Weak L-structures and dimension

by

Keiô Nagami (Matsuyama)

Abstract. A generalization of free L-structures is given and dimension theory is harmoniously developed for spaces admitting such weakened structures. For instance, such a space X has $\dim X \leqslant n$ if and only if X admits a σ-closure-preserving base $\mathfrak{B}$ with $\dim \partial B \leqslant n-1$ for each $B \in \mathfrak{B}$.

0. Introduction. In a previous paper [5] we study the dimension of spaces with free L-structures. The aim of this paper is to weaken the concept of free L-structures and to show that the dimension theory can still harmoniously be developed. Since I do not know whether the class of spaces with such weakened structures is a real generalization of the class of free L-spaces, I first hesitated the publication of such structures. I dare to publish hoping that the concept of weak L-structures would be one of the key words to clarify the very interesting but still wild plain between Lašnev spaces and M_i-spaces [1].

In this paper all spaces are assumed to be Hausdorff topological spaces, maps to be continuous onto, and images to be those under maps. The letter N denotes the positive integers. For undefined terminology and notation refer to [4] and [5].

1. Definition of weak L-structures.

1.1. Definition. Let X be a space, F a closed set of X, and $\mathcal{U}$ an anti-cover of F. An open neighborhood V of F is said to be *semi-canonical* if

$$F \cap \operatorname{Cl}\mathcal{U}(X-V) = \emptyset .$$

1.2. Definition. A pair $(\mathcal{F}, \{\mathcal{U}_F\colon F \in \mathcal{F}\})$ of a σ-locally finite closed cover $\mathcal{F}$ of a space X and anticover $\mathcal{U}_F$ of $F \in \mathcal{F}$ is said to be a *weak L-structure* if it satisfies the condition: For each point $x \in X$ and each neighborhood U of x there exist a finite subcollection $\{F_1, \ldots, F_k\}$ of $\mathcal{F}$ and semi-canonical neighborhoods U_i of F_i such that $x \in \bigcap_{i=1}^{k} F_i \subset \bigcap_{i=1}^{k} U_i \subset U$. A paracompact space admitting a weak L-structure is said to be a *weak L-space*.

The following is justified quite analogously to [5], Theorem 1.3.

1.3. Theorem. *To admit a weak L-structure is hereditary and countably productive property.*

2. Main theorem.

2.1. LEMMA *Let X be a semi-stratifiable normal space and $\{F_\alpha\colon \alpha\in A\}$ a closure-preserving closed cover of X with* $\dim F_\alpha\leqslant n$ *for each $\alpha\in A$. Then* $\dim X\leqslant n$.

Proof. Well order A. Set

$$\mathscr{H}_k=\{H_{k\alpha}=F_\alpha\cap(X-\bigcup\{F_\beta\colon \beta<\alpha\})_k\colon \alpha\in A\},$$

where k succeeding the parentheses indicates the kth index of semi-stratification. Then $\mathscr{H}_k$ is discrete and $\bigcup_{k=1}^{\infty}\mathscr{H}_k$ covers X. Thus $\dim X\leqslant n$ by the sum theorem. That completes the proof.

2.2. LEMMA. *Let $F_1,\ldots,F_k$ be closed sets of a space X and $\mathscr{U}_i=\{U_\alpha\colon \alpha\in A_i\}$ be their anti-covers which are locally finite* $\bmod F_i$ *(i.e. locally finite in $X-F_i$). Set*

$$A_i=\{\lambda\subset A_i\colon V_\lambda=F_i\cup(\bigcup\{U_\alpha\colon \alpha\in\lambda\})\ \text{is open}\},$$
$$\mathscr{V}_i=\{V_\lambda\colon \lambda\in A_i\},$$
$$\mathscr{W}=\{W(\lambda_1,\ldots,\lambda_k)=\bigcap_{i=1}^{k}V_{\lambda_i}\colon (\lambda_i)\in\prod A_i\}.$$

Then $\mathscr{W}$ is closure-preserving $\bmod F=\bigcap_{i=1}^{k}F_i$ *and, for some closed sequence $\{H_i\}$ with $\bigcup H_i=X-F$, the restriction of $\partial\mathscr{W}$ to each H_i is closure-preserving.*

