$$A_1 = \{y|\ \exists(\zeta_1, \zeta_2, \ldots) \in \mathcal{N} \text{ such that } y(\zeta_1, \ldots, \zeta_n) = 1\ \forall n\}$$

is BA. By Corollary 2.2, the function $f\colon X \to \{0, 1\}^{\Sigma}$ whose sth component is the indicator of $A(s)$ is BA. The result of operation (A) on the system $\{A(s)|\ s \in \Sigma\}$ is

$$\bigcup_{(\zeta_1,\zeta_2,\ldots)\in\mathcal{N}} \bigcap_{n=1}^{\infty} A(\zeta_1, \ldots, \zeta_n) = f^{-1}(A_1),$$

and this is BA by the remark following Theorem 5. Q.E.D.

If p is a BP function from $\{0, 1\}^{\infty}$ to $\{0, 1\}^{\infty}$ satisfying (4), then p_{ω_1} defined by (5), (6) can fail to be BP [9]. If $\{g_\alpha|\ \alpha < \omega_1\}$ is a BA approach to g, it is not known if g can fail to be BA. It is not known whether the BA σ-field properly contains the BP σ-field, nor whether the BA σ-field is properly contained in the σ-field of absolutely measurable sets. The relation between the BP sets, the BA sets and the R sets [8] has not been determined. Indeed, the cardinality of the class of BA sets is not known. A particularly intriguing question is whether the product of the BA σ-fields in X and Y is the BA σ-field in $X \times Y$.

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A stabilization property and its applications in the theory of sections

by

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Abstract. We introduce a stabilization property in descriptive set theory which generalizes the topological and measure theoretical situations. An associated theory of sections for measurable sets in products is developed.

I. Preliminaries. The aim of this section is to make the text more selfcontained. We will introduce the various classical notions and properties, which are the starting point of this work. They can also be found in [12].

DEFINITION 1.1. Let E be a set. A paving on E will be a class $\mathcal{E}$ of subsets of E containing the empty set. We will call $(E, \mathcal{E})$ a *paved set*.

DEFINITION 1.2. If $(E, \mathcal{E})$ is a paved set, we denote by $c\mathcal{E}$: the class of subsets A of E such that $E \setminus A$ belongs to $\mathcal{E}$, $b\mathcal{E} = \mathcal{E} \cap c\mathcal{E}$.

$\mathcal{E}^{\wedge}$ (resp. $\mathcal{E}^{\vee}$, $\mathcal{E}^{-}$, $\mathcal{E}^{*}$): the stabilization of $\mathcal{E}$ for finite intersection (resp. finite union, finite intersection and finite union, countable intersection and countable union).

$\mathfrak{S}(\mathcal{E})$: the σ-algebra generated by $\mathcal{E}$.

DEFINITION 1.3. Let $(E_i, \mathcal{E}_i)_{i\in I}$ be a family of paved sets. The set $\mathcal{E}$ of subsets of $E = \prod_i E_i$ of the form $\prod_i A_i$, where $A_i \in \mathcal{E}_i$ for each $i \in I$, is called the *product paving* $\mathcal{E}_i \prod_i$.

PROPOSITION 1.4. *Let $(E_i, \mathcal{E}_i)_{i\in I}$ be paved sets such that $E_i \in \mathcal{E}_i$ for each $i \in I$. Then $\mathfrak{S}(\prod_i \mathcal{E}_i)$ contains the product σ-algebra $\oplus_i \mathfrak{S}(\mathcal{E}_i)$. If moreover I is countable, then $\mathfrak{S}(\prod_i \mathcal{E}_i) = \oplus_i \mathfrak{S}(\mathcal{E}_i)$.*

In fact, only finite and countable products will be involved here.

Let $(E, \mathcal{E})$ be a paved set and let $(K_i)_{i\in I}$ be a family of elements of $\mathcal{E}$. We will say that $(K_i)_{i\in I}$ has the finite intersection property provided $\bigcap_{i\in J} K_i \neq \emptyset$ whenever J is a finite subset of I.

DEFINITION 1.5. A paving $\mathscr{E}$ on a set E is said to be *compact* (resp. *semi-compact*) if every family (resp. every countable family) of elements of $\mathscr{E}$, possessing the finite intersection property, has nonempty intersection.

By a simple ultra-filter argument, we obtain

PROPOSITION 1.6. *If $\mathscr{E}$ is a compact (resp. semi-compact) paving on E, then also $\mathscr{E}^{\vee}$ is compact (resp. semi-compact).*

The following proposition is immediate

PROPOSITION 1.7. *Let $(E_i, \mathscr{E}_i)_{i\in I}$ be a family of paved sets. If each $\mathscr{E}_i$ is compact (resp. semi-compact), then $\prod_i \mathscr{E}_i$ on $\prod_i E_i$ is compact (resp. semi-compact).*

We now pass to a proposition which will be often used later (especially in product situations).

PROPOSITION 1.8. *Let $(E, \mathscr{E})$ be a paved set and f an application of E into a set F. We assume that, for each $x \in F$, the paving consisting of the sets $f^{-1}(\{x\}) \cap A$, $A \in \mathscr{E}$, is semi-compact. If $(A_n)_n$ is a decreasing sequence in $\mathscr{E}$, then $f(\bigcap_n A_n) = \bigcap_n f(A_n)$.*

Proof. It is clear that if $x \in \bigcap_n f(A_n)$, then the family $f^{-1}(\{x\}) \cap A_n$ has the finite intersection property. By hypothesis, the set $f^{-1}(\{x\}) \cap \bigcap_n A_n$ contains some point $y \in E$. Hence $x = f(y) \in f(\bigcap_n A_n)$, completing the proof.

N will denote the set of all positive integers 1, 2, ... Let $\mathscr{R} = \bigcup_{k=1}^{\infty} N^k$, consisting of the finite complexes of integers. Take $\mathscr{R}^* = \mathscr{R} \cup \{\emptyset\}$. If $c \in \mathscr{R}^*$, let $|c|$ be the length of c. If $c, d \in \mathscr{R}^*$, we write $c < d$ if c is an initial section of d. Let $\mathscr{N} = N^N$. If $v \in \mathscr{N}$ and $c \in \mathscr{R}^*$, we write $c < v$ if c is an initial section of v.

DEFINITION 1.9. Let $(E, \mathscr{E})$ be a paved set. A Souslin scheme $(A_c)_{c\in\mathscr{R}}$ on $\mathscr{E}$ will be a mapping of $\mathscr{R}$ into $\mathscr{E}$. The scheme $(A_c)_{c\in\mathscr{R}}$ is said to be *regular* if $A_c \supset A_d$ whenever $c < d$. The result of the scheme $(A_c)_{c\in\mathscr{R}}$ is the set $\bigcup_{v} \bigcap_{c<v} A_c = \bigcup_{v} \bigcap_{k=1}^{\infty} A_{v|k}$, where v runs over $\mathscr{N}$.

Let $(E, \mathscr{E})$ be a paved set and $(A_c)_{c\in\mathscr{R}}$ a scheme on $\mathscr{E}$. For each complex $c \in N^k$, we introduce the following sets:

$$A_{[c]} = \bigcup_{\substack{n_1 \leqslant c_1\\ \vdots \\ n_k \leqslant c_k}} A_{n_1,\dots,n_k},$$

$$A(c) = \bigcup_{c<v} \bigcap_{k=1}^{\infty} A_{v|k},$$

where v runs over $\mathscr{N}_c = \{v \in \mathscr{N};\, c < v\}$,

$$A[c] = \bigcup_{\substack{n_1 \leqslant c_1\\ \vdots \\ n_k \leqslant c_k}} A(n_1, \dots, n_k).$$

Obviously, the following properties hold

PROPOSITION 1.10. *If $c \in \mathscr{R}$, then*

$$A_{[c]} \in \mathscr{E}^{\vee},$$
$$A(c) \subset A_c,$$
$$A[c] \subset A_{[c]},$$
$$A(c) \doteq \bigcup_{n=1}^{\infty} A(c, n),$$
$$A[c] = \bigcup_{n=1}^{\infty} A[c, n],$$

$\{\emptyset\} \cup \{\mathscr{N}_c;\, c \in \mathscr{R}\}$ is a paving on $\mathscr{N}$, which we denote by $\underline{\mathscr{N}}$.

The reader will easily verify

PROPOSITION 1.11. *$\underline{\mathscr{N}}$ is a compact paving on $\mathscr{N}$.*

The following result is basic in the theory of analytic sets.

PROPOSITION 1.12. *Let $(E, \mathscr{E})$ be a paved set and $(A_c)_{c\in R}$ a regular scheme on $\mathscr{E}$, with result A. If $v \in \mathscr{N}$, then $\bigcap_k A_{[v|k]} \subset A$.*

Proof. Suppose $x \in \bigcap_k A_{[v|k]}$. For each $k \in N$, we introduce the set

$$K_k = \{\mu \in \mathscr{N};\, \mu_1 \leqslant v_1, \dots, \mu_k \leqslant v_k \text{ and } x \in A_{\mu_1,\dots,\mu_k}\},$$

which is clearly a nonempty member of $\underline{\mathscr{N}}^{\vee}$. By the regularity of the scheme, the sequence $(K_k)_k$ is decreasing.

Since, by 1.6, also $\underline{\mathscr{N}}^{\vee}$ is compact, we obtain some $\mu \in \bigcap_k K_k$. It follows that $x \in \bigcap_{k=1}^{\infty} A_{\mu|k} \subset A$, completing the proof.

DEFINITION 1.13. Let $(E, \mathscr{E})$ be a paved set. A subset A of E is said to be *$\mathscr{E}$-analytic* if it is the result of a Souslin scheme on $\mathscr{E}$. Let $\mathscr{A}(\mathscr{E})$ denote the class of all $\mathscr{E}$-analytic subsets of E. The members of $\mathrm{c}\mathscr{A}(\mathscr{E})$ (resp. $\mathrm{b}\mathscr{A}(\mathscr{E})$) are called *$\mathscr{E}$-coanalytic* (resp. *$\mathscr{E}$-bianalytic*).

The main property of $\mathscr{A}(\mathscr{E})$ is the following:

PROPOSITION 1.14. $\mathscr{A}(\mathscr{A}(\mathscr{E})) = \mathscr{A}(\mathscr{E})$.

In fact the proof of this property consists in the reduction of a scheme of schemes to a single scheme. Although the idea is quite simple, its working-out is rather complicated. For the details, we refer the reader to [15] for instance.

