$\varrho(A_2, A_1) < \frac{1}{3} r_1$. Désignons par r_2 la distance du point A_2 à l'ensemble fermé Q_3 (fermé dans le cercle S_1) et remarquons que le cercle ouvert $S_2 = S(A_2, r_2) \subset Q_2 \cap S_1$. Mais, il existe dans la frontière du cercle S_2 un point $A_3 \in Q_1$, en contradiction, d'après le lemme 1, avec la propriété (P_2) de la fonction f.

De la même façon on obtient le théorème suivant:

THÉORÈME 3. *Si une fonction mesurable $f: R^2 \to R$ a la propriété* (P_3), *elle a également la propriété de Darboux.*

Remarque. Il existe une fonction $f: R^2 \to R$ approximativement continue (donc ayant les propriétés (P_2) et (P_3)) qui n'est pas connexe, c'est-à-dire telle qu'il existe un ensemble connexe $P \subset R^2$ pour lequel l'image $f(P)$ n'est pas connexe.

Travaux cités

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Non standard interpretations of higher order theories

by

Paweł Zbierski (Warszawa)

Abstract. We prove the existence of some types of nonstandard interpretations of higher order arithmetic (or higher order set theory) in itself.

Section 0. Classifying interpretations. Interpreting a theory in itself or in another theory is one of most basic tools in foundational research. There is a number of papers devoted to general theory of interpretations. In Szczerba [8] and Pinter [7] it is proved that some logical and model theoretical notions are preserved under interpretations satisfying suitable conditions. In Szczerba and Setti [9], we find an algebraic characterization for a functor from the class of models of T_1 to the class of models of T_2 to be determined by an interpretation of T_2 in T_1. Here we turn our attention to the problem of classification of interpretations according to semantical notions preserved. We do this in the case of nth order arithmetic A_n or some consistent extensions of A_n. Similar results hold in the case of nth order set theory M_n (see Marek and Zbierski [4]). We distinguish the following classes of interpretations:

(1) β-interpretations or standard interpretations. These are interpretations preserving well-orderings. More precisely, for an interpretation I let p^I denote the formula corresponding to p under I. Then I is a β-interpretation if the formula

$$I(x) \mathbin{\&} \mathrm{Bord}^I(x) \to \mathrm{Bord}(x)$$

is provable (in the theory in which we interpret), where $\mathrm{Bord}(x)$ stands for "x is a well-ordering".

(2) k-β-(or k-standard) interpretations. We assume $1 \leqslant k < n$ and then the interpretations are those which preserve well-orderings up to the kth type, but for some x of type $k+1$ we have

$$\vdash I(x) \mathbin{\&} \mathrm{Bord}^I(x) \mathbin{\&} \neg\mathrm{Bord}(x)\,.$$

Standard (k-standard) interpretations preserve natural numbers. Hence

(3) Non ω-interpretations (or ω-nonstandard interpretations). Those under which the natural numbers are (provably) nonstandard.

Examples of β-interpretations are constructible sets and the ramified analytic hierarchy. They were extensively studied in Gandy [1], Marek [2], Marek and Mostowski [3], Vetulani [10], Zbierski [11].

In Zbierski [12] we prove the existence of a standard interpretation of the second order arithmetic (with choice) in arithmetic without choice. As a corollary we derive that the system of second order arithmetic with ω-rule (or β-rule) is not finitely axiomatizable.

In this paper we prove the existence of interpretations of classes (2) and (3). In Section 1 we briefly describe the system A_n and related notions; in Section 2 we recall the interpretation by trees. In Section 3 we prove the existence of an internal ω-nonstandard interpretation of A_2 plus the axiom of constructibility. Finally, in Section 4, we prove that for each $1 \leqslant k < n$ there is a k-standard interpretation of A_n in A_n plus there is a β-model of A_n. As was remarked earlier, all these results hold with A_n replaced by M_n.

There are, of course, other classifications of interpretations. For instance, if K is a class of formulas, then I is called a *K-interpretation* if all formulas in K are absolute in I. It is known that β-interpretations of A_2 coincide with Π^1_1-interpretations (see Mostowski [5]). We shall not pursue this matter here.