Proof. Choose an arbitrary index set $\Xi\subset\prod A_i$. Set

$$W=\bigcup\{W(\xi)\colon \xi\in\Xi\}$$

and pick a point $x\in\partial W$. Set

$$M=\{1,\ldots,k\},$$
$$M'=\{i\in M\colon x\notin F_i\},$$
$$M''=\{i\in M\colon x\in F_i\}.$$

Let D be an open neighborhood of x with

$$D\cap(\bigcup\{F_i\colon i\in M'\})=\emptyset$$

such that

$$B_i=\{\alpha\in A_i\colon D\cap U_\alpha\neq\emptyset\}$$

is finite for each $i\in M'$. Set

$$\mathscr{U}'_i=\{U_\alpha\colon \alpha\in B_i\},\qquad i\in M',$$
$$\mathscr{U}=\bigwedge\{\mathscr{U}'_i\colon i\in M'\}.$$

Then $\bigwedge\{\mathscr{U}_i\colon i\in M'\}|\ D=\mathscr{U}|D$. Set

$$W'(\lambda_1,\ldots,\lambda_k)=\bigcap\{V_{\lambda_i}\colon i\in M'\},$$
$$W''(\lambda_1,\ldots,\lambda_k)=\bigcap\{V_{\lambda_i}\colon i\in M''\},$$
$$W'=\bigcup\{W'(\xi)\colon \xi\in\Xi\}.$$

Assume that $x\in W'(\xi)$ for some $\xi\in\Xi$. Then $x\in W'(\xi)\cap(\bigcap\{F_i\colon i\in M''\})\subset W'(\xi)\cap W''(\xi)=W(\xi)$. This contradiction implies that $x\notin W'(\xi)$ for any $\xi\in\Xi$ and hence that $x\notin W'$. Let

$$\{\mathscr{T}_\gamma\colon \gamma\in\Gamma\}$$

be the family of all subcollection of $\mathscr{U}$. For each $\xi\in\Xi$, $W'(\xi)\cap D=\mathscr{T}_\gamma^{\#}\cap D$ for some $\gamma\in\Gamma$. Thus there exists an index $\delta\in\Gamma$ with $W'\cap D=\mathscr{T}_\delta^{\#}\cap D$. Since $x\notin W'$ and $x\in\partial W$, then $x\in\partial W'$. Since Γ is finite, there exists an index $\eta\in\Xi$ with $x\in\partial W'(\eta)$. Pick an arbitrary neighborhood E of x. Then $W''(\eta)\cap E$ is again a neighborhood of x. Thus $W(\eta)\cap E=W'(\eta)\cap(W''(\eta)\cap E)\neq\emptyset$, which implies that $x\in\partial W(\eta)$. $\mathscr{W}$ is therefore closure-preserving.

To prove the rest let M be the collection of all non-empty subsets of $\{1,\ldots,k\}$. Set $T_a=\bigcap\{F_i\colon i\notin a\}-\bigcup\{F_i\colon i\in a\}$, $a\in M$. Then $X-F=\bigcup\{T_a\colon a\in M\}$. The restriction of $\partial\mathscr{W}$ to each T_a is closure-preserving and each T_a is F_σ. That completes the proof.

This argument proves essentially the following.

2.3. LEMMA. *Assume the same as in the preceding lemma. Then for each $\xi=(\lambda_1,\ldots,\lambda_k)\in\prod A_i$ and each point x of $\partial W(\xi)$, there exist i and $\alpha\in\lambda_i$ such that $x\in\partial U_\alpha$.*

2.4. THEOREM. *For a weak L-space X the following conditions are equivalent.*

(1) $\dim X\leqslant n$.

(2) *X is the image of a weak L-space Z with* $\dim Z\leqslant 0$ *under a closed map f with order* $f\leqslant n+1$.

(3) *X is the sum of $n+1$ subsets Z_i, $i=1,\ldots,n+1$, with* $\dim Z_i\leqslant 0$ *for each i.*

(4) $\operatorname{Ind} X\leqslant n$.