The class of the analytic sets is stable under projection in the following sense:

PROPOSITION 1.15. *Let $(E, \mathscr{E})$ and $(F, \mathscr{F})$ be paved sets, such that the paving $\mathscr{F}$ is semi-compact. If $A \subset E \times F$ belongs to $\mathscr{A}(\mathscr{E} \times \mathscr{F})$, then $\pi(A)$ is a member of $\mathscr{A}(\mathscr{E})$, if π: $E \times F \to E$ is the projection.*

Proof. Let A be the result of the scheme $(E_c \times F_c)_{c\in\mathscr{R}}$ on $\mathscr{E} \times \mathscr{F}$, where $E_c \in \mathscr{E}$ and $F_c \in \mathscr{F}$ for each $c \in \mathscr{R}$. We define a scheme $(B_c)_{c\in\mathscr{R}}$ on $\mathscr{E}$ by taking $B_c = E_c$

if $\bigcap_{k=1}^{|c|} F_{c|k} \neq \emptyset$ and $B_c = \emptyset$ otherwise. Since for each $\nu \in \mathcal{N}$ we obtain that

$$\pi\Big(\bigcap_k E_{\nu|k} \times \bigcap_k F_{\nu|k}\Big) = \bigcap_k B_{\nu|k},$$

the result of the scheme $(B_c)_{c\in\mathcal{R}}$ is precisely $\pi(A)$.

To each subset R of $\mathcal{R}^*$ we associate a transfinite system $(R_\alpha)_{\alpha<\omega_1}$, which we define inductively as following

$$R_0 = R,$$
$$R_{\alpha+1} = \{c \in R;\ \text{there exists } d \in R_\alpha \text{ with } c<d \text{ and } c \neq d\}.$$

If γ is a limit ordinal, take $R_\gamma = \bigcap_{\alpha<\gamma} R_\alpha$. It is easily verified that the sequence $(R_\alpha)_{\alpha<\omega_1}$ is decreasing. Because R is at most countable, the sequence stabilizes. Let $i(R) = \inf\{\alpha<\omega_1\colon R_\alpha = R_{\alpha+1}\}$, which is called the ordinal of R.

We are now able to introduce the Lusin–Sierpiński index, which is of fundamental importance in the study of Souslin schemes.

Definition 1.16. Let $(E, \mathcal{E})$ be a paved set and $(A_c)_{c\in\mathcal{R}}$ a regular scheme on $\mathcal{E}$. Suppose $x \in E$ and consider $R(x) = \{\emptyset\} \cup \{c \in \mathcal{R};\ x \in A_c\}$. Let $\eta = i(R(x))$. If $R(x)_\eta = \emptyset$, let $i(x) = \eta$. If $R(x)_\eta \neq \emptyset$, let $i(x) = \omega_1$. The ordinal $i(x)$ is called the *Lusin–Sierpiński index of the scheme* $(A_c)_{c\in\mathcal{R}}$ *in the point* x.

Remark that a predecessor of a member of $R(x)_\alpha$ is also in $R(x)_\alpha$ and in particular $R(x)_\alpha \neq \emptyset$ if and only if $\emptyset \in R(x)_\alpha$.

Proposition 1.17. *If A is the result of the regular scheme $(A_c)_{c\in\mathcal{R}}$, then $i(x) = \omega_1$ if and only if $x \in A$.*

Proof. 1. If $x \in A$, then $x \in \bigcap_{c<\nu} A_c$ for some $\nu \in \mathcal{N}$. It is easily verified, using induction, that for each $\alpha<\omega_1$ the set $R(x)_\alpha$ contains every initial section of ν.

2. If $\eta = i(R(x))$, then $R(x)_\eta = R(x)_{\eta+1}$ and therefore every element of $R(x)_\eta$ has a strict successor in $R(x)_\eta$. Assume $R(x)_\eta \neq \emptyset$. Then we find some $\nu \in \mathcal{N}$ so that $\nu|k \in R(x)_\eta$ for each $k \in N$. Hence also $\nu|k \in R(x)$ for each $k \in N$, implying $x \in \bigcap_k A_{\nu|k} \subset A$.

Proposition 1.18. *If $i(x)<\omega_1$, then $i(x)$ is never a limit ordinal.*

Proof. If $\eta = i(x)$ would be a limit ordinal, we would obtain that $R(x)_\eta = \bigcap_{\alpha<\eta} R(x)_\alpha$. For each $\alpha<\eta$ we have that $R(x)_\alpha \neq R(x)_{\alpha+1}$ and hence $R(x)_\alpha \neq \emptyset$. It follows that $\emptyset \in R(x)_\eta$, which is a contradiction.

Definition 1.19. Let $(E, \mathcal{E})$ be a paved set and $(A_c)_{c\in\mathcal{R}}$ a regular scheme on $\mathcal{E}$. If $x \in E$, then we define for each $c \in \mathcal{R}^*$ a subset $R(c, x)$ of $\mathcal{R}^*$ and an ordinal $i(c, x)$ by taking

$$R(\emptyset, x) = R(x),$$
$$R(c, x) = \{d \in \mathcal{R}^*;\ x \in A_{c,d}\} \text{ if } c \neq \emptyset.$$

If $\eta = i(R(c, x))$, let $i(c, x) = \eta$ if $R(c, x)_\eta = \emptyset$ and $i(c, x) = \omega_1$ if $R(c, x)_\eta \neq \emptyset$.

Of course $i(\emptyset, x) = i(x)$. If $c \neq \emptyset$, then $i(c, x)$ is the Lusin–Sierpiński index of the scheme $(A_{c,d})_{d\in\mathcal{R}}$ if $x \in A_c$. In virtue of 1.17 and 1.18, we obtain that $i(c, x) = \omega_1$ if and only if $x \in \bigcup_{c<\nu} \bigcap_k A_{\nu|k}$ and otherwise $i(c, x)$ is never a limit ordinal.

Proposition 1.20. *If $\alpha<\omega_1$ and $c, d \in \mathcal{R}^*$, then $d \in R(c, x)_\alpha$ if and only if $(c, d) \in R(x)_\alpha$.*

Proof. If $c = \emptyset$, there is nothing to prove. If $c \neq \emptyset$, we proceed again by induction on $\alpha<\omega_1$.

Proposition 1.21. *If $c \in \mathcal{R}^*$, then $i(c, x) = \inf\big(\omega_1, \sup_n i((c, n), x)+1\big)$.*

Proof. If $i(c, x) = \omega_1$, then $R(c, x)$ contains every initial section of some sequence $\nu \in \mathcal{N}$. Therefore $R((c, \nu_1), x)$ contains every section of the sequence μ defined by $\mu_k = \nu_{k+1}$. It follows that $i((c, \nu_1), x) = \omega_1$.

Assume now $i(c, x)<\omega_1$. Then also $i((c, n), x)<\omega_1$ for each $n \in N$.

1. If $n \in N$ and $\alpha<i((c, n), x)$, then $R((c, n), x)_\alpha \neq \emptyset$ and thus contains $\emptyset$. It follows that $n \in R(c, x)_\alpha$ and thus $\emptyset \in R(c, x)_{\alpha+1}$. Therefore $i(c, x)>\alpha+1$. Since $i((c, n), x)$ is not a limit ordinal, it follows that $i(c, x)>i((c, n), x)$. Because $i(c, x)$ is not a limit ordinal, $i(c, x)>\sup_n i((c, n), x)$.

2. If $\alpha = \sup_n i((c, n), x)$, then $R((c, n), x)_\alpha = \emptyset$ whenever $n \in N$. Suppose $d \in R(c, x)_\alpha$ and $d \neq \emptyset$. Then $d = (n, d')$ for some $n \in N$ and $d' \in \mathcal{R}^*$. We obtain that $d' \in R((c, n), x)_\alpha$, a contradiction. Hence $R(c, x)_\alpha \subset \{\emptyset\}$ and $R(c, x)_{\alpha+1} = \emptyset$, implying $i(c, x) \leqslant \alpha+1$. This completes the proof.

Proceeding by induction, we deduce easily from 1.21

Proposition 1.22. *If $(A_c)_{c\in\mathcal{R}}$ is a regular scheme on $\mathcal{E}$, then $\{x \in E;\ i(c, x)>\alpha\}$ is a member of $\mathcal{E}^*$ whenever $c \in \mathcal{R}^*$ and $\alpha<\omega_1$.*

II. A stabilization property. The topic of this section is to define a stabilization property, which we will call (S). It will provide us a generalization of various situations, especially the topological and measure-theoretical case.

Definition 2.1. Let E be a set and $\mathcal{E}$, $\mathfrak{N}$ pavings on E. We agree to say that $(E, \mathcal{E}, \mathfrak{N})$ is basic, if:

1. $\mathcal{E}$ is stable under finite intersection.
2. If $A \in \mathfrak{N}$ and $B \subset A$, then also $B \in \mathfrak{N}$.

Definition 2.2. Let $(E, \mathcal{E}, \mathfrak{N})$ be basic. We say that $(E, \mathcal{E}, \mathfrak{N})$ *has property* (S) if moreover the following is true:

Let $(A_c)_{c\in\mathcal{R}}$ be a regular scheme on $\mathcal{E}$ with index i. Then either the result of the scheme is nonempty or $\{x \in E\colon i(x)>\alpha\} \in \mathfrak{N}$ for some $\alpha<\omega_1$ (and hence for the succeeding countable ordinals). It is clear that (S) is preserved if $\mathcal{E}$ decreases and $\mathfrak{N}$ increases. The following proposition will provide us a more explicit formulation of property (S).

Proposition 2.3. *Let $(E, \mathcal{E}, \mathfrak{N})$ be basic. Then the following properties are equivalent*:

I. *Let for each* $c \in \mathcal{R}^*$ *a transfinite system* $(A_c^\alpha)_{\alpha<\omega_1}$ *of sets in* $\mathcal{E}^*$ *be given, verifying*:

1. $(A_c^0)_{c\in\mathcal{R}}$ *is a regular scheme on* $\mathcal{E}$,
2. $A_c^\alpha \supset A_c^\beta$ *if* $\alpha<\beta$,
3. $A_c^{\alpha+1} \subset \bigcup_{n=1}^{\infty} A_{c,n}^\alpha$.

Then either $(A_c^0)_{c\in\mathcal{R}}$ *has a nonempty result or* $A_\emptyset^\alpha \in \mathfrak{N}$ *for some* $\alpha<\omega_1$.

II. $(E, \mathcal{E}, \mathfrak{N})$ *has property* (S).

III. *The same as* (I), *but where* $\mathcal{E}^*$ *is replaced by* 2^E.

Proof. I $\Rightarrow$ II. Assume $(A_c)_{c\in\mathcal{R}}$ a regular scheme on $\mathcal{E}$ and define $A_c^\alpha = \{x\in E;\ i(c,x)>\alpha\}$, which belongs to $\mathcal{E}^*$. Applying 1.21, we see that the conditions of (I) are satisfied. Therefore either $(A_c)_{c\in\mathcal{R}}$ has nonempty result or $A_\emptyset^\alpha = \{x\in E;\ i(x)>\alpha\}\in\mathfrak{N}$ for some $\alpha<\omega_1$.