Section 1. The system A_n. We briefly describe the theory A_n and related notions. More details about A_n or M_n can be found in Marek and Zbierski [4] and Zbierski [11].

The first order language of A_n contains unary predicates $S_1, \ldots, S_n$, for types ($S_1(x)$ stands for "x is a natural number" and $S_2(x)$ for "x is a set of numbers etc.); two ternary predicates for addition and multiplication of numbers and a binary one $\in$ — for membership. The axioms of A_n are: Peano axioms relativized to S_1, comprehension and choice schemata in each type S_k, $2 \leqslant k < n$, the axiom asserting that the universe splits into a disjoint union of types $S_1, \ldots, S_n$ and axioms

$$x \in y \,\&\, S_{k+1}(y) \rightarrow S_k(x) \quad \text{for} \quad 1 \leqslant k < n.$$

Structures (models) of A_n are represented in the form

$$M = \langle S_1^M, \ldots, S_n^M, +^M, \cdot^M, \varepsilon^M \rangle,$$

where the reduct $\langle S_1^M, +^M, \cdot^M \rangle$ is a structure (model) of Peano arithmetic. Each structure M is isomorphic to one with standard membership: we put

$$f(x) = x \text{ for } x \in S_1^M; \quad f(a) = \{f(x) \colon x \varepsilon^M a\},$$

for $a \in S_2^M$ and, similarly, extend f to higher types. Note that if M is definable, so is $f(M)$.

The formulas

(1.1) $$\exists a [S_n a \,\&\, x \in a]; \ \exists a, b [S_n(a) \,\&\, b \in a \,\&\, x \in b] \ \ldots$$

define the types $S_{n-1}, S_{n-2}, \ldots$, respectively.

Thus, all types are determined by the highest one.

The pairing functions J_k are definable at each type S_k. Thus, every set M in S_{k+1} determines a family consisting of S_{k+1}-sets $M^{(x)}$, where $M^{(x)} = \{y \colon J(x, y) \in M\}$ and x runs over S_k. Iterating the singleton operation definable in A_n we see that each type S_k, $2 \leqslant k \leqslant n$, contains a definable copy of natural numbers. Hence, the symbol $M^{(n)}$, for n in S_1, is meaningful and M can be treated as a countable family. In particular, some M of the highest type can be understood as codes for structures of A_n denoted also by M. Namely, S_n^M consists of all $M^{(n)}$'s, lower types are defined with help of formulae (1.1) and $M^{(0)}$, $M^{(1)}$ are codes for addition and multiplication: we assume that $M^{(0)}$ and $M^{(1)}$ arise from a, b (a, b in S_2) by iterating the singleton $n-1$ times. Then a and b split into ternary relations are addition and multiplication. Thus, M is a code for a structure with standard membership if all the determined relations are non-empty. In a similar way, we may define codes for structures with non-standard membership. We shall not deal with details, since this coding procedure is commonly known.

Assume that the formal language has been gödelized in A_n. We let the letters p, q run over (Gödel numbers of) formulas of A_n.

As shown in Mostowski [5], there is a formula of A_n expressing the satisfaction relation $M \models p[a_1, \ldots, a_k]$, where M is a (code of) structure of A_n, p — a Gödel number of a formula and $a_1 \ldots a_n$-elements of a structure (coded by) M. It follows that there are formulas of A_n saying: "M" is a code of a model of A_n; "M is a β-model", etc.

Section 2. Trees. In the theory A_n some higher rank sets can be encoded with the help of trees. A similar technique works also in the case of higher order set theory, see Marek [2] and Marek and Zbierski [4]. We recall below the basic notions of this technique.

A tree t (denoted also $\leqslant_t$) on an infinite set X is a partial ordering with $\mathrm{Dm}(t) \subseteq X$ having the following properties:

(i) There is a greatest element denoted by $w(t)$.

(ii) The set of successors of an element x is linearly ordered by $\leqslant_t$, for all $x \in \mathrm{Dm}(t)$.

(iii) Each nonempty subset of the domain of t has both maximal and minimal elements.