(5) *X has a σ-closure-preserving base $\mathscr{B}$ with* $\dim(\partial\mathscr{B})^{\#}\leqslant n-1$ *such that* $(\partial\mathscr{B})^{\#}$ *is F_σ.*

(6) *X has a σ-closure-preserving base $\mathscr{B}$ with* $\dim\partial B\leqslant n-1$ *for each $B\in\mathscr{B}$.*

(7) *X admits a stratification $U\to\{U_i\}$ such that* $\dim\partial U_i\leqslant n-1$ *for each open U and each $i\in N$.*

Proof. To prove that (1) implies (2) let $(\mathscr{F},\{\mathscr{U}_F\})$ be a weak L-structure of X. Let $\mathscr{F}=\bigcup_{i=1}^{\infty}\mathscr{F}_i$, $\mathscr{F}_i=\{F_\alpha\colon \alpha\in A_i\}$, $A_1\subset A_2\subset\ldots$, where each $\mathscr{F}_i$ is locally finite. For simplicity $\mathscr{U}_\alpha$ stands for $\mathscr{U}_{F_\alpha}$ in the sequel. Set

$$B_{ij}=\{\xi\subset A_i\colon |\xi|=j\},$$
$$\mathscr{F}_{ij}=\{F_{ij}(\xi)=\bigcap\{F_\alpha\colon \alpha\in\xi\}\colon \xi\in B_{ij}\}.$$

Well order B_{ij}. As in Lemma 2.1 define discrete closed collections

$$\mathscr{H}_{ijk}=\{H_{ijk}(\xi)\colon \xi\in B_{ij}\}$$

in the following manner.

$$H_{ijk}\ (\text{the first}) = F_{ij}\ (\text{the first}),$$
$$H_{ijk}(\xi) = F_{ij}(\xi) \cap \left(X - \bigcup\{F_{ij}(\eta):\ \eta<\xi\}\right)_k,$$

where the index k in the right hand side denotes the kth index of semi-stratification. Let $D_{ijt}(\xi)$, $t = 1, 2, 3$, be open neighborhoods of $H_{ijk}(\xi)$ such that

$$D_{ijkt}(\xi) \supset \mathrm{Cl}\, D_{ijkt+1}(\xi)$$

and

$$\mathcal{D}_{ijk} = \{D_{ijkl}(\xi):\ \xi \in B_{ij}\}$$

is discrete. Set

$$\mathcal{E}_{ijk} = \{X - \bigcup\{\mathrm{Cl}\, D_{ijk2}(\xi'):\ \xi' \in B_{ij}\}\} \cup$$
$$\cup \{D_{ijkl}(\xi) - \mathrm{Cl}\, D_{ijk3}(\xi),\ D_{ijk2}(\xi) - H_{ijk}(\xi):\ \xi \in B_{ij}\}.$$

For each i, j, k and $\alpha \in A_i$ let $\mathcal{V}_{ijk\alpha}$ be an anti-cover of F_α such that (i) $\mathcal{V}_{ijk\alpha} < \mathcal{E}_{ijk}|(X-F_\alpha)$, (ii) $\mathrm{Cl}\,\mathcal{V}_{ijk\alpha} < \mathcal{U}_\alpha$, (iii) $\mathcal{V}_{ijk\alpha}$ is locally finite $\mathrm{mod}\,F_\alpha$ and σ-discrete in X. Set

$$\mathcal{V}_{ijk\alpha} = \bigcup_{t=1}^{\infty} \mathcal{V}_{ijk\alpha t},$$
$$\mathcal{V}_{ijk\alpha t} = \{V_{ijk\alpha t}(\lambda):\ \lambda \in \Lambda_{ijk\alpha t}\},$$

where each $\mathcal{V}_{ijk\alpha t}$ is discrete in X. Let $\{K_{ijk\alpha t}(\lambda):\ \lambda \in \Lambda_{ijk\alpha t}\}$ be a closed cover of $X - F_\alpha$ such that

$$K_{ijk\alpha t}(\lambda) \subset V_{ijk\alpha t}(\lambda), \qquad \lambda \in \Lambda_{ijk\alpha t}.$$

Set

$$\Lambda_{ijk\alpha t\xi} = \{\lambda \in \Lambda_{ijk\alpha t}:\ V_{ijk\alpha t}(\lambda) \subset D_{ijkl}(\xi)\}, \qquad \xi \in B_{ij},$$
$$V_{ijk\alpha t\xi} = \bigcup\{V_{ijk\alpha t}(\lambda):\ \lambda \in \Lambda_{ijk\alpha t\xi}\}, \qquad \xi \in B_{ij},$$
$$K_{ijk\alpha t\xi} = \bigcup\{K_{ijk\alpha t}(\lambda):\ \lambda \in \Lambda_{ijk\alpha t\xi}\}, \qquad \xi \in B_{ij}.$$

