta_k)_{k \in \mathbb{N}}$ in $\mathcal{K}$ such that $\bigcap_{c \in \mathcal{K}} W_{c_1, ..., c_k} \in \mathcal{K}$ contains some point

1 Fundamenta Mathematicae CXII
$x \in E$. Therefore $\bigcap_{k} (Y_{\nu_k}(x) \times K^{\nu_k}) \neq \emptyset$ for each $k \in N$. By the semi-compactness of the paving $X^{\nu}$ on $K^{\nu} = \prod_{k} K_{\nu_k}$, we get

$$\bigcap_{k} \bigcap_{\nu_k} (Y_{\nu_k}(x) \times K^{\nu_k}) = \bigcap_{k} \bigcap_{\nu_k} (Y_{\nu_k}(x) \times K^{\nu_k}) \neq \emptyset$$

and thus contains a point $(y_{\nu_k})_{k} \in \prod_{k} K_{\nu_k}$. For each integer $l$, we have

$$(x, y_1, \ldots, y_l) \in \bigcap_{\nu_k} Y_{\nu_k}(x) \subset X^{\nu_l},$$

completing the proof.

We pass to the following first corollary.

**Proposition 2.8.** If $(E, \mathcal{A}, \mathcal{B})$ has property $(S)$, then also $(E, \mathcal{A}(\mathcal{B}), \mathcal{B})$ has property $(S)$.

**Proof.** Let for each $e \in \mathcal{A}(\mathcal{B})$ a transfinite system $(\mathcal{A}(\mathcal{B}))_{\alpha+\nu}$ of subsets of $E$ be given, satisfying

1. $(\mathcal{A}(\mathcal{B}))_{\alpha+\nu}$ is a regular scheme on $\mathcal{A}(\mathcal{B})$,
2. $\mathcal{A}(\mathcal{B}) \supseteq \mathcal{A}(\mathcal{B})$, if $\alpha < \beta$,
3. $\mathcal{A}(\mathcal{B}) \subseteq \bigcup \mathcal{A}(\mathcal{B})$.

Take $K = N$ and let $\mathcal{A} = \emptyset \cup \{a; n \in N\}$, which is a compact paving on $K$, stable under finite intersection. For each $k \in N$ and $x < a_0$, we define $X^a_k = \{x, \nu \in \mathcal{A} \times \mathcal{B}; x \in \mathcal{A}(\mathcal{B})\}$, which clearly satisfy the conditions (1), (2), (3) of 2.7. Therefore we have one of the following 2 possibilities:

1. There is $a < a_0$ so that $p_1(X^a_\nu) \in \mathcal{B}$. But $\mathcal{A}(\mathcal{B}) \subseteq \bigcup \mathcal{A}(\mathcal{B}) = \mathcal{B}$, implying $\mathcal{A}(\mathcal{B}) \cap \mathcal{B} = \emptyset$.

2. There is $x \in E$ and $\nu \in \mathcal{A}$ such that $(x, \nu) \in X^a_\nu$ for each $k \in N$.

Theorem 2.9. Assume $(E, \mathcal{A}, \mathcal{B})$ with property $(S)$ and let $(\mathcal{A}, \mathcal{B})$ be a sequence in $\mathcal{A}(\mathcal{B})$ such that $\bigcap \mathcal{A} = \emptyset$. Then there is a sequence $(\mathcal{A}, \mathcal{B})$ in $\mathcal{A}(\mathcal{B})$ so that $\mathcal{A}(\mathcal{B}) \cap \mathcal{B}$ for each $n$ and $\bigcap \mathcal{B} = \emptyset$.

**Proof.** Each set $\mathcal{A}$ is the result of a regular scheme on $\mathcal{A}$ with index $\nu$. Let $K = \mathcal{A}$ and $X^a_\nu \cap \mathcal{A}(\mathcal{B})$, which is a compact paving on $K$, stable under finite intersection. For each $k \in N$ and $x < a_0$, we define $X^a_\nu = \{x, \nu \in \mathcal{A}(\mathcal{B}); x \in \mathcal{A}(\mathcal{B})\}$, which again satisfy the conditions (1), (2), (3) of 2.7 (cfr. 1.21). Thus there are 2 alternatives:

1. There is $x < a_0$ so that $p_1(X^a_\nu) \in \mathcal{B}$. If we let $\mathcal{B}_0 = \{x \in E; l(x) > a + 1\}$, then $\mathcal{B}_0$ belongs to $\mathcal{A}(\mathcal{B}) \cap \mathcal{B}$. Moreover $\bigcap \mathcal{B}_0 = \bigcap \{x \in E; l(x) > a + 1\}$.

2. There is $a < a_0$ so that $x \in X^a_\nu$. Since $\bigcap \mathcal{B}_0 = \bigcap \{x \in E; l(x) > a + 1\}$ and $D = \{x \in E; l(x) > a\}$, then $\bigcap \mathcal{B}_0 = \bigcap \{x \in E; l(x) > a\}$.

In particular, we obtain the Novikov separation result (see [24]):

**Proposition 2.10.** Let $(E, \mathcal{D})$ be a paved set where $\mathcal{D} = \mathcal{D}$ is semi-compact. If $p_{\mathcal{D}}$ is a sequence in $\mathcal{D}$ such that $\bigcap \mathcal{D} = \emptyset$, then there is a sequence $(\mathcal{D}, \mathcal{B})$ in $\mathcal{D}$ so that $\bigcap \mathcal{D} = \emptyset$.