From this definition it follows that each element $x \in \mathrm{Dm}(t)$, $x \neq w(t)$, has exactly one immediate successor (denoted by x^+), for each x there is a minimal (in t) element $y \leqslant_t x$ and each nonminimal x has (possibly many) immediate predecessors.

To each tree t we assign a set $Z(t)$ by means of the following inductive conditions:

$$Z(t, x) = \varnothing, \quad \text{for minimal } x,$$
$$Z(t, x) = \{Z(t, y) \colon y^+ = x\}, \quad \text{for nonminimal } x,$$
$$Z(t) = Z(t, w(t)).$$

One can easily see that the family $\{Z(t)\colon t \text{ is a tree on } X\}$ coincides with the family $H(|X|^+)$ of all sets of hereditary cardinality less than $|X|^+$.

Furthermore, there is a relation $r(t, s, x, y)$ (where t, s are trees and x, y are elements of domains of t and s, respectively) defined by the inductive condition:

$$r(t,s,x,y) \equiv \forall u\{u^+ = x \to \exists v[v^+ = y \,\&\, r(t,s,u,v)]\} \,\&$$
$$\&\, \forall v\{v^+ = y \to \exists u[u^+ = x \,\&\, r(t,s,u,v)]\}\,.$$

A (generalized) isomorphism of t, s is defined by

$$t \simeq s \quad \text{iff} \quad r(t,s,w(t),w(s))\,.$$

We also put

$$t\varepsilon s \quad \text{iff} \quad \exists y[y^+ = w(s) \,\&\, r(t,s,w(t),y)]\,.$$

One shows that $t\simeq s$ iff $Z(t) = Z(s)$ and $t\,\varepsilon\, s$ iff $Z(t)\in Z(s)$. Moreover, $\simeq$ is an equivalence congruent with ε.

All the above notions can be defined in A_n. Thus there are formulas T, I, E of A_n defining trees and relations $\simeq$ and ε, respectively. In particular, we have

$$M \models T[t] \quad \text{iff} \quad t \text{ is a tree on } \underbrace{P \ldots P}_{n-2}(\omega)\,,$$
$$M \models I[t,s] \quad \text{iff} \quad t\simeq s \text{ iff } Z(t) = Z(s)\,,$$
$$M \models E[t,s] \quad \text{iff} \quad t\,\varepsilon\, s \text{ iff } Z(t)\in Z(s)\,.$$

Let ZFC_n denote the fragment of ZFC set theory obtained from ZFC by deleting the power set and assuming its first $n-2$ steps (i.e., there exists a $\underbrace{P \ldots P}_{n-2}(\omega)$). One adopts also the choice schema

$$(\forall x)_a \exists p \to \exists f (\forall x)_a \exists y [\mathrm{Fnc}(f) \,\&\, y = f(x) \,\&\, p] \quad \text{for an arbitrary formula } p.$$

The theory ZFC_n thus defined is interpretable in A_n: we interpret sets as trees, $\in$ as E and the equality $=$ as I (hence the interpretation of equality is nonstandard). Let p^T be the formula of A_n corresponding to p under this interpretation.

We have

$$A_n \vdash p^T \quad \text{for all } p \text{ in } ZFC_n$$

and

$$A_n \vdash (V = H(J^+_{n-2}))^T\,,$$

where $V = H(I^+_{n-2})$ means: each set has hereditary cardinality less than I^+_{n-2}.

Analogous results hold also in the case of higher order set theory M_n replacing A_n. The interpreted theory ZFC_n is then enriched by the axiom "there is a family which is inaccessible in the sense of Tarski".

Section 3. ω-nonstandard interpretation. In this section we show that some consistent extension of A_n has an internal ω-nonstandard interpretation. We add to A_n

a suitable form of the axiom of constructibility. In such an extension it is possible to define a proper nonprincipal ultrafilter over natural numbers. The interpretation in question is then obtained by defining the ultrapower construction.

Obviously, the main case of interest is $n = 2$, i.e. the second order arithmetic. First, we do this in set theory. Hence let Z denote ZFC_2 plus $V = HCL$ (all sets are constructible and hereditarily countable).