Set

$$\xi = \{\alpha_1(\xi), \ldots, \alpha_j(\xi)\}, \qquad \xi \in B_{ij},$$
$$V_{ijk[u]t\cdot} = \bigcup\{V_{ijk\alpha_u(\xi)t\xi}:\ \xi \in B_{ij}\},$$
$$K_{ijk[u]t\cdot} = \bigcup\{K_{ijk\alpha_u(\xi)t\xi}:\ \xi \in B_{ij}\},$$
$$D_{ijkt} = \bigcup\{D_{ijkt}(\xi):\ \xi \in B_{ij}\}, \qquad t = 1, 2, 3.$$

Then $V_{ijk[u]t\cdot}$ is open and $K_{ijk[u]t\cdot}$ is closed.

Let $\varrho:\ X \to \varrho X$ be a contraction of X to a metric space ϱX with $\dim \varrho X \leqslant \dim X$ such that

$$\varrho(X - V_{ijk[u]t\cdot}), \quad \varrho(X - D_{ijkt}), \quad \varrho K_{ijk[u]t\cdot}, \quad \varrho \bar{D}_{ijkt}$$

are closed in ϱX for each i, j, k, t, u. Consider the diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & X \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\varrho} \\ \sigma Z & \xrightarrow[\ g\ ]{} & \varrho X \end{array}$$

where σZ is a metric space with $\dim \sigma Z \leqslant 0$, g is a closed map with order $g \leqslant n+1$, Z is the pullback with the natural maps σ and f. Then f is a closed map with order $f \leqslant n+1$. It is almost obvious that Z is a weak L-space. The only one thing to be proved is the assertion: $\dim Z \leqslant 0$. Let

$$\mathcal{G} = \bigcup_{i=1}^{\infty} \mathcal{G}_i, \qquad \mathcal{G}_i = \{G_{i\gamma}:\ \gamma \in \Gamma_i\},$$

be a base of σZ such that $\mathcal{G}_i > \mathcal{G}_{i+1}$, mesh $\mathcal{G}_i \leqslant 1/i$ and each $\mathcal{G}_i$ is a disjoint open cover of Z. Set

$$S_\delta = f^{-1}(H_{ijk}(\xi)) \cap \sigma^{-1}(G_{s\gamma}), \qquad \xi \in B_{ij},\ \gamma \in \Gamma_s,$$
$$T_\delta = f^{-1}(D_{ijkl}(\xi)) \cap \sigma^{-1}(G_{s\gamma}), \qquad \xi \in B_{ij},\ \gamma \in \Gamma_s,$$

where $\delta = \delta(i, j, k, \xi, s, \gamma)$.

Set

$$\mathcal{S} = \bigcup\{f^{-1}(\mathcal{H}_{ijk}) \wedge \sigma^{-1}(\mathcal{G}_s):\ i, j, k, s \in N\} = \{S_\delta:\ \delta \in \Delta\},$$
$$\mathcal{T} = \bigcup\{f^{-1}(\mathcal{D}_{ijk}) \wedge \sigma^{-1}(\mathcal{G}_s):\ i, j, k, s \in N\} = \{T_\delta:\ \delta \in \Delta\}.$$

Then $\mathcal{S}$ and $\mathcal{T}$ are σ-discrete and $S_\delta \subset T_\delta$ for each $\delta \in \Delta$.

To prove that $\dim Z \leqslant 0$ let $\mathcal{W}$ be an arbitrary binary open cover of Z. Set

$$\Delta' = \{\delta \in \Delta:\ S_\delta \subset W_\delta \subset T_\delta,\ W_\delta < \mathcal{W} \text{ for some clopen set } W_\delta\}.$$

To prove $\dim Z \leqslant 0$ it suffices to prove $\bigcup\{S_\delta:\ \delta \in \Delta'\} = Z$, since this equality implies that $\mathcal{W}$ admits a σ-discrete refinement $\{W_\delta:\ \delta \in \Delta'\}$ each element of which are clopen.