**III. Applications in section theory.**

**A. Classes of sets.** The starting point will be a paved set $(X, \mathcal{A})$ such that:

1. $X \in \mathcal{A}$.
2. $\mathcal{A}$ is stable under finite union and finite intersection.
3. $\mathcal{A}$ is a transitive scheme (i.e. $\mathcal{A} \in b \mathcal{A}(\mathcal{A})$).
4. $\mathcal{A}$ is a $\sigma$-ideal.
5. If $A \in \mathcal{A}$, then there is $B \in \mathcal{A}$ satisfying $B \subset A$.
6. $(X, \mathcal{A}, \mathcal{B})$ is a class of subsets of $X$ satisfying $B = B \cap \mathcal{A}$ for all $B \in \mathcal{A}$.

**Definition 3.1.** If $\mathcal{A}$ is a class of subsets of $X$, we let $\mathcal{A}^*$ consist of the $A \subset X$ such that there is $B \in \mathcal{A}$ with $AB \in \mathcal{A}$.

**Proposition 3.2.** If $A \in \mathcal{A}$, then there exist $B, C \in b \mathcal{A}(\mathcal{A})$ satisfying $B \subset A$, $A \subset C$ and $A \subset B \in \mathcal{A}$, $A \subset C \in \mathcal{A}$.

**Proof.** Take $A_1 \in \mathcal{A}$ so that $A = A_1 \cap \mathcal{A}$ and consider $D \in \mathcal{A}$ with $A = A_1 \cap D$. It is easily seen that $D = A \cap D$ and $C = A_1 \cup D$ satisfy.

**Proposition 3.3.** $(X, \mathcal{A}, \mathcal{B})$ has property $(S)$.

**Proof.** It is clear that $(X, \mathcal{A}, \mathcal{B})$ is basic. It follows from 3.2 that if $(\mathcal{A}, \mathcal{B})$ is a regular scheme on $X$, then there is a regular scheme $(\mathcal{A}, \mathcal{B})$ on $b \mathcal{A}(\mathcal{A})$ such that $\mathcal{A} \subset A_1$ and $A \subset B \in \mathcal{B}$ for each $e \in \mathcal{A}$. Hence $D = \bigcup (A \setminus B)$ is still a member of $\mathcal{B}$.

Let $l$ and $i$ be the indices of the schemes $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$, respectively. By induction and using 1.21, we see that $(x \in X; l(x) > a; f(x) < a)$ is contained in $D$ for each $e \in \mathcal{A}$ and $x < a_0$. Since, by 2.8, also $(X, b \mathcal{A}(\mathcal{A}), \mathcal{B})$ has property $(S)$, there are 2 possibilities:

1. The scheme $(\mathcal{A}, \mathcal{B})$ and hence certainly $(\mathcal{A}, \mathcal{B})$ have a nonempty result.
2. There is $x \in X$ so that $(x \in X; f(x) > a)$ is a member of $\mathcal{B}$. Since

$$\{x \in X; f(x) > a\} \subset \{x \in X; f(x) > a\} \cup D$$
also \( \{ x \in X; f(x) > x \} \in \mathcal{B} \).

So the proof is complete.

**PROPOSITION 3.4.** \((X)'^* = (X)^*\).

**Proof.** 1. Since \( X \subseteq (X)' \) and \((X)' \) is stable under countable union and countable intersection, \((X)'^* \subseteq (X)^*\).

2. Define \( \mathcal{A} = \{ A \in \mathcal{X}; \langle A \rangle \subseteq (X)'^* \} \) which of course contains \( X \). Moreover \( \mathcal{A} \) is stable under countable union and countable intersection. We give the details for the intersection, the argument for the union being similar.

Let \( \{ A_n \} \) be a sequence in \( \mathcal{A} \), \( A = \bigcap \{ A_n \} \) and \( B \) some set in \( \{ A \} \). If, for each \( n \) we take \( B_n = \{ A_n \setminus (A \setminus B) \} \cup (B \setminus A) \), then \( B_n \) is in \( \{ A \} \) and hence in \((X)'^*\). Thus also \( B = \bigcap B_n \) is in \((X)^*\). So we proved that \( A \in \mathcal{A} \). Therefore \( X \subseteq \mathcal{A} \), implying \((X)'^* \subseteq (X)^*\).

The following is left as an exercise for the reader.

**PROPOSITION 3.5.** \( a(\mathcal{X}')^* = a(\mathcal{X})' \).

**PROPOSITION 3.6.** \( b(\mathcal{X})' = b(\mathcal{X}') \).

**Proof.** It follows from 3.5 that \( b(\mathcal{X}')(X)' = b(\mathcal{X}')(X)' \). If \( A \in b(\mathcal{X})' \), then \( A \subseteq (X)' = (X)' \), and we obtain \( B \subseteq (X)' = (X)' \) so that \( A \subseteq B \), \( X \setminus A \subseteq C \) and \( B \setminus C \subseteq X \), and \( A \subseteq b(\mathcal{X})' \).

Finally \( (X)' = b(\mathcal{X})' \), since \( X \) is bi-analytic.