II $\Rightarrow$ III. Let for each $c\in\mathcal{R}^*$ a transfinite system $(A_c)_{\alpha<\omega_1}$ of subsets of E be given, satisfying (1), (2), (3). We consider the scheme $(A_c^0)_{c\in\mathcal{R}}$ on $\mathcal{E}$. The reader will easily verify by induction on $\alpha<\omega_1$ that $A_c^\alpha \subset \{x\in E;\ i(c,x)>\alpha\}$.

If $(A_c^0)_{c\in\mathcal{R}}$ has an empty result, then $\{x\in E;\ i(x)>\alpha\}$ and hence $A_\emptyset^\alpha$ belongs to $\mathfrak{N}$ for some $\alpha<\omega_1$.

III $\Rightarrow$ I. This is obvious.

It is clear that if $(E,\mathcal{E},\mathfrak{N})$ has (S), then also $(E,\mathcal{E},\mathfrak{N}_1)$ has (S), where $\mathfrak{N}_1 = \{A\subset E;\ A\subset A_1 \text{ with } A_1\in\mathfrak{N}\cap\mathcal{E}^*\}$. Some examples are in order. The first example requires the notion of a capacity.

Definition 2.4. Let $(E,\mathcal{E})$ be a paved set such that $\mathcal{E}$ is stable under finite union and finite intersection. An $\mathcal{E}$-capacity on E will be a real valued function I defined on 2^E, verifying the following conditions:

1. I is increasing: $A\subset B \Rightarrow I(A)\leqslant I(B)$.
2. If $(A_n)_n$ is an increasing sequence of subsets of E, then $I(\bigcup_n A_n) = \sup_n I(A_n)$.
3. If $(A_n)_n$ is a decreasing sequence in $\mathcal{E}$, then $I(\bigcap_n A_n) = \inf_n I(A_n)$.

Example I. Let $(E,\mathcal{E})$ be a paved set such that $\mathcal{E}$ is stable under finite union and finite intersection. Let I be an $\mathcal{E}$-capacity with $I(\emptyset)=0$. If we take $\mathfrak{N} = \{A\subset E;\ I(A)=0\}$, then $(E,\mathcal{E},\mathfrak{N})$ has property (S).

Proof. Let for each $c\in\mathcal{R}^*$ a transfinite system $(A_c^\alpha)_{\alpha<\omega_1}$ of subsets of E be given, such that (1), (2), (3) of Proposition 2.3 are satisfied.

If $c\in\mathcal{R}$ with $|c|=k$ and $\alpha<\omega_1$, let

$$A_{[c]}^\alpha = \bigcup_{\substack{n_1\leqslant c_1\\ \vdots \\ n_k\leqslant c_k}} A_{n_1,\ldots,n_k}^\alpha .$$

Assume $A_\emptyset^\alpha\notin\mathfrak{N}$ for each $\alpha<\omega_1$. Then there is some $\varepsilon>0$ with $I(A_\emptyset^\alpha)>\varepsilon$ for each $\alpha<\omega_1$. By induction on k, we construct a sequence $(n_k)_k$ of integers satisfying $I(A_{[n_1,\ldots,n_k]}^\alpha)>\varepsilon$ for each $\alpha<\omega_1$ and $k\in N$.

For each $\alpha<\omega_1$ we have that $I(A_\emptyset^{\alpha+1})>\varepsilon$ and $A_\emptyset^{\alpha+1}\subset \overset{\uparrow}{\bigcup_n} A_{[n]}^\alpha$. Therefore there must be some $n_1\in N$ so that $I(A_{[n_1]}^\alpha)>\varepsilon$ for each $\alpha<\omega_1$. Suppose $n_1,\ldots,n_k$ obtained verifying $I(A_{[n_1,\ldots,n_k]}^\alpha)>\varepsilon$ for each $\alpha<\omega_1$.

For each $\alpha<\omega_1$, we have that $A_{[n_1,\ldots,n_k]}^{\alpha+1}\subseteq \overset{\uparrow}{\bigcup_n} A_{[n_1,\ldots,n_k,n]}^\alpha$. Therefore there must be again some $n_{k+1}\in N$ so that $I(A_{[n_1,\ldots,n_k,n_{k+1}]}^\alpha)>\varepsilon$ for each $\alpha<\omega_1$.

So the construction is complete.

Since in particular $(A_{[n_1,\ldots,n_k]}^0)_k$ is a decreasing sequence in $\mathcal{E}$ and $I(A_{[n_1,\ldots,n_k]}^0)>\varepsilon$ for each $k\in N$, we find that $\bigcap_k A_{[n_1,\ldots,n_k]}^0\neq\emptyset$. But, by 1.12, this set is contained in the result of the scheme $(A_c^0)_{c\in\mathcal{R}}$, which is therefore also nonempty.

Example II. Let $(E,\mathcal{E})$ be a paved set such that $\mathcal{E}$ is semi-compact and stable under finite union and finite intersection. If $\mathfrak{N}=\{\emptyset\}$, then $(E,\mathcal{E},\mathfrak{N})$ has property (S).

Proof. We define I on 2^E by taking $I(\emptyset)=0$ and $I(A)=1$ if $A\neq\emptyset$. Clearly I is an $\mathcal{E}$-capacity. We obtain a special case of Example I.

The following example is of different nature.

Example III. Let $(E,\mathcal{E})$ be a paved set such that $\mathcal{E}$ is stable under countable union and countable intersection. Let $\mathfrak{N}$ be a class of subsets of E, such that:

1. $\mathfrak{N}$ is a σ-ideal.
2. If $(A_\alpha)_{\alpha<\omega_1}$ is decreasing in $\mathcal{E}$, then there is some $\eta<\omega_1$ so that $A_\eta\setminus A_\alpha\in\mathfrak{N}$ whenever $\alpha>\eta$.

Then $(E,\mathcal{E},\mathfrak{N})$ has property (S).

Proof. Let for each $c\in\mathcal{R}^*$ a transfinite system $(A_c^\alpha)_{\alpha<\omega_1}$ of subsets in $\mathcal{E}^*=\mathcal{E}$ be given, such that (1), (2), (3) of Proposition 2.3 are satisfied. There exists $\eta<\omega_1$ so that $A_c^\eta\setminus A_c^\alpha\in\mathfrak{N}$ for each $c\in\mathcal{R}^*$ and $\alpha>\eta$. Remark that $\bigcup_{c\in\mathcal{R}^*}(A_c^\eta\setminus A_c^{\eta+1})\in\mathfrak{N}$. If $A_\emptyset^\eta\notin\mathfrak{N}$, then there is x in $A_\emptyset^\eta$ not belonging to $\bigcup_{c\in\mathcal{R}^*}(A_c^\eta\setminus A_c^{\eta+1})$. By induction on k we construct a sequence $(n_k)_k$ of integers satisfying $x\in A_{n_1,\ldots,n_k}^\eta$ for each $k\in N$.

Since $x\in A^\eta$ and $x\notin A_\emptyset^\eta\setminus A_\emptyset^{\eta+1}$, we obtain that $x\in A_\emptyset^{\eta+1}\subset\bigcup_n A_n^\eta$. Thus there is $n_1\in N$ with $x\in A_{n_1}^\eta$.

Suppose $n_1,\ldots,n_k$ obtained such that $x\in A_{n_1,\ldots,n_k}^\eta$. Since $x\notin A_{n_1,\ldots,n_k}^\eta\setminus A_{n_1,\ldots,n_k}^{\eta+1}$, we obtain $x\in A_{n_1,\ldots,n_k}^{\eta+1}\subset\bigcup_n A_{n_1,\ldots,n_k,n}^\eta$. Thus there is $n_{k+1}\in N$ with $x\in A_{n_1,\ldots,n_k,n_{k+1}}^\eta$, completing the construction.

In particular $x\in A_{n_1,\ldots,n_k}^0$ for each $k\in N$. Hence x belongs to the result of the scheme $(A_c^0)_{c\in\mathcal{R}}$.

The following example reduces as well to (I) as to (III):

Example IV. Let $(E,\mathcal{E},\mu)$ be a probability space and take

$$\mathfrak{N} = \{A\subset E;\ \mu^*(A)=0\}.$$

Then $(E, \mathscr{E}, \mathfrak{N})$ has property (S). Also the following example, which is an application of (III), is worth to be mentioned.

EXAMPLE V. Let E be a separable metric space, $\mathscr{E}$ the Baire σ-algebra and $\mathfrak{N}$ the class of first category sets. Then $(E, \mathscr{E}, \mathfrak{N})$ has property (S).

PROPOSITION 2.5. *Assume* $(E, \mathscr{E}, \mathfrak{N})$ *with property* (S) *and let* $(K, \mathscr{K})$ *be a paved set such that* $\mathscr{K}$ *is semi-compact and stable under finite intersection. Let* $\pi\colon E\times K\to E$ *be the projection and consider* $\pi^{-1}(\mathfrak{N}) = \{A\subset E\times K;\ \pi(A)\in\mathfrak{N}\}$. *Then* $(E\times K, \mathscr{E}\times\mathscr{K}, \pi^{-1}(\mathfrak{N}))$ *has property* (S).

Proof. First, remark that $(E\times K, \mathscr{E}\times\mathscr{K}, \pi^{-1}(\mathfrak{N}))$ is basic. For each $c\in\mathscr{R}^*$, let $(A_c^\alpha)_{\alpha<\omega_1}$ be a transfinite system of subsets of $E\times K$ satisfying (1), (2), (3) of 2.3. Then the subsets $\pi(A_c^\alpha)$ of E also satisfy (1), (2), (3) of 2.3, with respect to the paving $\mathscr{E}$. Suppose there is $\nu\in\mathscr{N}$ so that $\bigcap_{c<\nu}\pi(A_c^0)\neq\emptyset$. Since $\bigcap_{c<\nu}\pi(A_c^0) = \pi(\bigcap_{c<\nu}A_c^0)$, by 1.8, we see that also $(A_c^0)_{c\in\mathscr{R}}$ has a nonempty result. Otherwise $A_\emptyset^\alpha\in\pi^{-1}(\mathfrak{N})$ for some $\alpha<\omega_1$.