Pick an arbitrary point $z \in Z$. Then

$$f^{-1}(E) \cap \sigma^{-1}(G_{s\gamma}) < \mathcal{W}$$

for some open neighborhood E of $f(z)$, some s, and some $\gamma \in \Gamma_s$. Choose a finite subset ξ of A and semi-canonical neighborhoods U_α, $\alpha \in \xi$, of F_α such that

$$f(z) \in \bigcap_{\alpha \in \xi} F_\alpha \subset \bigcap_{\alpha \in \xi} U_\alpha \subset E.$$

Let i be the minimum with $\xi \subset A_i$. Set

$$\eta = \{\alpha \in A_i:\ f(z) \in F_\alpha\}, \qquad |\eta| = j,$$
$$\alpha_u(\eta) = \beta_u, \qquad u = 1, \ldots, j.$$

Then $\xi \subset \eta$ and $\eta \in B_{ij}$. Since $f(z) \in F_{ij}(\eta)$ and $f(z) \notin F_{ij}(\eta')$ for any $\eta' \in B_{ij}$ with $\eta' \neq \eta$, then

$$f(z) \in F_{ij}(\eta) \cap \big(X - \bigcup \{F_{ij}(\eta'):\ \eta' < \eta\}\big).$$

Thus there exists an index k with

$$f(z) \in F_{ij}(\eta) \cap \big(X - \bigcup \{F_{ij}(\eta'):\ \eta' < \eta\}\big)_k = H_{ijk}(\eta).$$

Since $\dim \sigma Z \leqslant 0$ and the diagram is commutative, there exist clopen sets $R_{ijk \cdot t \cdot u}$ and M_{ijk} of Z such that

$$f^{-1}(K_{ijk[u]t \cdot}) \subset R_{ijk \cdot t \cdot u} \subset f^{-1}(V_{ijk[u]t \cdot}), \qquad t, u \in N,$$
$$f^{-1}(\bar{D}_{ijk3}) \subset M_{ijk} \subset f^{-1}(D_{ijk2}).$$

Set

$$R_{ijk\beta_u t\eta}(\lambda) = f^{-1}(V_{ijk\beta_u t}(\lambda)) \cap R_{ijk[u]t \cdot},$$
$$\lambda \in \Lambda_{ijk\beta_u t\eta}, \qquad u = 1, \ldots, j,$$
$$M_{ijk\eta} = f^{-1}(D_{ijk2}(\eta)) \cap M_{ijk}.$$

Set

$$\mathscr{L}_{ijk\beta_u} = f^{-1}\{V_{ijk\beta_u t}(\lambda):\ \lambda \in \Lambda_{ijk\beta_u t} - \Lambda_{ijk\beta_u t\eta},\ t \in N\},$$
$$\mathscr{M}_{ijk\beta_u} = \{R_{ijk\beta_u t\eta}(\lambda):\ \lambda \in \Lambda_{ijk\beta_u t\eta}:\ t \in N\},$$
$$\mathscr{W}_{ijk\beta_u} = \mathscr{L}_{ijk\beta_u} \cup \mathscr{M}_{ijk\beta_u}.$$

Then $\mathscr{W}_{ijk\beta_u}$ is an anti-cover of $f^{-1}(F_{\beta_u})$ refining $f^{-1}(\mathscr{V}_{ijk\beta_u})$. Set

$$P_u = \big(\bigcup \{P \in \mathscr{W}_{ijk\beta_u}:\ P \subset f^{-1}(U_{\beta_u})\}\big) \cup f^{-1}(F_{\beta_u}),$$
$$Q_u = \big(\bigcup \{Q \in \mathscr{M}_{ijk\beta_u}:\ Q \subset f^{-1}(U_{\beta_u})\}\big) \cup f^{-1}(F_{\beta_u}).$$

Then P_u is an open neighborhood of $f^{-1}(F_{\beta_u})$, since $f^{-1}(U_{\beta_u})$ is a semi-canonical neighborhood of $f^{-1}(F_{\beta_u})$ with respect to $f^{-1}(\mathscr{U}_{\beta_u})$ and hence with respect to $\mathscr{W}_{ijk\beta_u}$. Set

$$Q = \bigcap_{u=1}^{j} P_u.$$

Then Q is an open neighborhood of z. Since $\bigcap_{\alpha \in \xi} U_\alpha \subset E$ and $\xi \subset \eta = \{\beta_1, \ldots, \beta_j\}$ then

$$Q \subset \bigcap_{\alpha \in \xi} f^{-1}(U_\alpha) \subset f^{-1}(E).$$

Set

$$\delta_0 = \delta(i, j, k, \eta, s, \gamma),$$
$$W_{\delta_0} = Q \cap M_{ijk\eta} \cap G_{s\gamma}.$$