We let \( \mathcal{U} = \mathcal{U}(X, X, \mathcal{X}) \) be the \( \sigma \)-algebra \( b(\mathcal{X}') \).

**DEFINITION 3.7.** If \( Y \) is a Polish (i.e. a complete metric space which is separable), let \( B_\mathcal{X} \) denote its Borel field. The \( \sigma \)-algebra \( b(\mathcal{X}) \) is the union of the classes \( G_\alpha \) (\( \alpha \in \mathcal{G} \)), where:

(i) \( F_\alpha \) is the family of the closed sets and \( G_\alpha \) the open sets in \( Y \).

(ii) The sets of the family \( F_\beta \) are countable intersections or unions of sets belonging to \( F_\alpha \) with \( \alpha < \beta \) according to whether \( \beta \) is even or odd. The sets of the family \( G_\beta \) are countable unions or intersections of sets belonging to \( G_\alpha \) with \( \alpha < \beta \) according to whether \( \beta \) is even or odd.

The families \( F_\alpha \) with even indices as well as the families \( G_\alpha \) with odd indices form the multiplicative class \( c_n \), the families \( F_\alpha \) with odd indices and the families \( G_\alpha \) with even indices the additive class \( d_n \); (for more details, we refer to [21], p. 345).

We let \( \mathcal{A} = \mathcal{A}(X, Y) = \{ A \subseteq X; A \subseteq X \text{ and } F \text{ closed in } Y \} \).

**PROPOSITION 3.8.** \( \mathcal{M} \otimes \mathcal{A} = \mathcal{P}^* \).

**Proof.** This follows from the fact that \( \mathcal{M} = (X)^* \), \( \mathcal{A} = F_0^* \), and monotonicity arguments.

Let \( A \subseteq X \times Y \), \( x \in X \) and \( y \in Y \). Define

\( A(x) = \{ y \in Y; (x, y) \in A \} \) and \( A(y) = \{ x \in X; (x, y) \in A \} \).

Such sets will be called sections of \( A \).

From 3.8, we deduce the following result

**PROPOSITION 3.9.** If \( A \in \mathcal{M} \otimes \mathcal{A}_0 \), then the sections \( A(x) \), where \( x \) is taken in \( X \), are of bounded Baire class.

**DEFINITION 3.10.** For each \( \alpha < \gamma \), let \( \mathcal{A}_\gamma = \mathcal{A}_\gamma(X, Y) \) be the class of those \( A \in \mathcal{M} \otimes \mathcal{A}_0 \) such that \( A(x) \) is an \( F_0 \)-set for each \( x \in X \) and \( \mathcal{A}_\gamma = \mathcal{A}_\gamma(X, Y) \) the class of the \( A \in \mathcal{M} \otimes \mathcal{A}_0 \) such that \( A(x) \) is a \( G_\gamma \)-set for each \( x \in X \). Hence \( \mathcal{A}_\gamma = \mathcal{O}_\gamma \).

**PROPOSITION 3.9** can be reformulated as following

**PROPOSITION 3.11.** \( \mathcal{M} \otimes \mathcal{A}_\gamma = \bigcup \mathcal{A}_\gamma \bigcup \mathcal{A}_\alpha \), \( \alpha < \gamma \).

**DEFINITION 3.12.** For each \( \alpha < \omega_1 \), we introduce a class \( \mathcal{A}_\alpha = \mathcal{A}_\alpha(X, Y) \) and a class \( \mathcal{A}_\alpha = \mathcal{A}_\alpha(X, Y) \) as follows:

(i) \( \mathcal{A}_0 = \mathcal{A}_0 \) and \( \mathcal{A}_{\alpha} = \mathcal{A}_0 \).

(ii) The sets of the family \( \mathcal{A}_\alpha \) are countable intersections or unions of sets belonging to \( \mathcal{A}_\beta \) with \( \alpha < \beta \) according to whether \( \beta \) is even or odd.

The sets of the family \( \mathcal{A}_\alpha \) are countable unions or intersections of sets belonging to \( \mathcal{A}_\beta \) with \( \alpha < \beta \) according to whether \( \beta \) is even or odd.

By induction, we verify that \( \mathcal{A}_\alpha = \mathcal{A}_\alpha \).

It is clearly seen that \( \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \) for all \( \alpha < \omega_1 \). In fact the following deep property holds

**THEOREM 3.12.** \( \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \) and \( \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta \) for each \( \alpha < \omega_1 \).

**Proof.** We remark that \( \mathcal{A} \) is a \( \sigma \)-algebra on \( X \) satisfying \( \mathcal{A} = b(\mathcal{A}) \). Then the theorem follows from recent results in descriptive set theory obtained by A. Louveau (see [22]).
tively on $a$. If $a = 0$, then every section $A(x)$ of $A$ is closed and hence $A = \mathcal{X}$. Since for every $n \in \mathbb{N}$, $n \in \mathbb{N}$ the set $A_{n \in \mathbb{N}} \in \mathcal{B}$, and hence $A = \mathcal{F}$. Let the property be true for every $a < b$ and assume $A(x)$ is an $\mathcal{F}$ set for each $x \in X$. Clearly there is a sequence $(A_n)$ of subsets of $X \times Y$ such that $\pi_x(A_n) = \pi_y(A)$ for each $n$, the set $A_n(x)$ is in $\mathcal{F}_x$, for each $n$ and each $x \in X$ and $A = \bigcap A_n$ if $\beta$ is even, $A = \bigcup A_n$ if $\beta$ is odd.