The next result requires the following lemma, which is more technical than basically difficult

PROPOSITION 2.6. *Assume* $(E, \mathscr{E}, \mathfrak{N})$ *with property* (S). *Let for each* $k\in N$ *and* $(c_1, \ldots, c_k)\in(\mathscr{R}^*)^k$ *a set* $W_{c_1,\ldots,c_k}$ *in* $\mathscr{E}$ *and a transfinite system* $(V_{c_1,\ldots,c_k})_{\alpha<\omega_1}$ *of subsets of* E *be given, so that following properties are satisfied*:

1. $W_{c_1,\ldots,c_k}\supset W_{d_1,\ldots,d_k}$ *if* $c_1<d_1, \ldots, c_k<d_k$,
2. $W_{c_1,\ldots,c_k,\emptyset}\subset W_{c_1,\ldots,c_k}$,
3. $V^0_{c_1,\ldots,c_k}\subset W_{c_1,\ldots,c_k}$,
4. $V^\alpha_{c_1,\ldots,c_k}\supset V^\beta_{c_1,\ldots,c_k}$ *if* $\alpha<\beta$,
5. $V_{c_1,\ldots,c_k} = \bigcup_n V^\alpha_{(c_1,n),c_2,\ldots,c_k} = \ldots = \bigcup_n V^\alpha_{c_1,\ldots,c_{k-1},(c_k,n)}$,
6. $V^{\alpha+1}_{c_1,\ldots,c_k}\subset V^\alpha_{c_1,\ldots,c_k,\emptyset}$.

Then one of the following 2 alternatives must occur

1. $V_\emptyset^\alpha\in\mathfrak{N}$ *for some* $\alpha<\omega_1$.
2. *There is a sequence* $(\nu^k)_k$ *in* $\mathscr{N}$ *such that* $\bigcap_k W_{\nu^1|k,\ldots,\nu^k|k}\neq\emptyset$.

Proof. The Cantor enumeration of $N\times N$ induces a map

$$\mathscr{R}\to\bigcup_k\mathscr{R}^k\colon\ c\mapsto(d_c^1, \ldots, d_c^{k_{|c|}}),$$

where the number $k_{|c|}$ of complexes is of course only dependent on $|c|$. This map is extended to $\mathscr{R}^*$ by taking $k_0 = 1$ and $d_\emptyset^1 = \emptyset$.

For each $c\in\mathscr{R}^*$, we define

$$A_c^0 = W_{d_c^1,\ldots,d_c^{k_{|c|}}}\quad\text{and}\quad A_c^\alpha = V_{d_c^1,\ldots,d_c^{k_{|c|}}}\quad\text{if }\alpha>0.$$

We show that the conditions (1), (2), (3) of 2.3 are verified.

(1) To see that the scheme $(A_c^0)_{c\in\mathscr{R}}$ on $\mathscr{E}$ is regular, take $c', c''\in\mathscr{R}$ with $c'<c''$.

Then $k_{|c'|}\leqslant k_{|c''|}$ and $d_{c'}^1<d_{c''}^1, \ldots, d_{c'}^{k_{|c'|}}<d_{c''}^{k_{|c'|}}$. We only have to apply properties 1 and 2.

(2) This follows immediately from properties 3 and 4.

(3) Assume $c\in\mathscr{R}^*$ and $|c| = r$. We distinguish 2 cases

Case I. $k_r = k_{r+1}$. There is some $k = 1, \ldots, k_r$ so that $d_{c,n}^l = d_c^l$ if $l\neq k$ and $d_{c,n}^k = (d_c^k, n)$, whenever $n\in N$. We find

$$A_c^{\alpha+1} = \bigcup_n V^{\alpha+1}_{d_c^1,\ldots,(d_c^k,n),\ldots,d_c^{k_r}} = \bigcup_n V^{\alpha+1}_{d_{c,n}^1,\ldots,d_{c,n}^{k_{r+1}}} = \bigcup_n A_{c,n}^{\alpha+1}\subset\bigcup_n A_{c,n}^\alpha.$$

Case II. $k_{r+1} = k_r+1$. Then $d_{c,n}^l = d_c^l$ if $1\leqslant l\leqslant k_r$ and $d_{c,n}^{k_r+1} = n$, whenever $n\in N$. We obtain

$$A_c^{\alpha+1} = V^{\alpha+1}_{d_c^1,\ldots,d_c^{k_r}}\subset V^\alpha_{d_c^1,\ldots,d_c^{k_r},\emptyset} = \bigcup_n V^\alpha_{d_c^1,\ldots,d_c^{k_r},n} = \bigcup_n A_{c,n}^\alpha.$$

Since $(E, \mathscr{E}, \mathscr{N})$ possesses (S), either $A_\emptyset^\alpha\in\mathscr{N}$ for some $\alpha<\omega_1$ or there is $\nu\in\mathscr{N}$ with $\bigcap_r A^0_{\nu|r}\neq\emptyset$. Remark that $A_\emptyset^\alpha = V_\emptyset^\alpha$. If $\nu\in\mathscr{N}$, then there is a sequence $(\nu^k)_k$ in $\mathscr{N}$ such that $d^k_{\nu|r}<\nu^k$ whenever $r\in N$ and $k\leqslant k_r$. If $k\in N$ is fixed, then there exists $r\in N$ with $k\leqslant k_r$ and $\nu^l|k<d^l_{\nu|r}$ for each $l=1, \ldots, k$. Then $A^0_{\nu|r} = W_{d_\nu^1|r,\ldots,d_\nu^{k_r}|r}\subset W_{\nu^1|k,\ldots,\nu^k|k}$. This completes the proof.

THEOREM 2.7. *Assume* $(E, \mathscr{E}, \mathfrak{N})$ *with property* (S) *and* $(K, \mathscr{K})$ *a paved set such that* $\mathscr{K}$ *is semi-compact and stable under finite intersection. We consider the projections* $\pi_k\colon E\times K^{k+1}\to E\times K^k$ *and* $p_k\colon E\times K^k\to E$. *For each* $k\in N$, *let* $(X_k^\alpha)_{\alpha<\omega_1}$ *be a transfinite system of subsets of* $E\times K^k$, *so that following properties are satisfied*:

1. X_k^0 *is* $(\mathscr{E}\times\mathscr{K}^k)$*-analytic in* $E\times K^k$,
2. $X_k^\alpha\supset X_k^\beta$ *if* $\alpha<\beta$,
3. $X_k^{\alpha+1}\subset\pi_k(X_{k+1}^\alpha)$.

Assume $p_1(X_1^\alpha)\notin\mathfrak{N}$ *for each* $\alpha<\omega_1$. *Then there exist* $x\in E$ *and* $(y_k)_k$ *in* K^N *such that* $(x, y_1, \ldots, y_k)\in X_k^0$ *for each* $k\in N$.

Proof. Let X_k^0 be the result of a regular scheme $(Y_c^k)_{c\in\mathscr{R}}$ on $\mathscr{E}\times\mathscr{K}^k$. For each $k\in N$ and $(c_1, \ldots, c_k)\in(\mathscr{R}^*)^k$, define

$$W_{c_1,\ldots,c_k} = p_k\Big(\bigcap_{l=1}^k(Y_{c_l}^l\times K^{k-l})\Big)$$

and

$$V^\alpha_{c_1,\ldots,c_k} = p_k\Big(\bigcap_{l=1}^k\big(Y^l(c_l)\times K^{k-l}\big)\cap X_k^\alpha\Big).$$

The reader will easily make out that (1) → (6) of Proposition 2.6 are verified. Hence there are 2 possibilities:

I. There is $\alpha<\omega_1$ such that $V_\emptyset^\alpha = p_1(X_1^\alpha)\in\mathfrak{N}$.

II. There is a sequence $(\nu^k)_k$ in $\mathscr{N}$ such that $\bigcap_k W_{\nu^1|k,\ldots,\nu^k|k}$ contains some point

$x \in E$. Therefore $\bigcap_{l=1}^{k} (Y^l_{\nu^l|k}(x) \times K^{k-l}) \neq \emptyset$ for each $k \in N$. By the semi-compactness of the paving $\mathscr{K}^N$ on $K^N = \prod_k K_k$, we get

$$\bigcap_k \bigcap_{l \leqslant k} (Y^l_{\nu^l|k}(x) \times \prod_{m>l} K_m) = \bigcap_l (\bigcap_k Y^l_{\nu^l|k}(x) \times \prod_{m>l} K_m)) \neq \emptyset$$

and thus contains a point $(y_k)_k$ of $\prod_k K_k$. For each integer l, we have

$$(x, y_1, \ldots, y_l) \in \bigcap_k Y^l_{\nu^l|k} \subset X^0_l,$$

completing the proof.

We pass to the following first corollary

Proposition 2.8. *If $(E, \mathscr{E}, \mathfrak{N})$ has property (S), then also $(E, \mathscr{A}(\mathscr{E}), \mathfrak{N})$ has property (S).*

Proof. Let for each $c \in \mathscr{R}^*$ a transfinite system $(A^\alpha_c)_{\alpha<\omega_1}$ of subsets of E be given, satisfying

1. $(A^0_c)_{c\in\mathscr{R}}$ is a regular scheme on $\mathscr{A}(\mathscr{E})$,
2. $A^\alpha_c \supset A^\beta_c$ if $\alpha<\beta$,
3. $A^{\alpha+1}_c \subset \bigcup_n A^\alpha_{(c,n)}$.

Take $K = N$ and let $\mathscr{K} = \{\emptyset\} \cup \{\{n\};\ n \in N\}$, which is a compact paving on K, stable under finite intersection. For each $k \in N$ and $\alpha<\omega_1$, we define $X^\alpha_k = \{(x, c) \in E \times K^k;\ x \in A^\alpha_c\}$, which clearly satisfy the conditions (1), (2), (3) of 2.7. Therefore we have one of the following 2 possibilities:

I. There is $\alpha<\omega_1$ so that $p_1(X^\alpha_1) \in \mathfrak{N}$. But $A^{\alpha+1}_\emptyset \subset \bigcup_n A^\alpha_n = p_1(X^\alpha_1)$, implying $A^{\alpha+1}_\emptyset \in \mathfrak{N}$.

II. There is $x \in E$ and $\nu \in \mathscr{N}$ such that $(x, \nu|k) \in X^0_k$ for each $k \in N$. Then $x \in \bigcap_k A^0_{\nu|k}$ and thus in the result of the scheme $(A^0_c)_{c\in\mathscr{R}}$. So the proof is given.

Theorem 2.9. *Assume $(E, \mathscr{E}, \mathfrak{N})$ with property (S) and let $(A_n)_n$ be a sequence in $\mathscr{A}(\mathscr{E})$ such that $\bigcap_n A_n = \emptyset$. Then there is a sequence $(B_n)_n$ in $\mathscr{E}^*$ so that $A_n \subset B_n$ for each n and $\bigcap_n B_n \in \mathfrak{N}$.*

Proof. Each set A_n is the result of a regular scheme on $\mathscr{E}$ with index i_n. Let $K = \mathscr{N}$ and $\mathscr{K} = \underline{\mathscr{N}}$, which is a compact paving on K, stable under finite intersection. For each $k \in N$ and $\alpha<\omega_1$, we define $X^\alpha_k = \bigcap_n \{(x, \nu^1, \ldots, \nu^k);\ i_n((\nu^1_n, \ldots, \nu^k_n), x)>\alpha\}$, which again satisfy the conditions (1), (2), (3) of 2.7 (cfr. 1.21). Thus there are 2 alternatives:

I. There is $\alpha<\omega_1$ so that $p_1(X^\alpha_1) \in \mathfrak{N}$. If we let $B_n = \{x \in E;\ i_n(x)>\alpha+1\}$, then B_n belongs to $\mathscr{E}^*$ and $A_n \subset B_n$. Moreover $\bigcap_n B_n = \bigcap_n \{x \in E;\ \exists \nu_n \in N$ such that $i_n(\nu_n, x)>\alpha\} = \{x \in E;\ \exists \nu \in \mathscr{N}$ such that $(x, \nu) \in X^\alpha_1\} = p_1(X^\alpha_1)$, thus a member of $\mathfrak{N}$.