Then

$$S_{\delta_0} = f^{-1}(H_{ijk}(\eta)) \cap G_{s\gamma} \subset W_{\delta_0} \subset f^{-1}(D_{ijk1}(\eta)) \cap G_{s\gamma} = T_{\delta_0}.$$

Since $M_{ijk\eta}$ and $G_{s\gamma}$ are clopen,

$$\partial W_{\delta_0} = \partial Q \cap M_{ijk\eta} \cap G_{s\gamma}.$$

Assume that ∂W_{δ_0} contains a point p. Since each element of $\bigcup_{u=1}^{j} \mathscr{L}_{ijk\beta_u}$ cannot meet $M_{ijk\eta}$, then by Lemma 2.3 some element R of $\bigcup_{u=1}^{j} \mathscr{M}_{ijk\beta_u}$ contains p, which contradicts to $\partial R = \emptyset$. Thus $\partial W_{\beta_0} = \emptyset$ and $\delta_0 \in \Delta'$.

After the implication (1) $\to$ (2) has proved, the conditions (1), (2), (3), (4) can be proved to be equivalent as in [5], Theorem 2.3.

Let us show that the relation $\dim X = \operatorname{Ind} X \leqslant n$ implies (5) using the same notation as in the above. Choose $i, j, k, \xi = \{\alpha_1, \ldots, \alpha_j\} \in B_{ij}$. Let $E_{ijk}(\xi)$ be an open set with

$$H_{ijk}(\xi) \subset E_{ijk}(\xi) \subset D_{ijk3}(\xi), \qquad \dim \partial E_{ijk}(\xi) \leqslant n-1.$$

Set

$$\mathscr{V}_{ijk\alpha_u} = \{V_{ijk\alpha_u}(\lambda):\ \lambda \in \Lambda_{ijk\alpha_u}\}.$$

Let $\{J_{ijk\alpha_u}(\lambda):\ \lambda \in \Lambda_{ijk\alpha_u}\}$ be an anti-cover of F_{α_u} such that

$$\operatorname{Cl} J_{ijk\alpha_u}(\lambda) \subset V_{ijk\alpha_u}(\lambda),$$
$$\dim \partial J_{ijk\alpha_u}(\lambda) \leqslant n-1, \qquad \lambda \in \Lambda_{ijk\alpha_u}.$$

Set

$$\Theta_{ijk\alpha_u} = \{\theta \subset \Lambda_{ijk\alpha_u}:\ B_\theta = F_{\alpha_u} \cup \big(\bigcup \{J_{ijk\alpha_u}(\lambda):\ \lambda \in \theta\}\big) \text{ is open}\},$$
$$\mathscr{B}_{ijk\alpha_u} = \{B_\theta:\ \theta \in \Theta_{ijk\alpha_u}\},$$
$$\mathscr{B}'_{ijk}(\xi) = \bigwedge_{u=1}^{j} \mathscr{B}_{ijk\alpha_u},$$
$$\mathscr{B}_{ijk}(\xi) = \mathscr{B}'_{ijk}(\xi) | J_{ijk}(\xi).$$

Then $\mathscr{B}_{ijk}(\xi)$ is closure-preserving by Lemma 2.2. Set

$$\mathscr{B} = \bigcup \{\mathscr{B}_{ijk}(\xi):\ \xi \in B_{ij},\ i, j, k \in N\}.$$

Since $\{\mathscr{B}_{ijk}(\xi):\ \xi \in B_{ij}\}$ is closure-preserving, $\mathscr{B}$ is a σ-closure-preserving base. Let

$$B = B_{\theta_1} \cap \ldots \cap B_{\theta_j} \cap J_{ijk}(\xi), \qquad \theta_u \in \Theta_{ijk\alpha_u},\ u = 1, \ldots, j,$$

be a generic element of $\mathscr{B}_{ijk}(\xi)$. Since $\dim \partial B_{\theta_u} \leqslant n-1$, $u = 1, \ldots, j$, by the locally finite sum theorem, then $\dim \partial B \leqslant n-1$.