Let $n \in \mathbb{N}$ be fixed. If for each $a < b$ we take $A_\ast = \bigcap A_n \in \mathcal{F}$, then $A_{a \in \mathbb{N}}(x)$ is precisely an $F_\sigma$ set, then $A_{a \in \mathbb{N}} = A \cap \bigcap A_n \in \mathcal{F}$, by induction hypothesis. It follows from 3.13 that $A_n = \bigcup A_n \in \mathcal{F}_x$. Hence also $A \in \mathcal{F}_x$, which completes the proof. $\ast$

**Proposition 3.18.** Let $A \in \mathcal{S}$. Then $A \in \mathcal{B} \Leftrightarrow A(x)$ is in $\mathcal{B}_x$ for any $x \in X$. Proof. The "only if" part is precisely 3.9. Assume $A \in \mathcal{S}$, then there exists some $B \in \mathcal{B} \Leftrightarrow \mathcal{B}_x$, such that $\pi_x(A) \in \mathcal{B}_x$. If the sections $A(x)$ are of bounded Baire class, then, again by 3.9, this is also true for the sections $(A \cap B)(x)$ of $A \cap B$ and $(\mathcal{B}_x \cap B_x)(x)$ of $\mathcal{B}_x \cap B_x$. It follows from 3.17 that $A \cap B$ and $\mathcal{B}_x \cap B_x$ are members of $\mathcal{B} \Leftrightarrow \mathcal{B}_x$. Hence $A = (A \cap B \cap \mathcal{B}_x) \in \mathcal{B} \Leftrightarrow \mathcal{B}_x$.

We introduce $\mathcal{A} = \mathcal{A}(X, Y)$ as the class of $\mathcal{P}(X, Y)$-analytic subsets of $X \times Y$. From 3.8 and the fact that $\mathcal{A} = \mathcal{A}^*$, we obtain immediately

**Proposition 3.19.** $\mathcal{B} \in \mathcal{A}(X, Y) \Leftrightarrow \mathcal{A}$. Proof. The following result is similar to 3.17.

**Proposition 3.20.** Let $A \subset X \times Y$ and suppose $\pi_x(A) \not\in \mathcal{B}$, then the section $A(x)$ is analytic in $X$ for each $x \in X$, then $A \not\in \mathcal{A}$. Proof. For each $x \in X$, $A(x)$ is the result of a Souslin scheme $(F_n)_{n \in \mathbb{N}}$ on the paving of the closed subsets of $Y$. For each $n \in \mathbb{N}$, define $F_n \subset X \times Y$ by $F_n = F_n^x$ if $x \in \pi_x(A)$ and $F_n = \emptyset$ if $x \not\in \pi_x(A)$. By 3.19, we find that $F_n \in \mathcal{F}_x$. Because $A(x)$ is the result of the scheme $(F_n)_{n \in \mathbb{N}}$, $\mathcal{A}$, we find $A \not\in \mathcal{A}$.

**Proposition 3.21.** $\mathcal{A}(X, Y) \subset \mathcal{A}(X, Y)$. Proof. Let $A \subset (X \times Y)$, and take $B \subset \mathcal{B} \Leftrightarrow \mathcal{B}_x$, satisfying $\pi_x(A \cap B \subset \mathcal{B}_x) \subset \mathcal{B}_x$. Since $B_1 = B \cap \bigcap (X \times \pi_x(A \cap B \subset \mathcal{B}_x)) \in \mathcal{B} \Leftrightarrow \mathcal{B}_x$, $A_1 = A \cap \bigcap (\pi_x(A \cap B \subset \mathcal{B}_x)) \subset \mathcal{A}(X, Y)$,

by 3.20 and $A \not\in \mathcal{B}_x \cup A_1$, it follows that $A \not\in \mathcal{A}(X, Y)$.

**B. Separation results.** In this section, we will apply the general separation theorems obtained in the preceding chapter to more concrete situations. We start with the following well-known fact.

**Proposition 3.22.** Every Polish space is homeomorphic to a $G_\delta$-subset of $[0, 1]^n$, where $[0, 1]$ is the unit-interval. A proof can be found in [26], Ch. 1.

We assume $Y$ a fixed Polish space. By 3.22, $Y$ is homeomorphic to a $G_\delta$ subset of a compact metric space $K$. Let $x$ be the paving on $K$ consisting of the closed sets, which is of course compact.

**Proposition 3.23.** If $(A_n)$ is a sequence of analytic subsets of $Y$ such that $\bigcap A_n = \emptyset$, then there is a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\bigcap B_n = \emptyset$. Proof. $Y$ can clearly be assumed a $G_\delta$ subspace of $K$. Since $(A_n)$ is also a sequence of $x$-analytic subsets, we obtain by 2.10 a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\bigcap B_n = \emptyset$. We only have to take $B_\ast = B_\ast \subset Y$.

In the remainder of this section, we assume $(X, Y, \pi_x)$ satisfying (1) $\rightarrow$ (6) of III, A.

**Proposition 3.24.** If $A \in \mathcal{A}(X, Y)$, then $\pi_x(A) \subset \mathcal{A}$. Proof. It is clear that $Y$ can be assumed a $G_\delta$ subset of $K$. Because $A$, considered as subset of $X \times Y$, is $x$-analytic, $\pi_x(A)$ is $x$-analytic, by 1.15.