II. There is $x \in E$ and a sequence $(\nu^k)_k$ in $\mathscr{N}$ so that $(x, \nu^1, \ldots, \nu^k) \in X^0_k$ for each $k \in N$. Let $n \in N$ be fixed. We find that $i_n((\nu^1_n, \ldots, \nu^k_n), x)>0$ for every $k \in N$, implying $x \in A_n$. Hence $x \in \bigcap_n A_n$, which is a contradiction.

In particular, we obtain the Novikov separation result (see [24]):

Proposition 2.10. *Let $(E, \mathscr{E})$ be a paved set where $\mathscr{E} = \mathscr{E}^-$ is semi-compact. If $(A_n)_n$ is a sequence in $\mathscr{A}(\mathscr{E})$ such that $\bigcap_n A_n = \emptyset$, then there is a sequence $(B_n)_n$ in $\mathscr{E}^*$ so that $A_n \subset B_n$ for each n and $\bigcap_n B_n = \emptyset$.*

III. Applications in section theory.

A. Classes of sets. The starting point will be a paved set $(X, \mathfrak{X})$ such that:

1. $X \in \mathfrak{X}$.
2. $\mathfrak{X}$ is stable under finite union and finite intersection.
3. $\mathfrak{X}$ is bianalytic (i.e. $\mathfrak{X} \subset b\mathscr{A}(\mathfrak{X})$).

Let further $\mathfrak{N}$ be a class of subsets of X satisfying

4. $\mathfrak{N}$ is a σ-ideal.
5. If $A \in \mathfrak{N}$, then there is $B \in \mathfrak{N} \cap \mathfrak{X}^*$ so that $A \subset B$.
6. $(X, \mathfrak{X}, \mathfrak{N})$ has property (S).

Definition 3.1. If $\mathscr{F}$ is a class of subsets of X, we let $\mathscr{F}'$ consist of the $A \subset X$ such that there is $B \in \mathscr{F}$ with $A \Delta B \in \mathfrak{N}$. It is clear that $(\mathscr{F}')' = \mathscr{F}'$.

Proposition 3.2. *If $A \in \mathfrak{X}'$, then there exist $B, C \in b\mathscr{A}(\mathfrak{X})$ satisfying $B \subset A$, $A \subset C$ and $A \setminus B \in \mathfrak{N}$, $C \setminus A \in \mathfrak{N}$.*

Proof. Take $A_1 \in \mathfrak{X}$ so that $A \Delta A_1 \in \mathfrak{N}$ and consider $D \in \mathfrak{N} \cap \mathfrak{X}^*$ with $A \Delta A_1 \subset D$. It is easily seen that $B = A_1 \setminus D$ and $C = A_1 \cup D$ satisfy.

Proposition 3.3. *$(X, \mathfrak{X}', \mathfrak{N})$ has property (S).*

Proof. It is clear that $(X, \mathfrak{X}', \mathfrak{N})$ is basic. It follows from 3.2 that if $(A_c)_{c\in\mathscr{R}}$ is a regular scheme on $\mathfrak{X}'$, then there is a regular scheme $(B_c)_{c\in\mathscr{R}}$ on $b\mathscr{A}(\mathfrak{X})$ such that $B_c \subset A_c$ and $A_c \setminus B_c \in \mathfrak{N}$ for each $c \in \mathscr{R}$. Hence $D = \bigcup_c (A_c \setminus B_c)$ is still a member of $\mathfrak{N}$. Let i and j be the indices of the schemes $(A_c)_{c\in\mathscr{R}}$ and $(B_c)_{c\in\mathscr{R}}$ respectively. By induction and using 1.21, we see that $\{x \in X;\ i(c, x)>\alpha,\ j(c, x) \leqslant \alpha\}$ is contained in D for each $c \in \mathscr{R}^*$ and $\alpha<\omega_1$. Since, by 2.8, also $(X, b\mathscr{A}(\mathfrak{X}), \mathfrak{N})$ has property (S), there are 2 possibilities:

1. The scheme $(B_c)_{c\in\mathscr{R}}$ and hence certainly $(A_c)_{c\in\mathscr{R}}$ have a nonempty result.
2. There is $\alpha<\omega_1$ so that $\{x \in X;\ j(x)>\alpha\} \in \mathfrak{N}$. Since

$$\{x \in X;\ i(x)>\alpha\} \subset \{x \in X;\ j(x)>\alpha\} \cup D$$

also
$$\{x \in X;\ i(x) > \alpha\} \in \mathfrak{N}.$$
So the proof is complete.

PROPOSITION 3.4. $(\mathfrak{X}')^* = (\mathfrak{X}^*)'$.

Proof. 1. Since $\mathfrak{X}' \subset (\mathfrak{X}^*)'$ and $(\mathfrak{X}^*)'$ is stable under countable union and countable intersection, $(\mathfrak{X}')^* \subset (\mathfrak{X}^*)'$.

2. Define $\mathscr{Y} = \{A \in \mathfrak{X}^*;\ \{A\}' \subset (\mathfrak{X}')^*\}$ which of course contains $\mathfrak{X}$. Moreover $\mathscr{Y}$ is stable under countable union and countable intersection. We give the details for the intersection, the argument for the union being similar.

Let thus $(A_n)_n$ be a sequence in $\mathscr{Y}$, $A = \bigcap_n A_n$ and B some set in $\{A\}'$. If for each n we take $B_n = [A_n \setminus (A \setminus B)] \cup (B \setminus A)$, then B_n is in $\{A_n\}'$ and hence in $(\mathfrak{X}')^*$. Thus also $B = \bigcap_n B_n$ is in $(\mathfrak{X}')^*$. So we proved that $A \in \mathscr{Y}$. Therefore $\mathfrak{X}^* \subset \mathscr{Y}$, implying that $(\mathfrak{X}^*)' \subset (\mathfrak{X}')^*$.

The following is left as an exercise for the reader.

PROPOSITION 3.5. $\mathscr{A}(\mathfrak{X}') = \mathscr{A}(\mathfrak{X})'$.

PROPOSITION 3.6. $b\mathscr{A}(\mathfrak{X})' = b\mathscr{A}(\mathfrak{X}') = (\mathfrak{X}^*)'$.

Proof. It follows from 3.5 that $b\mathscr{A}(\mathfrak{X}') \subset b(\mathscr{A}(\mathfrak{X})') = b\mathscr{A}(\mathfrak{X}')$. If $A \in b\mathscr{A}(\mathfrak{X}')$, then $A \in \mathscr{A}(\mathfrak{X}')$, $X \setminus A \in \mathscr{A}(\mathfrak{X}')$ and we obtain $B, C \in (\mathfrak{X}')^* = (\mathfrak{X}^*)'$ so that $A \subset B$, $X \setminus A \subset C$ and $B \cap C \in \mathfrak{N}$, applying 3.3 and 2.9. Since $B \setminus A \subset B \cap C$, also $A \in (\mathfrak{X}^*)'$. Finally $(\mathfrak{X}^*)' \subset b\mathscr{A}(\mathfrak{X})'$, since $\mathfrak{X}$ is bianalytic.

We let $\mathfrak{M} = \mathfrak{M}(X, \mathfrak{X}, \mathfrak{N})$ be the σ-algebra $b\mathscr{A}(\mathfrak{X}')$.

DEFINITION 3.7. If Y is a Polish (a.e. a complete metric space which is separable), let B_Y denote its Borel field. The σ-algebra B_Y is the union of the classes F_α and also the union of the classes G_α ($\alpha < \omega_1$), where:

(i) F_0 is the family of the closed sets and G_0 of the open sets in Y.

(ii) The sets of the family F_β are countable intersections or unions of sets belonging to F_α with $\alpha < \beta$ according to whether β is even or odd. The sets of the family G_β are countable unions or intersections of sets belonging to G_α with $\alpha < \beta$ according to whether β is even or odd.

The families F_α with even indices as well as the families G_α with odd indices form the multiplicative class α, the families F_α with odd indices and the families G_α with even indices the additive class α (for more details, we refer to [21], p. 345).

We let $\mathscr{P} = \mathscr{P}(X, Y) = \{A \times F;\ A \in \mathfrak{X}' \text{ and } F \text{ closed in } Y\}$.

PROPOSITION 3.8. $\mathfrak{M} \otimes \mathscr{B}_Y = \mathscr{P}^*$.

Proof. This follows from the fact that $\mathfrak{M} = (\mathfrak{X}')^*$, $\mathscr{B}_Y = F_0^*$ and monotonicity arguments.

Let $A \subset X \times Y$, $x \in X$ and $y \in Y$. Define
$$A(x) = \{y \in Y;\ (x, y) \in A\} \quad\text{and}\quad A(y) = \{x \in X;\ (x, y) \in A\}.$$
Such sets will be called sections of A.

From 3.8, we deduce the following result

PROPOSITION 3.9. *If $A \in \mathfrak{M} \otimes \mathscr{B}_Y$, then the sections $A(x)$, where x is taken in X, are of bounded Baire class.*

DEFINITION 3.10. For each $\alpha < \omega_1$, let $\mathscr{S}_\alpha = \mathscr{S}_\alpha(X, Y)$ be the class of those $A \in \mathfrak{M} \otimes \mathscr{B}_Y$ such that $A(x)$ is an F_α-set for each $x \in X$ and $\mathscr{T}_\alpha = \mathscr{T}_\alpha(X, Y)$ the class of the $A \in \mathfrak{M} \otimes \mathscr{B}_Y$ such that $A(x)$ is a G_α-set for each $x \in X$. Hence $\mathscr{T}_\alpha = c\mathscr{S}_\alpha$.

Proposition 3.9 can be reformulated as following

PROPOSITION 3.11. $\mathfrak{M} \otimes \mathscr{B}_Y = \bigcup_{\alpha < \omega_1} \mathscr{S}_\alpha = \bigcup_{\alpha < \omega_1} \mathscr{T}_\alpha$.

DEFINITION 3.12. For each $\alpha < \omega_1$, we introduce a class $\mathscr{F}_\alpha = \mathscr{F}_\alpha(X, Y)$ and a class $\mathscr{G}_\alpha = \mathscr{G}_\alpha(X, Y)$ as follows:

(i) $\mathscr{F}_0 = \mathscr{S}_0$ and $\mathscr{G}_0 = \mathscr{T}_0$.