To prove that $(\partial \mathscr{B})^*$ is an F_σ-set with $\dim (\partial \mathscr{B})^* \leqslant n-1$ it suffices to prove that $(\partial \mathscr{B}_{ijk}(\xi))^*$ is an F_σ-set with $\dim (\partial \mathscr{B}_{ijk}(\xi))^* \leqslant n-1$ for each i, j, k and $\xi \in B_{ij}$. Let Φ be the family of all subsets of ξ. Set

$$C_\emptyset = X - \bigcup \{F_\alpha:\ \alpha \in \xi\},$$
$$C_\eta = \bigcap \{F_\alpha:\ \alpha \in \eta\} - \bigcup \{F_\alpha:\ \alpha \in \xi - \eta\}, \qquad \eta \in \Phi.$$

Since each C_η is F_σ, we can set

$$C_\eta = \bigcup_{t=1}^{\infty} C_{\eta t}, \quad \eta \in \Phi,$$

where each $C_{\eta t}$ is a closed set. As we recognized in Lemma 2.2 each point $x \in C_{\eta t}$ has a neighborhood U_x such that

$$U_x \cap (\partial \mathscr{B}_{ijk}(\xi)|C_{\eta t})^{\#} = U_x \cap (\partial \mathscr{B}'|C_{\eta t})$$

for some finite subcollection $\mathscr{B}'$ of $\mathscr{B}_{ijk}(\xi)$. This fact implies that $(\partial \mathscr{B}_{ijk}(\xi))^{\#} \cap C_{\eta t}$ is closed and that

$$\dim((\partial \mathscr{B}_{ijk}(\xi))^{\#} \cap C_{\eta t}) \leqslant n-1 .$$

Since $\{C_{\eta t}\colon \eta \in \Phi, t \in N\}$ is a countable closed cover of X, then $(\partial \mathscr{B}_{ijk}(\xi))^{\#}$ is an F_σ-set with $\dim(\partial \mathscr{B}_{ijk}(\xi))^{\#} \leqslant n-1$.

The implication $(5) \rightarrow (6)$ is evident.

To prove that (6) implies (7) let $\mathscr{B} = \bigcup \mathscr{B}_i$, with each $\mathscr{B}_i$ closure-preserving, be a base such that $\dim \partial B \leqslant n-1$ for each $B \in \mathscr{B}$. For each open set U set

$$U_i = \bigcup \{B \in \mathscr{B}_i\colon \bar{B} \subset U\} .$$

Then $\{U_i\}$ gives a stratification. Set

$$U_i = \bigcup \{B_\alpha \in \mathscr{B}_i\colon \alpha \in A\} .$$

Then $\{\partial U_i \cap \bar{B}_\alpha\colon \alpha \in A\}$ is a closure-preserving closed cover of ∂U_i. Since $\partial U_i \cap \bar{B}_\alpha = \partial U_i \cap \partial B_\alpha$, then $\dim(\partial U_i \cap \bar{B}_\alpha) \leqslant \dim(\partial U_i \cap \partial B_\alpha) \leqslant \dim \partial B_\alpha \leqslant n-1$. Thus $\dim \partial U_i \leqslant n-1$ by Lemma 2.1.

That (7) implies (1) is a well known fact (cf. [3], Theorem 11.12). The proof is thus completed.

3. Corollaries.

3.1. COROLLARY. *Let X and Y be weak L-spaces at least one of which is non-empty. Then* $\dim(X \times Y) \leqslant \dim X + \dim Y$.

Proof. Let $\mathscr{B}$ and $\mathscr{B}'$ be respectively σ-closure-preserving bases of X and of Y with $\dim \partial B \leqslant \dim X - 1$ for each $B \in \mathscr{B}$ and with $\dim \partial B' \leqslant \dim Y - 1$ for each $B' \in \mathscr{B}'$. Then $\mathscr{B} \times \mathscr{B}'$ is a σ-closure-preserving base of $X \times Y$. If $B \in \mathscr{B}$ and $B' \in \mathscr{B}'$, then

$$\partial(B \times B') \subset (\partial B \times \bar{B}') \cup (\bar{B} \times \partial B') .$$

Thus

$$\dim \partial(B \times B') \leqslant \dim X + \dim Y - 1$$

if we pose an induction hypothesis on $\dim X + \dim Y$. That completes the proof, since the case when $\dim X + \dim Y = -1$ assures that the product theorem is trivially true.

3.2. COROLLARY. *Let X be a weak L-space with $\dim X = n > 1$. Let k be an arbitrary integer with $0 < k < n$. Then there exists a weak L-space Y with $\dim Y = k$ such that X is the image of Y under a closed map f with* $\operatorname{order} f = n-k+1$.