**Proposition 3.25.** If $(A_n)$ is a sequence in $\mathcal{A}(X, Y)$ such that $\bigcap A_n = \emptyset$, then there is a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\pi_x(B_n) \subset \mathcal{B}_x$.

Proof. Again we may assume $Y$ a $G_\delta$ subset of $K$. Remark that each set $A_n$ is $x \times Y$-analytic. Since by 3.3 and 2.5, $(X \times Y, x \times Y, \pi_x^1(\mathcal{B}))$ has property (S), 2.9 yields us a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\pi_x(B_n) \subset \mathcal{B}_x$. If we take $B_\ast = B_\ast \subset Y$, the required sequence $(B_n)$ is obtained.

**Theorem 3.26.** If $(A_n)$ is a sequence in $\mathcal{A}(X, Y)$ such that $\bigcap A_n = \emptyset$, then there is a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\bigcap B_n = \emptyset$.

Proof. By 3.25, there is a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\pi_x(B_n) \subset \mathcal{B}_x$. Applying 3.23, we find on the other side for each $x \in X$ a sequence $(B_n)$ in $B$ satisfying $A_n \subset B_n$ for each $n$ and $\bigcap B_n = \emptyset$. The sets $B_n$ are introduced by taking $B_n = B_n(x)$ if $x \not\in \mathcal{B}_x$ and $B_n = B_n$ if $x \in \mathcal{B}_x$. Because $\pi_x(B_n) \subset \mathcal{B}_x$, each set $B_n$ belongs to $\mathcal{B}(X, Y)$ and it follows from the construction that $A_n \subset B_n$ for each $n$ and $\bigcap B_n = \emptyset$.

The following 2 corollaries are straightforward.

**Proposition 3.27.** Disjoint sets in $\mathcal{A}(X, Y)$ can be separated by sets in $\mathcal{B}(X, Y)$.

**Proposition 3.28.** $\mathcal{B}(X, Y) = \mathcal{B}(X, Y)$.

**C. Stable mappings.** We still assume $(X, Y, \pi_x)$ with properties (1) $\rightarrow$ (6) of III, A. From 3.29 to 3.36, $Y$ and $Z$ will be fixed Polish spaces and $D \in \mathcal{B}(X, Y)$.

**Definition 3.29.** A mapping $\varphi : D \rightarrow X \times Z$ will be called stable, if $\pi_x \circ \varphi = \pi_x$ ($\varphi$ preserves the first coordinate).

Obviously $\varphi$ is determined by $\pi_x \circ \varphi$, which we denote by $\varphi_1$. 
**Definition 3.30.** Let $\varphi: D \to X \times Z$ be a stable mapping. We will say that $\varphi$ is measurable if $\varphi$ is $\Sigma(X, Y) \in \Sigma(X, Z)$ measurable.

**Proposition 3.31.** A stable mapping $\varphi: D \to X \times Z$ is measurable if and only if $\varphi_2: D \to Z$ is $\Sigma(X, Y) \in \Sigma_2$ measurable.

**Proof.** 1. Suppose $\varphi$ measurable. Since $\pi_2: X \times Z \to Z$ is $\Sigma(X, Z) \in \Sigma_2$ measurable, it follows that $\varphi_2$ is $\Sigma(X, Y) \in \Sigma_2$ measurable.

2. Assume now $\varphi_2$ is $\Sigma(X, Y) \in \Sigma_2$ measurable. First we verify that $\varphi$ is $\Sigma(X, Y) \in \Sigma(\Sigma_2)$ measurable. Take then $A \in \Sigma(X, Z)$ and consider $B \in \Sigma_2 \Sigma_2$ satisfying $\varphi_2^{-1}(A \cap B) \in \Sigma_2$. Clearly

$$\pi_2^{-1}(A)(x) = \varphi_2^{-1}(A \cap B)(x) \in \Sigma_2.$$

and furthermore

$$\varphi^{-1}(A)(x) = \varphi_2^{-1}(A \cap B)(x) \in \Sigma_2.$$

Hence $\varphi^{-1}(A) \in \Sigma(X, Y)$.

**Definition 3.32.** If $\varphi: D \to X \times Z$ is a stable mapping, then the graph of $\varphi$ will be the set $\Gamma(\varphi) = \{(x, y, \varphi_2(x, y)): (x, y) \in D\}$.

**Proposition 3.33.** If $\varphi: D \to X \times Z$ is stable and measurable, then $\Gamma(\varphi)$ is a member of $\Sigma(X, Y \times Z)$.

**Proof.** Let $\psi: D \times Z \to Z \times Z$ be given by $\psi(x, y, z) = (\varphi_2(x, y), z)$. Then $\psi$ is $\Sigma(X, Y \times Z) \in \Sigma_2 \Sigma_2$ measurable. Indeed, $\pi_2: D \times Z \to Z$ is $\Sigma(X, Y \times Z) \in \Sigma_2$ measurable and $\pi_2 \Sigma_2: D \times Z \to D \times \Sigma_2 \Sigma_2 \Sigma_2 \Sigma_2$ is $\Sigma(X, Y \times Z)$ measurable. The diagonal $A$ of $X \times Z$ belongs to $\Sigma_2 \Sigma_2$, since it is closed. The fact that $\Gamma(\varphi) = \psi^{-1}(A \cap B)$ completes the proof.