(ii) The sets of the family $\mathscr{F}_\beta$ are countable intersections or unions of sets belonging to $\mathscr{F}_\alpha$ with $\alpha < \beta$ according to whether β is even or odd.

The sets of the family $\mathscr{G}_\beta$ are countable unions or intersections of sets belonging to $\mathscr{G}_\alpha$ with $\alpha < \beta$ according to whether β is even or odd.

By induction, we verify that $\mathscr{G}_\alpha = c\mathscr{F}_\alpha$.

It is easily seen that $\mathscr{F}_\alpha \subset \mathscr{S}_\alpha$ and $\mathscr{G}_\alpha \subset \mathscr{T}_\alpha$ for all $\alpha < \omega_1$. In fact the following deep property holds

THEOREM 3.12. *$\mathscr{F}_\alpha = \mathscr{S}_\alpha$ and $\mathscr{G}_\alpha = \mathscr{T}_\alpha$ for each $\alpha < \omega_1$.*

Proof. We remark that $\mathfrak{M}$ is a σ-algebra on X satisfying $\mathfrak{M} = b\mathscr{A}(\mathfrak{M})$. Then the theorem follows from recent results in descriptive set theory obtained by A. Louveau (see [22]).

The following proposition is easily established by induction

PROPOSITION 3. 13. *Let $(X_n)_n$ be a sequence of disjoint sets in $\mathfrak{M}$. If $\alpha < \omega_1$ and $(A_n)_n$ is a sequence in $\mathscr{F}_\alpha$ (resp. $\mathscr{G}_\alpha$), then also $A = \bigcup_n [A_n \cap (X_n \times Y)]$ is in $\mathscr{F}_\alpha$ (resp. $\mathscr{G}_\alpha$).*

DEFINITION 3.14. $\mathfrak{S} = \mathfrak{S}(X, Y)$ will be the class of the subsets A of $X \times Y$ so that $A(x) \in \mathscr{B}_Y$ for each $x \in X$ and there exists $B \in \mathfrak{M} \otimes \mathscr{B}_Y$ satisfying $\pi_X(A \Delta B) \in \mathfrak{N}$.

Obviously we have

PROPOSITION 3.15. *$\mathfrak{S}$ is a σ-algebra.*

DEFINITION 3.16. If $A \subset X \times Y$, then $\bar{A}^s \subset X \times Y$ is defined by $\bar{A}^s(x) = \overline{A(x)}$, where $^-$ denotes the closure operation.

The following description of $\bar{A}^s$ will be useful. If $y \in Y$ and $\varepsilon > 0$, then $B(y, \varepsilon)$ is the open ball with midpoint y and radius ε. Let now $(y_n)_n$ be a dense sequence in Y. If for each $n \in N$ and $k \in N$ we take $X_{n,k} = \pi_X[A \cap (X \times B(y_n, 1/k))]$, then $\bar{A}^s = \bigcap_k \bigcup_n (X_{n,k} \times B(y_n, 1/k))$.

PROPOSITION 3.17. *Let $A \subset X \times Y$ and suppose $\pi_X(A) \in \mathfrak{N}$. If $\alpha < \omega_1$ and the section $A(x)$, where x is taken in X, are F_α (resp. G_α) sets, then $A \in \mathscr{F}_\alpha$ (resp. $\mathscr{G}_\alpha$).*

Proof. It is clearly enough to prove only the first property. We proceed inductively on α. If $\alpha = 0$, then every section $A(x)$ of A is closed and hence $A = \bar{A}'$. Since for every $n \in N$, $k \in N$ the set $X_{n,k} \in \mathfrak{N}$, $A \in \mathfrak{M} \otimes \mathscr{B}_Y$ and hence $A \in \mathscr{F}_0$. Now let the property be true for every $\alpha < \beta$ and assume $A(x)$ an $\mathscr{F}_\beta$ set for each $x \in X$. Clearly there is a sequence $(A_n)_n$ of subsets of $X \times Y$ such that $\pi_X(A_n) = \pi_X(A)$ for each n, the set $A_n(x)$ is in $\bigcup_{\alpha<\beta} \mathscr{F}_\alpha$ for each n and each $x \in X$ and $A = \bigcap_n A_n$ if β is even, $A = \bigcup_n A_n$ if β is odd.

Let $n \in N$ be fixed. If for each $\alpha < \beta$ we take $X_{n,\alpha} = \{x \in X;\ A_n(x)$ is precisely an F_α set$\}$, then $A_{n,\alpha} = A_n \cap (X_{n,\alpha} \times Y) \in \mathscr{F}_\alpha$, by induction hypothesis. It follows from 3.13 that $A_n = \bigcup_{\alpha<\beta} A_{n,\alpha} \in \mathscr{F}_\beta$. Hence also $A \in \mathscr{F}_\beta$, which completes the proof.

PROPOSITION 3.18. *Let $A \in \mathfrak{S}$. Then $A \in \mathfrak{M} \otimes \mathscr{B}_Y$ if and only if the sections $A(x)$, where x is taken in X, are of bounded Baire class.*

Proof. The "only if" part is precisely 3.9. Assume $A \in \mathfrak{S}$, then there exists some $B \in \mathfrak{M} \otimes \mathscr{B}_Y$ such that $\pi_X(A \Delta B) \in \mathfrak{N}$. If the sections $A(x)$ are of bounded Baire class, then, again by 3.9, this is also true for the sections $(A \setminus B)(x)$ of $A \setminus B$ and $(B \setminus A)(x)$ of $B \setminus A$. It follows from 3.17 that $A \setminus B$ and $B \setminus A$ are members of $\mathfrak{M} \otimes \mathscr{B}_Y$. Hence $A = (B \setminus (B \setminus A)) \cup (A \setminus B) \in \mathfrak{M} \otimes \mathscr{B}_Y$.

We introduce $\mathscr{A} = \mathscr{A}(X, Y)$ as the class of $\mathscr{P}(X, Y)$-analytic subsets of $X \times Y$. From 3.8 and the fact that $\mathscr{A} = \mathscr{A}^*$, we obtain immediately

PROPOSITION 3.19. $\mathfrak{M} \otimes \mathscr{B}_Y \subset \mathscr{A}(X, Y)$.

The following result is similar to 3.17.

PROPOSITION 3.20. *Let $A \subset X \times Y$ and suppose $\pi_X(A) \,\hat{\in}\, \mathfrak{N}$. If the section $A(x)$ is analytic in Y for each $x \in X$, then $A \in \mathscr{A}$.*

Proof. For each $x \in X$, $A(x)$ is the result of a Souslin scheme $(F_c^x)_{c \in \mathscr{R}}$ on the paving of the closed subsets of Y. For each $c \in \mathscr{R}$, define $F_c \subset X \times Y$ by $F_c(x) = F_c^x$ if $x \in \pi_X(A)$ and $F_c(x) = \emptyset$ if $x \notin \pi_X(A)$. By 3.19, we find that $F_c \in \mathscr{F}_0$. Because A is the result of the scheme $(F_c)_{c \in \mathscr{R}}$ and 1.14, we find $A \in \mathscr{A}$.

PROPOSITION 3.21. $\mathfrak{S}(X, Y) \subset \mathscr{A}(X, Y)$.

Proof. Let $A \in \mathfrak{S}$ and take $B \in \mathfrak{M} \otimes \mathscr{B}_Y$ satisfying $\pi_X(A \Delta B) \in \mathfrak{N}$. Since

$$B_1 = B \cap [(X \setminus \pi_X(A \Delta B)) \times Y] \in \mathfrak{M} \otimes \mathscr{B}_Y,$$
$$A_1 = A \cap [\pi_X(A \Delta B) \times Y] \in \mathscr{A}(X, Y)$$

by 3.20 and $A \triangleq B_1 \cup A_1$, it follows that $A \in \mathscr{A}(X, Y)$.

B. Separation results. In this section, we will apply the general separation theorems obtained in the preceding chapter to more concrete situations. We start with the following well-known fact.

PROPOSITION 3.22. *Every Polish space is homeomorphic to a G_δ-subset of $[0, 1]^N$, where $[0, 1]$ is the unit-interval.*

A proof can be found in [26], Ch. I.

We assume Y a fixed Polish space. By 3.22, Y is homeomorphic to a G_δ subset of a compact metric space K. Let $\mathscr{K}$ be the paving on K consisting of the closed sets, which is of course compact.

PROPOSITION 3.23. *If $(A_n)_n$ is a sequence of analytic subsets of Y such that $\bigcap_n A_n = \emptyset$, then there is a sequence $(B_n)_n$ in $\mathscr{B}_Y$ satisfying $A_n \subset B_n$ for each n and $\bigcap_n B_n = \emptyset$.*

Proof. Y can clearly be assumed a G_δ subspace of K. Since $(A_n)_n$ is also a sequence of $\mathscr{K}$-analytic subsets, we obtain by 2.10 a sequence $(B'_n)_n$ in $\mathscr{B}_K$ satisfying $A_n \subset B'_n$ for each n and $\bigcap_n B'_n = \emptyset$. We only have to take $B_n = B'_n \cap Y$.

In the remainder of this section, we assume $(X, \mathfrak{X}, \mathfrak{N})$ satisfying (1) $\to$ (6) of III, A.

PROPOSITION 3.24. *If $A \in \mathscr{A}(X, Y)$, then $\pi_X(A) \in \mathscr{A}(\mathfrak{X}')$.*

Proof. It is clear that Y can be assumed a G_δ subset of K. Because A, considered as subset of $X \times K$, is $\mathfrak{X}' \times \mathscr{K}$-analytic, $\pi_X(A)$ is $\mathfrak{X}'$-analytic, by 1.15.

PROPOSITION 3.25. *If $(A_n)_n$ is a sequence in $\mathscr{A}(X, Y)$ such that $\bigcap_n A_n = \emptyset$, then there is a sequence $(B_n)_n$ in $\mathfrak{M} \otimes \mathscr{B}_Y$ with $A_n \subset B_n$ for each n and $\pi_X(\bigcap_n B_n) \in \mathfrak{N}$.*

Proof. Again we may assume Y a G_δ subset of K. Remark that each set A_n is $\mathfrak{X}' \times \mathscr{K}$-analytic. Since by 3.3 and 2.5, $(X \times K, \mathfrak{X}' \times \mathscr{K}, \pi_X^{-1}(\mathfrak{N}))$ has property (S), 2.9 yields us a sequence $(B'_n)_n$ in $(\mathfrak{X}' \times \mathscr{K})^* = \mathfrak{M} \otimes \mathscr{B}_k$ so that $A_n \subset B'_n$ for each n and $\pi_X(\bigcap_n B'_n) \in \mathfrak{N}$. If we take $B_n = B'_n \cap (X \times Y)$, the required sequence $(B_n)_n$ is obtained.