Proof. Consider the diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{f'} & Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\varrho} \\ \sigma Z & \xrightarrow[g']{} & \tau Y & \xrightarrow[g]{} & \varrho X \end{array}$$

where ϱ and ϱX are the same as was constructed in Theorem 2.4. Then $\dim \varrho X = \dim X$. τY and g are the same as was constructed by the author [2], § 6, such that $\dim \tau Y = k$ and g is a closed map with order $g = n-k+1$. Y is the pull-back of the right hand half. Then Y is a weak L-space and f is a closed map with order $f = n-k+1$. Let τY be the image of a metric space σZ with $\dim \sigma Z = 0$ under a closed map g' with order $g' = k+1$. Z is the pull-back of the left hand half. Then Z is a weak L-space and f' is a closed map with order $f' = k+1$.

By an analogous argument as in Theorem 2.4, we can prove $\dim Z = 0$ which implies that $\dim Y \leqslant k$. On the other hand

$$n = \dim X \leqslant \dim Y + \operatorname{order} f - 1 = \dim Y + n - k$$

and hence $\dim Y \geqslant k$. Thus $\dim Y = k$. That completes the proof.

3.3. COROLLARY. *Let X be a weak L-space and Y be a subset of X with $\dim Y \leqslant n$. Then we get the following.*

(1) *There exists a σ-closure-preserving base $\mathscr{B}$ of X such that $(\partial \mathscr{B})^{\#}$ is F_σ and $\dim \partial(B \cap Y) \leqslant n-1$ for each $B \in \mathscr{B}$.*

(2) *There exists a G_δ-set Y' with $Y \subset Y'$ and with $\dim Y' \leqslant n$.*

Proof. (1) Let Y_i, $i = 1, \ldots, n+1$, be subsets of Y with $\dim Y_i \leqslant 0$. If we replace the conditions $\dim \partial J_{ijk\alpha u}(\lambda) \leqslant n-1$ and $\dim \partial E_{ijk}(\xi) \leqslant n-1$ in the proof of Theorem 2.4 with $\partial J_{ijk\alpha u}(\lambda) \cap Y_1 = \emptyset$ and $\partial E_{ijk}(\xi) \cap Y_1 = \emptyset$ respectively, we can obtain the desired base $\mathscr{B}_1$ such that $(\partial \mathscr{B}_1)^{\#} \cap Y_1 = \emptyset$.

(2) For each $i = 1, \ldots, n+1$ let $\mathscr{B}_i$ be a σ-closure-preserving base such that $(\partial \mathscr{B}_i)^{\#} \cap Y_i = \emptyset$. Set

$$Y_i' = X - (\partial \mathscr{B}_i)^{\#}, \quad Y' = \bigcup_{i=1}^{n+1} Y_i' .$$

Then Y_i' is a G_δ-set with $\dim Y_i' \leqslant 0$ and with $Y_i \subset Y_i'$. Y' is therefore the desired. That completes the proof.

3.4. COROLLARY. *Each weak L-space is the image of a weak L-space Z with $\dim Z \leqslant 0$ under a perfect map.*

This can be verified analogously to Theorem 2.4. The following two are also essentially proved in Theorem 2.4.

3.5. Corollary. *Each weak L-space X admits a countable family of disjoint pairs of closed sets determining the dimension of all subsets of X.*

3.6. Corollary. *Each weak L-space admits a σ-closure-preserving base $\mathscr{B}$ such that $(\partial\mathscr{B})^*$ is F_σ and such that for each subset S of X $\mathscr{B}|S$ is also a σ-closure-preserving base of S.*

References

[1] J. G. Ceder, *Some generalizations of a metric space*, Pacific J. Math. 11 (1961), pp. 105–126.
[2] K. Nagami, *Mappings of finite order and dimension theory*, Japan. J. Math. 30 (1960), pp. 25–54.
[3] — *Dimension Theory*, Academic Press, New York 1970.
[4] — *The equality of dimensions*, Fund. Math. 106 (1980), pp. 239–246.
[5] — *Dimension of free L-spaces*, Fund. Math. 108 (1980), pp. 211–224.

Accepté par la Rédaction 23. 4. 1979

Errata to the paper "Remarks of the elementary theories of formal and convergent power series" Fundamenta Mathematicae 105 (1980), pp. 229–239

by

Joseph Becker and **Leonard Lipshitz** (West Lafayette, Ind.)

Page	For	Insert
229^2 abstract	$\prec$	$\not\prec$
234^{12}	$\prec$	$\not\prec$