**Proposition 3.34.** If $\varphi: D \to X \times Z$ is stable and measurable and $A \in \Sigma(X, Y \times Z)$, then $\varphi(A \cap D) \in \Sigma(X, Z)$.

**Proof.** We may assume $Y$ a $G_\delta$-subset of a compact metric space $X$ with paving $\mathcal{X}$ of its compact subsets. Let $\mathcal{F}$ be the paving on $Z$ consisting of the closed sets. By 3.33, $\Gamma(\varphi) \in \Sigma(X, Y \times Z)$ and hence, by 3.21, $\Gamma(\varphi) \cap (A \times Z \times Z) \in \Sigma(X, Y \times Z)$. Since $\Gamma(\varphi) \cap (A \times Z)$, considered as subset of $X \times X \times Z$, is $X \times X \times \mathcal{F}$-analytic, we obtain, by 1.15, that $\varphi(A \cap D) = \pi_2 \Sigma_2(\Gamma(\varphi) \cap (A \times Z))$ is $X \times \mathcal{X}$-analytic. Thus $\varphi(A \cap D) \in \Sigma(X, Z)$.

**Definition 3.35.** We will say that a stable mapping $\varphi: D \to X \times Z$ is continuous provided the partial map $(\varphi_2)_x: D(x) \to Z$ is continuous for each $x \in X$.

**Proposition 3.36.** If $D \in \Sigma(\Sigma_2)$ and $\varphi: D \to X \times Z$ is a stable and continuous map, then $\varphi$ is $\Sigma \Sigma_2$ measurable.

**Proof.** Let $B$ be a member of $\Sigma \Sigma_2$. Applying 3.18, we only have to show that the sections $\varphi^{-1}(B)(x) = ((\varphi_2)_x^{-1})(B(x))$ are of bounded Baire class. But this follows immediately from 3.2.6 and the fact that each $(\varphi_2)_x$ is continuous.

- Obvious; the following composition results hold:

**Proposition 3.37.** Let $Y, Z, W$ be Polish spaces, $D \in \Sigma(X, Y)$, $E \in \Sigma(Z, X)$, $\varphi: D \to X \times Z$ and $\psi: E \to X \times W$ mappings so that $\varphi(D) = E$. If $\varphi$ and $\psi$ are stable, then $\psi \circ \varphi$ is stable. If moreover $\varphi$ and $\psi$ are measurable (continuous) then also $\psi \circ \varphi$ is measurable (continuous).

**Proposition 3.38.** If $Y$ is a Polish space and $A \in \Sigma(X, Y)$, then there exists a set $D \in \Sigma(X, Y)$ and a continuous map $\varphi: D \to Y$ so that $\varphi(D) = A(x)$ for all $x \in X$.

**Proof.** Let $A$ be the result of a regular scheme $(M_k \times F_k)_{k \in \mathbb{N}}$ on $\mathcal{F}(X, Y)$. It is easily seen that we may assume $F_k \not= \emptyset, F_k \supseteq F_{k+1}$, $k \in \mathbb{N}$, diam $F_k \leq \epsilon |c|$ where the diameter is taken with respect to a complete metric. Obviously the set

$$D = \bigcup_{k \in \mathbb{N}} (M_k \times F_k) = \bigcup_{k \in \mathbb{N}} (M_k \times F_k)$$

belongs to $\mathcal{F}(X, Y)$. The map $\varphi$ on $D$ will be given by $\varphi(x) = \bigcap_{c \in F_k}$, which is a unique point of $Y$. It is clear that $\varphi$ is continuous. Moreover

$$D(x) = \{x \in \bigcap_{c \in F_k} M_k \}$$

and hence $\varphi(D(x)) = \bigcap_{c \in F_k} F_k$, which is precisely $A(x)$.

Our next aim is to establish the following result

**Proposition 3.39.** If $Y$ is a Polish space and $A \in \Sigma(X, Y)$, then there exists a set $D \in \Sigma(X, A)$ and an injective, stable, measurable and continuous map $\varphi: D \to X \times Y$ onto $A$.

We need the following lemma

**Proposition 3.40.** Let $(X_k)$ be a sequence of Polish spaces and let $Y = \prod_{k \in \mathbb{N}} Y_k$. We consider for each $n \in \mathbb{N}$ a member $D_n \in \Sigma \Sigma_2$. Then the subset $D = \prod_{k \in \mathbb{N}} D_n$ belongs to $\Sigma \Sigma_2$.

**Proof.** It is easily verified that for each $n$ the set $D_n = \{(x, y) \in X \times Y: (x, y) \in D_n\}$ is a member of $\Sigma \Sigma_2$. Since $D = \prod_{k \in \mathbb{N}} D_n$, the proof is clear.

The main step in the proof of 3.39 is the following

**Proposition 3.41.** Let $B$ be the class of subsets $A$ of $X \times Y$ with the property that there is a set $D \in \Sigma(X, A)$ and an injective, stable, measurable and continuous map $\varphi: D \to X \times Y$ satisfying $\varphi(D) = A$. Then:

1. $B$ is stable under countable disjoint union.
2. $B$ is stable under countable intersection.