THEOREM 3.26. *If $(A_n)_n$ is a sequence in $\mathscr{A}(X, Y)$ such that $\bigcap_n A_n = \emptyset$, then there is a sequence $(B_n)_n$ in $\mathfrak{S}(X, Y)$ with $A_n \subset B_n$ for each n and $\bigcap_n B_n = \emptyset$.*

Proof. By 3.25, there is a sequence $(B'_n)_n$ in $\mathfrak{M} \otimes \mathscr{B}_Y$ such that $A_n \subset B'_n$ for each n and $N = \pi_X(\bigcap_n B'_n) \in \mathfrak{N}$. Applying 3.23, we find on the other side for each $x \in X$ a sequence $(B_n^x)_n$ in $\mathscr{B}_Y$ satisfying $A_n(x) \subset B_n^x$ for each n and $\bigcap_n B_n^x = \emptyset$. The sets B_n are introduced by taking $B_n(x) = B'_n(x)$ if $x \notin N$ and $B_n(x) = B_n^x$ if $x \in N$. Because $\pi_X(B_n \Delta B'_n) \subset N$, each set B_n belongs to $\mathfrak{S}(X, Y)$ and it follows from the construction that $A_n \subset B_n$ for each n and $\bigcap_n B_n = \emptyset$.

The following 2 corollaries are straightforward

PROPOSITION 3.27. *Disjoint sets in $\mathscr{A}(X, Y)$ can be separated by sets in $\mathfrak{S}(X, Y)$.*

PROPOSITION 3.28. $b\mathscr{A}(X, Y) = \mathfrak{S}(X, Y)$.

C. Stable mappings. We still assume $(X, \mathfrak{X}, \mathfrak{N})$ with properties (1) $\to$ (6) of III, A. From 3.29 to 3.36, Y and Z will be fixed Polish spaces and $D \in \mathfrak{S}(X, Y)$.

DEFINITION 3.29. A mapping $\varphi: D \to X \times Z$ will be called *stable*, if $\pi_X \circ \varphi = \pi_X$ (φ preserves the first coordinate).

Obviously φ is determined by $\pi_Z \circ \varphi$, which we denote by φ_2.

DEFINITION 3.30. Let $\varphi\colon D\to X\times Z$ be a stable mapping. We will say that φ is *measurable* if φ is $\mathfrak{S}(X, Y)-\mathfrak{S}(X,Z)$ measurable.

PROPOSITION 3.31. *A stable map $\varphi\colon D\to X\times Z$ is measurable if and only if $\varphi_2\colon D\to Z$ is $\mathfrak{S}(X, Y)-\mathscr{B}_Z$ measurable.*

Proof. 1. Suppose φ measurable. Since $\pi_Z\colon X\times Z\to Z$ is $\mathfrak{S}(X,Z)-\mathscr{B}_Z$ measurable, it follows that φ_2 is $\mathfrak{S}(X, Y)-\mathscr{B}_Z$ measurable.

2. Assume now φ_2 is $\mathfrak{S}(X, Y)-\mathscr{B}_Z$ measurable. First we verify that φ is $\mathfrak{S}(X, Y)-\mathfrak{M}\otimes\mathscr{B}_Z$ measurable. Take then $A\in\mathfrak{S}(X,Z)$ and consider $B\in\mathfrak{M}\otimes\mathscr{B}_Z$ satisfying $\pi_X(A\,\Delta\, B)\in\mathfrak{N}$. Clearly

$$\pi_X(\varphi^{-1}(A)\,\Delta\,\varphi^{-1}(B))\subset\pi_X(A\,\Delta\, B)$$

and furthermore

$$\varphi^{-1}(A)(x) = \varphi_2^{-1}(A(x))(x)\in\mathscr{B}_Y.$$

Hence $\varphi^{-1}(A)\in\mathfrak{S}(X, Y)$.

DEFINITION 3.32. If $\varphi\colon D\to X\times Z$ is a stable mapping, then the graph of φ will be the set $\Gamma(\varphi) = \{(x, y, \varphi_2(x, y));\ (x, y)\in D\}$.

PROPOSITION 3.33. *If $\varphi\colon D\to X\times Z$ is stable and measurable, then $\Gamma(\varphi)$ is a member of $\mathfrak{S}(X, Y\times Z)$.*

Proof. Let $\psi\colon D\times Z\to Z\times Z$ be given by $\psi(x, y, z) = (\varphi_2(x, y), z)$. Then ψ is $\mathfrak{S}(X, Y\times Z)-\mathscr{B}_Z\otimes\mathscr{B}_Z$ measurable. Indeed, $\pi_Z\colon D\times Z\to Z$ is $\mathfrak{S}(X, Y\times Z)-\mathscr{B}_Z$ measurable and $\pi_{X\times Y}\colon D\times Z\to D$ is $\mathfrak{S}(X, Y\times Z)-\mathfrak{S}(X, Y)$ measurable. The diagonal Δ of $Z\times Z$ belongs to $\mathscr{B}_Z\otimes\mathscr{B}_Z$, since it is closed. The fact that $\Gamma(\varphi) = \psi^{-1}(\Delta)$ completes the proof.

PROPOSITION 3.34. *If $\varphi\colon D\to X\times Z$ is stable and measurable and $A\in\mathscr{A}(X, Y)$, then $\varphi(A\cap D)\in\mathscr{A}(X, Z)$.*

Proof. We may assume Y a G_δ-subset of a compact metric space $\mathscr{K}$ with paving $\mathscr{K}$ of its compact subsets. Let $\mathscr{F}$ be the paving on Z consisting of the closed sets. By 3.33, $\Gamma(\varphi)\in\mathfrak{S}(X, Y\times Z)$ and hence, by 3.21, $\Gamma(\varphi)\cap(A\times Z)\in\mathscr{A}(X, Y\times Z)$. Since $\Gamma(\varphi)\cap(A\times Z)$, considered as subset of $X\times K\times Z$, is $\mathfrak{X}'\times\mathscr{K}\times\mathscr{F}$-analytic, we obtain, by 1.15, that $\varphi(A\cap D) = \pi_{X\times Y}(\Gamma(\varphi)\cap(A\times Z))$ is $\mathfrak{X}'\times\mathscr{F}$-analytic. Thus $\varphi(A\cap D)\in\mathscr{A}(X, Z)$.

DEFINITION 3.35. We will say that a stable map $\varphi\colon D\to X\times Z$ is *continuous* provided the partial map $(\varphi_2)_x\colon D(x)\to Z$ is continuous for each $x\in X$.

PROPOSITION 3.36. *If $D\in\mathfrak{M}\otimes\mathscr{B}_Y$ and $\varphi\colon D\to X\times Z$ is a stable, measurable and continuous map, then φ is $\mathfrak{M}\otimes\mathscr{B}_Y-\mathfrak{M}\otimes\mathscr{B}_Z$ measurable.*

Proof. Let B be a member of $\mathfrak{M}\otimes\mathscr{B}_Z$. Applying 3.18, we only have to show that the sections $\varphi^{-1}(B)(x) = ((\varphi_2)_x)^{-1}(B(x))$ are of bounded Baire class. But this follows immediately from 3.9 and the fact that each $(\varphi_2)_x$ is continuous.

Obviously, the following composition results hold:

PROPOSITION 3.37. *Let Y, Z, W be Polish spaces, $D\in\mathfrak{S}(X, Y)$, $E\in\mathfrak{S}(X,Z)$, $\varphi\colon D\to X\times Z$ and $\psi\colon E\to X\times W$ mappings so that $\varphi(D)\subset E$. If φ and ψ are stable, then $\psi\circ\varphi$ is stable. If moreover φ and ψ are measurable (continuous) then also $\psi\circ\varphi$ is measurable (continuous).*

PROPOSITION 3.38. *If Y is a Polish space and $A\in\mathscr{A}(X, Y)$, then there exist a set D in $\mathscr{F}_0(X,\mathscr{N})$ and a continuous map $\varphi\colon\mathscr{N}\to Y$ so that $\varphi(D(x)) = A(x)$ or each $x\in X$.*

Proof. Let A be the result of a regular scheme $(M_c\times F_c)_{c\in\mathscr{R}}$ on $\mathscr{P}(X, Y)$. It is easily seen that we may assume $F_c\neq\emptyset$, $F_c\supset F_d$ if $c<d$ and $\operatorname{diam} F_c\leqslant 1/|c|$, where the diameter is taken with respect to a complete metric. Obviously the set

$$D = \bigcup_\nu\bigcap_{c<\nu}(M_c\times\mathscr{N}_c) = \bigcap_k\bigcup_{|c|=k}(M_c\times\mathscr{N}_c)$$

belongs to $\mathscr{F}_0(X,\mathscr{N})$. The map φ on $\mathscr{N}$ will be given by $\varphi(\nu) = \bigcap_{c<\nu}F_c$, which is a unique point of Y. It is clear that φ is continuous. Moreover

$$D(x) = \{\nu\in\mathscr{N};\ x\in\bigcap_{c<\nu}M_c\}$$

and hence $\varphi(D(x)) = \bigcup_{\nu\in D(x)}\bigcap_{c<\nu}F_c$, which is precisely $A(x)$.

Our next aim is to establish the following result

PROPOSITION 3.39. *If Y is a Polish space and $A\in\mathfrak{S}(X, Y)$, then there exists a set $D\in\mathscr{F}_0(X,\mathscr{N})$ and an injective, stable, measurable and continuous map $\varphi\colon D\to X\times Y$ onto A.*

We need the following lemma

PROPOSITION 3.40. *Let $(Y_n)_n$ be a sequence of Polish spaces and let $Y = \prod_n Y_n$. We consider for each $n\in N$ a member D_n of $\mathfrak{M}\otimes\mathscr{B}_{Y_n}$. Then the subset D of $X\times Y$ defined by $D(x) = \prod_n D_n(x)$ belongs to $\mathfrak{M}\otimes\mathscr{B}_Y$.*

Proof. It is easily verified that for each n the set

$$\hat{D}_n = \{(x, y)\in X\times Y;\ (x, y_n)\in D_n\}$$

is a member of $\mathfrak{M}\otimes\mathscr{B}_Y$. Since $D = \bigcap_n\hat{D}_n$, the proof is clear.

The main step in the proof of 3.39 is the following

PROPOSITION 3.41. *Let $\mathscr{D}$ be the class of subsets A of $X\times Y$ with the property that there is a set $D\in\mathscr{F}_0(X,\mathscr{N})$ and an injective, stable, measurable and continuous map $\varphi\colon D\to X\times Y$ satisfying $\varphi(D) = A$. Then:*

1. *$\mathscr{D}$ is stable under countable disjoint union.*
2. *$\mathscr{D}$ is stable under countable intersection.*

Hence $\mathscr{D}\cap c\mathscr{D}$ is stable under countable union.