Hence $B \cap cB$ is stable under countable union.

**Proof.** It is clear that in the definition of $B$ above, the space $\mathcal{A}$ can be replaced by a homeomorphic Polish space.

1. Let $(A_n)$ be a sequence of disjoint members of $B$. For each $n$, we obtain a set $D_n \in \Sigma(X, A_n)$ and an injective, stable, measurable and continuous map $\varphi_n: D_n \to X \times Y$.
\( \varphi: D \rightarrow X \times Y \) satisfying \( \varphi_0(D_0) = A_4 \). Obviously \( D = \bigcup D_0 \) is a member of \( \mathcal{F}_d(X, \mathcal{A}) \). Define \( \varphi \) on \( D \) by taking \( \varphi|_{D_0} = \varphi_0 \). Then \( \varphi \) satisfies the required properties and has image \( \bigcup A_4 \).

2. Let \( (A_n) \) be a sequence of members of \( \mathcal{A} \). For each \( n \), let \( D_n \in \mathcal{F}_d(X, \mathcal{A}) \) and \( \varphi_n: D_n \rightarrow X \times Y \) an injective, stable, measurable and continuous map so that \( \varphi_0(D_0) = A_4 \). Let \( S = \mathcal{A} \). From 3.40 we know that the subset \( D \) of \( X \times S \) defined by \( D(x) = \bigcap_n D_n(x) \) belongs to \( \mathcal{M} \otimes \mathcal{B} \) and hence to \( \mathcal{F}_d(X, S) \). We consider the map \( \varphi: D \rightarrow X \times Y \) given by \( \varphi(x, s) = (\varphi_n(x, s_n) ) \), if \( s = (s_n) \). Using 3.31, we see that \( \varphi \) is measurable and continuous. If \( A \) is a diagonal of \( Y \), then \( D(\varphi^{-1} A) \in \mathcal{F}_d(X, S) \) and hence \( D \in \mathcal{F}_d(X, S) \), since \( D(x) \) is closed in \( D(x) \) for each \( x \in X \). Let \( \iota: A \rightarrow Y \) be the canonical isomorphism and \( \varphi: D \rightarrow X \times Y \) the stable map given by \( \varphi = \iota \circ (\varphi_n) \). It is easily checked that \( \varphi \) is injective, measurable and continuous. We also verify that \( \varphi(D) = \bigcap A_4 \). Because \( S \) and \( \mathcal{A} \) are homeomorphic, the proof is complete.

PROPOSITION 3.42. If \( Y \) is Polish, then every member of \( \mathcal{M} \) is the continuous injective image of a closed subset of \( \mathcal{A} \).

Proof. We refer to [22], p. 247, Th. 79 or [26], Ch. 1.

Proof of 3.39. Let \( \mathcal{B} \) be as in 3.41. It is enough to prove 3.39 if \( A \in \mathcal{M} \otimes \mathcal{B} \) and if \( A \in \mathcal{F}_d(X, Y) \) with \( \pi_2(A) \in \mathcal{B} \), since every element of \( \mathcal{F}_d(X, Y) \) is the disjoint union of such sets.

1. From 3.42, it follows that \( \mathcal{F}_d(X, Y) \subset \mathcal{B} \) and hence also \( \mathcal{F}_d(X, Y) \subset \mathcal{B} \subset \mathcal{A} \). Therefore \( \mathcal{F}_d(X, Y) \subset \mathcal{B} \subset \mathcal{A} \), thus certainly \( \mathcal{M} \subset \mathcal{A} \).

2. Assume now \( A \in \mathcal{F}_d(X, Y) \) and \( \pi_2(A) \in \mathcal{B} \). By 3.42 there exist for each \( x \in X \) a closed subset \( D_0 \) of \( A \) and a continuous injective map \( \varphi: D_0 \rightarrow Y \) onto \( A(x) \). Let \( D(x) = \bigcap D_0 \) if \( x \in \pi_2(A) \) and \( D(x) = \emptyset \) otherwise. Define \( \varphi: D \rightarrow X \times Y \) by \( \varphi(x, s) = (\varphi_n(x, s_n)) \). Clearly, by 3.17, \( D \in \mathcal{F}_d(X, \mathcal{A}) \) and \( \varphi \) is an injective, stable, measurable and continuous map with image \( A \).

This completes the proof.

We will now pass to the proof of the converse result, namely

THEOREM 3.43. Let \( Y, Z \) be Polish. If \( D \in \mathcal{M} \otimes \mathcal{B} \) and \( \varphi: D \rightarrow X \times Z \) is an injective, stable and measurable map, then \( \varphi(D) \in \mathcal{M} \otimes \mathcal{B} \).

PROPOSITION 3.44. Let \( Y \) be Polish and \( (A_n) \) a sequence of mutually disjoint elements of \( \mathcal{F}_d(X, Y) \). Then there is a sequence \( (B_n) \) of mutually disjoint members of \( \mathcal{M} \) such that \( A_n \subset B_n \) for all \( n \in \mathbb{N} \).