Proof. It is clear that in the definition of $\mathscr{D}$ above, the space $\mathscr{N}$ can be replaced by a homeomorphic Polish space.

1. Let $(A_n)_n$ be a sequence of disjoint members of $\mathscr{D}$. For each n, we obtain a set D_n in $\mathscr{F}_0(X,\mathscr{N}_n)$ and an injective, stable, measurable and continuous map

$\varphi_n\colon D_n \to X\times Y$ satisfying $\varphi_n(D_n) = A_n$. Obviously $D = \bigcup_n D_n$ is a member of $\mathscr{F}_0(X, \mathscr{N})$. Define φ on D by taking $\varphi | D_n = \varphi_n$. Then φ satisfies the required properties and has image $\bigcup_n A_n$.

2. Let $(A_n)_n$ be a sequence of members of $\mathscr{D}$. For each n, let $D_n \in \mathscr{F}_0(X, \mathscr{N})$ and $\varphi_n\colon D_n \to X\times Y$ an injective, stable, measurable and continuous map so that $\varphi_n(D_n) = A_n$. Let $S = \mathscr{N}^N$. From 3.40 we know that the subset $\tilde{D}$ of $X\times S$ defined by $\tilde{D}(x) = \prod_n D_n(x)$ belongs to $\mathfrak{M}\otimes\mathscr{B}_S$ and hence to $\mathscr{F}_0(X, S)$. We consider the map $\tilde{\varphi}\colon \tilde{D} \to X\times Y^N$ given by $\tilde{\varphi}_2(x, s) = (\varphi_{n,2}(x, s_n))_n$, if $s = (s_n)_n$. Using 3.31, we see that $\tilde{\varphi}$ is measurable and continuous. If Δ is the diagonal of Y^N, then $D = (\tilde{\varphi})^{-1}(X\times\Delta) \in \mathfrak{S}(X, S)$ and hence $D \in \mathscr{F}_0(X, S)$, since $D(x)$ is closed in $\tilde{D}(x)$ for each $x\in X$. Let $\iota\colon \Delta \to Y$ be the canonical isomorphism and $\varphi\colon D \to X\times Y$ the stable map given by $\varphi_2 = \iota\circ(\tilde{\varphi}_2 | D)$. It is easily checked that φ is injective, measurable and continuous. We also verify that $\varphi(D) = \bigcap_n A_n$. Because S and $\mathscr{N}$ are homeomorphic, the proof is complete.

PROPOSITION 3.42. *If Y is Polish, then every member of $\mathscr{B}_Y$ is the continuous injective image of a closed subset of $\mathscr{N}$.*

Proof. We refer to [12], p. 247, Th. 79 or [26], Ch. I.

Proof of 3.39. Let $\mathscr{D}$ be as in 3.41. It is enough to prove 3.39 if $A \in \mathfrak{M}\otimes\mathscr{B}_Y$ and if $A \in \mathfrak{S}(X, Y)$ with $\pi_X(A) \in \mathfrak{N}$, since every element of $\mathfrak{S}(X, Y)$ is the disjoint union of such sets.

1. From 3.42, it follows that $\mathscr{P}(X, Y) \subset \mathscr{D}$ and hence also $\mathscr{P}(X, Y) \subset \mathscr{D} \cap c\mathscr{D}$. Therefore $\mathfrak{M}\otimes\mathscr{B}_Y \subset \mathscr{D} \cap c\mathscr{D}$, thus certainly $\mathfrak{M}\otimes\mathscr{B}_Y \subset \mathscr{D}$.

2. Assume now $A \in \mathfrak{S}(X, Y)$ and $\pi_X(A) \in \mathfrak{N}$. Again by 3.42 there exist for each $x\in X$ a closed subset D^x of $\mathscr{N}$ and a continuous injective map $\varphi^x\colon D^x \to Y$ onto $A(x)$. Let $D(x) = D^x$ if $x \in \pi_X(A)$ and $D(x) = \emptyset$ otherwise. Define $\varphi\colon D \to X\times Y$ by $\varphi(x, v) = (x, \varphi^x(v))$. Clearly, by 3.17, $D \in \mathscr{F}_0(X, \mathscr{N})$ and φ is an injective, stable, measurable and continuous mapping with image A.

This completes the proof.

We will now pass to the proof of a converse result, namely

THEOREM 3.43. *Let Y, Z be Polish. If $D \in \mathfrak{S}(X, Y)$ and $\varphi\colon D \to X\times Z$ is an injective, stable and measurable mapping, then $\varphi(D) \in \mathfrak{S}(X, Z)$.*

PROPOSITION 3.44. *Let Y by Polish and $(A_n)_n$ a sequence of mutually disjoint elements of $\mathscr{A}(X, Y)$. Then there is a sequence $(B_n)_n$ of mutually disjoint members of $\mathfrak{S}(X, Y)$ such that $A_n \subset B_n$ for all $n\in N$.*

Proof. Since A_m and A_n are disjoint for $m \neq n$, A_1 and $\bigcup_{n\geq 2} A_n$ are disjoint members of $\mathscr{A}(X, Y)$. By 3.27 we can find disjoint sets B_1 and C_1 in $\mathfrak{S}(X, Y)$ such that $A_1 \subset B_1$ and $\bigcup_{n\geq 2} A_n \subset C_1$. We can then separate similarly A_2 and $\bigcup_{n\geq 3} A_n$ by sets B_2 and C_2 in $\mathfrak{S}(X, Y)$ such that $B_2 \subset C_1$ and $C_2 \subset C_1$. Repeating this, we complete the proof.

Proof of 3.43. By 3.39 and 3.37, we may assume $Y = \mathscr{N}$. For every $c \in \mathscr{R}$, define $E_c = \varphi(D \cap (X\times\mathscr{N}_c))$, which is a member of $\mathscr{A}(X, Z)$ by 3.34. The scheme $(E_c)_{c\in\mathscr{R}}$ is regular and since φ is injective, $E_{c'} \cap E_{c''} = \emptyset$ if $|c'| = |c''|$ and $c' \neq c''$. Applying 3.44, we obtain a regular scheme $(B_c)_{c\in\mathscr{R}}$ on $\mathfrak{S}(X, Z)$ so that $E_c \subset B_c$ and $B_{c'} \cap B_{c''} = \emptyset$ if $|c'| = |c''|$ and $c' \neq c''$. For each $c\in\mathscr{R}$, let $C_c = \{(x, v, z) \in X\times\mathscr{N}\times Z;\ c < v \text{ and } (x, z) \in B_c\}$, which clearly belongs to $\mathfrak{S}(X, \mathscr{N}\times Z)$. Hence also $\Gamma^* = \bigcap_k \bigcup_{|c|=k} C_c$ is in $\mathfrak{S}(X, \mathscr{N}\times Z)$.

It is easily seen that $\Gamma(\varphi) \subset \Gamma^*$.

If $x\in X$, $z\in Z$, then $\Gamma^*(x, z) = \{v \in \mathscr{N};\ (x, z) \in \bigcap_{c<v} B_c\}$ and thus consists of at most one point of $\mathscr{N}$. Furthermore

$$\pi_{X\times Z}(\Gamma^*) = \bigcup_v \bigcap_{c<v} B_c = \bigcap_k \bigcup_{|c|=k} B_c$$

and therefore in $\mathfrak{S}(X, Z)$. Since $\Gamma(\varphi) \in \mathfrak{S}(X, \mathscr{N}\times Z)$ by 3.33, the set

$$\pi_{X\times Z}(\Gamma^*\setminus\Gamma(\varphi)) = \pi_{X\times Z}(\Gamma^*)\setminus\varphi(D)$$

is a member of $\mathscr{A}(X, Z)$. It follows that $(X\times Z)\setminus\varphi(D)$ belongs also to $\mathscr{A}(X, Z)$ and thus, by 3.34 and 3.28, $\varphi(D) \in \mathfrak{S}(X, Z)$.

An obvious corollary of 3.43 is Kuratowski's isomorphism theorem:

PROPOSITION 3.45. *If Y, Z are Polish, $D \in \mathscr{B}_Y$* and *$\varphi\colon D \to Z$ is injective and Borel measurable, then $\varphi(D) \in \mathscr{B}_Z$.*

For a slightly different proof of 3.45, the reader is referred to [16].

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Generalized Archimedean fields and logics with Malitz quantifiers

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Abstract. A characterization of Archimedean fields in a particular interpretation of the logic with Malitz quantifiers suggests a generalization of such fields. The theory of the real closed version of these generalized Archimedean fields in other interpretations of the Malitz quantifier is found to allow elimination of quantifiers.

The reader should be familiar with the model theory of first order logic. Some knowledge of ultrapowers, for example, is assumed. The notation is for the most part similar to that used in [1] or [2]. Gothic letters range over structures with the corresponding Latin letters denoting their universes: A denotes the universe of $\mathfrak{A}$, B_i denotes the universe of $\mathfrak{B}_i$, etc. Cardinals are initial von Neumann ordinals. Write Card(A) for the cardinality of A, $\boldsymbol{P}$ for the set of positive integers, $\boldsymbol{Q}$ for the set of rational numbers, and $\boldsymbol{R}$ for the set of real numbers.

Logics with Malitz quantifiers. For each positive integer n and each infinite cardinal $\varkappa$, the logic $\mathfrak{Q}_\varkappa^n$ is obtained by adding a new quantifier Q^n which binds n distinct variables and the following formation rule to those of first order logic: If φ is a formula and if the variables $x_1, \ldots, x_n$ are distinct, then $Q^n x_1, \ldots, x_n \varphi$ is also a formula. The logic $\mathfrak{Q}_\varkappa^{<\omega}$ is obtained from first order logic by adding all the quantifiers Q^n together with the corresponding formation rules.

The interpretation of the quantifier Q^n depends on the cardinal $\varkappa$:

$$\mathfrak{A} \models_\varkappa Q^n x_1, \ldots, x_n \varphi [\vec{a}]$$

just in case there is a subset I of A such that (i) Card$(I) = \varkappa$ and (ii) whenever $a_1, \ldots, a_n$ are distinct elements of I, then $\mathfrak{A} \models_\varkappa \varphi[a_1/x_1, \ldots, a_n/x_n, \vec{a}]$. Here the notation indicates how each of the variables $x_1, \ldots, x_n$ is to be interpreted and $\vec{a}$ is an interpretation of the free variables in $Q^n x_1, \ldots, x_n \varphi$.

The logic $\mathfrak{Q}_\varkappa^1$ coincides with the logic with the cardinal quantifier, "There exist $\varkappa$ many ...". For $n \geqslant 2$, the logics $\mathfrak{Q}_{\aleph_0}^n$ and $\mathfrak{Q}_{\aleph_0}^{<\omega}$ are referred to as logics with Ramsey quantifiers because of the similarity between their semantics and the well-known