Proof. Since \( A_0 \) and \( A_n \) are disjoint for \( n \neq 0 \), and \( A_n \) are disjoint members of \( \mathcal{F}_d(X, Y) \). By 3.27 we can find disjoint sets \( B_1 \) and \( C_1 \) in \( \mathcal{M} \otimes \mathcal{B} \) such that \( A_1 \subset B_1 \) and \( \bigcup A_n \subset C_1 \). We can then separate similarly \( A_2 \) and \( \bigcup A_n \) by sets \( B_1 \) and \( C_2 \in \mathcal{M} \) such that \( B_2 \subset C_2 \) and \( C_2 \subset C_1 \). Repeating this, we complete the proof.

Proof of 3.43. By 3.39 and 3.37, we may assume \( Y = \mathcal{A} \). For every \( c \in \mathcal{A} \), define \( E_c = \varphi(D \cap (X \times \{c\})) \), which is a member of \( \mathcal{F}_d(X, Y) \) by 3.34. The scheme \( (E_0, \mathcal{A}) \) is regular and since \( \varphi \) is injective, \( E_0 \cap E_c = \emptyset \) if \( |c| = |c'| \) and \( c' \neq c \).

Applying 3.44, we obtain a regular scheme \( (E_0, \mathcal{A}) \) on \( \mathcal{F}_d(X, Z) \) so that \( E_0 \subset B_0 \) and \( B_0 \cap B_0 = \emptyset \) if \( |c| = |c'| \) and \( c' \neq c \). For each \( c \in \mathcal{A} \), let \( C_c = \{x \in X : x \in \mathcal{A} \} \), which clearly belongs to \( \mathcal{F}_d(X, \mathcal{A}) \). Hence also \( \mathcal{A} = \bigcap C_c \) is in \( \mathcal{F}_d(X, \mathcal{A}) \).

It is easily seen that \( \Gamma(\varphi) \subset \mathcal{A} \).

If \( x \in X, x \in Z \), then \( \mathcal{A} = \{x \in X, x \in Z \} \) and thus consists of at most one point of \( \mathcal{A} \). Furthermore

\[ \pi_2(\mathcal{A}) = \bigcap C_c = \bigcap B_c = \bigcap C_c \]

and therefore in \( \mathcal{F}_d(X, Z) \). Since \( \Gamma(\varphi) \subset \mathcal{A} \) by 3.33, the set

\[ \pi_2(\mathcal{A}) \cap \mathcal{B} = \pi_2(\mathcal{A}) \cap \mathcal{B} \]

is a member of \( \mathcal{F}_d(X, Z) \). It follows that \( \mathcal{A} \subset \mathcal{F}_d(X, Z) \) and thus, by 3.34 and 3.28, \( \mathcal{A} \subset \mathcal{F}_d(X, Z) \).

An obvious corollary of 3.43 is Kuratowski's isomorphism theorem:

PROPOSITION 3.45. If \( Y, Z \) are Polish, \( D \in \mathcal{B} \) and \( \varphi: D \rightarrow Z \) is injective and Borel measurable, then \( \varphi(D) \in \mathcal{B} \).

For a slightly different proof of 3.45, the reader is referred to [16].

References


Generalized Archimedean fields and logics with Malitz quantifiers

by

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Abstract. A characterization of Archimedean fields in a particular interpretation of the logic with Malitz quantifiers suggests a generalization of such fields. The theory of the real closed version of these generalized Archimedean fields in other interpretations of the Malitz quantifier is found to allow elimination of quantifiers.

The reader should be familiar with the model theory of first order logic. Some knowledge of ultrapowers, for example, is assumed. The notation is for the most part similar to that used in [1] or [2]. Gothic letters range over structures with the corresponding Latin letters denoting their universes: $A$ denotes the universe of $\mathfrak{A}$, $B_i$ denotes the universe of $\mathfrak{B}_i$, etc. Cardinals are initial von Neumann ordinals. Write $\text{Card}(A)$ for the cardinality of $A$, $P$ for the set of positive integers, $Q$ for the set of rational numbers, and $R$ for the set of real numbers.

Logics with Malitz quantifiers. For each positive integer $n$ and each infinite cardinal $\kappa$, the logic $L^n$ is obtained by adding a new quantifier $Q^n$ which binds $n$ distinct variables and the following formation rule to those of first order logic: If $\varphi$ is a formula and if the variables $x_1, \ldots, x_n$ are distinct, then $Q^n x_1 \ldots x_n \varphi$ is also a formula. The logic $L^{\omega}$ is obtained from first order logic by adding all the quantifiers $Q^n$ together with the corresponding formation rules.

The interpretation of the quantifier $Q^n$ depends on the cardinal $\kappa$:

$$\mathfrak{A}, \kappa, Q^n x_1, \ldots, x_n \varphi \models \overline{\tau}$$

just in case there is a subset $I$ of $\mathcal{A}$ such that (i) $\text{Card}(I) = \kappa$ and (ii) whenever $a_1, \ldots, a_n$ are distinct elements of $I$, then $\mathfrak{A}, a_1, a_2 \varphi \models \overline{\tau}$. Here the notation indicates how each of the variables $x_1, \ldots, x_n$ is to be interpreted and $\bar{x}$ is an interpretation of the free variables in $Q^n x_1 \ldots x_n \varphi$.

The logic $L^{\omega}$ coincides with the logic with the cardinal quantifier, "There exist $\kappa$ many ...". For $n \geq 2$, the logics $L^{\omega_n}$ and $L^{\omega}$ are referred to as logics with Ramsey quantifiers because of the similarity between their semantics and the well-known