LEMMA 3.1. *There is a formula F of Z defining in Z a proper, nonprincipal ultrafilter over ω, i.e. the following formulas are provable in Z:*

$$F(x) \to x \subseteq \omega; \quad F(x) \,\&\, F(y) \to F(x\cap y)\,;$$
$$F(x) \,\&\, x\subseteq y\subseteq \omega \to F(y); \quad F(x) \vee F(\omega \setminus x)\,;$$
$$F(\omega); \quad \neg F(\{n\}) \quad \text{for all } n\in\omega\,.$$

Proof. Let C_α denote the αth constructible set. Since the family

$$B = \{\omega\setminus\{n\}\colon n\in\omega\}$$

is definable in Z, there is a formula $\varphi(u)$ expressing the following:

"$u\subseteq P(\omega)$ and $B\cup u$ has finite intersection property".

Now let $\psi_1(\alpha, f)$ be

$$\mathrm{Fnc}(f) \,\&\, \mathrm{Dm}(f)\subseteq\omega \,\&\, \mathrm{Rg}(f)\subseteq\alpha \,\&\, \varphi(\{C_{f(n)}\colon n\in \mathrm{Dm}(f)\}\cup\{C_\alpha\}) \,\&$$
$$\&\, \forall\beta<\alpha[\beta\notin \mathrm{Rg} f \to \neg\varphi(\{C_{f(n)}\colon n\in \mathrm{Dm}(f)\}\cup\{C_\alpha\}\cup\{C_\beta\})]\,.$$

And next let $\psi(\alpha, f)$ be

$$\psi_1(\alpha,f) \,\&\, \forall\beta\in \mathrm{Rg} f\, \psi_1(\beta, f\cap\omega\times\beta)\,.$$

Now let $F(x)$ be the formula

$$\exists\alpha, f[x = C_\alpha \,\&\, \psi(\alpha, f)]\,.$$

Thus the formula $F(x)$ expresses the inductive process in which at each stage we add the first constructible set which forms with the previously chosen sets and the sets $\omega\setminus\{n\}$ a family with finite intersection property. Hence F defines a maximal family of constructible subsets of ω with finite intersection property. Since $V = L$ is assumed in Z, this family is an ultrafilter and the lemma is proved.

According to the ultrapower construction we fix the following formulae:

$$U(f) \quad \text{is} \quad \mathrm{Fnc}(f) \,\&\, \mathrm{Dm}(f) = \omega\,,$$
$$e(f,g) \quad \text{is} \quad \exists x\{F(x) \,\&\, \forall n[n\in x \equiv f(n)\in g(n)]\}\,,$$
$$i(f,g) \quad \text{is} \quad \exists x\{F(x) \,\&\, \forall n[n\in x \equiv f(n) = g(n)]\}\,.$$

The next lemma expresses the fact that i is an equivalence congruent with e.

LEMMA 3.2. *The following formulas are provable in Z*:

$$i(f,f);\quad i(f,g)\equiv i(g,f);\quad i(f,g)\,\&\,i(g,h)\rightarrow i(f,h);$$
$$i(f,g)\,\&\,e(f,h)\rightarrow e(g,h);\quad i(f,g)\,\&\,e(h,g)\rightarrow e(h,f).$$

For each formula p of Z let p^U be the formula obtained from p by relativizing all variables to U and replacing each occurrence of $\in$ and $=$ by e and i, respectively. The following lemma is an instance of the Łoś theorem:

LEMMA 3.3. *For every formula $p(x_1, \ldots, x_k)$ (with the free variables indicated) the following holds*:

$$Z\vdash p^U[f_1,\ldots,f_k]\equiv\exists x\{F(x)\,\&\,\forall n[n\in x\equiv p(f_1(n),\ldots,f_k(n))]\}.$$

Proof. By induction on the length of formula p. Treating the quantifier case, we use the choice schema of Z to find a suitable function.

As an immediate corollary we obtain

THEOREM 3.4. *The triple $U=\langle U, e, i\rangle$ is an nonstandard interpretation of Z in Z.*

Proof. Indeed, from Lemma 3.3 we obtain $Z\vdash p^U$, for each theorem p of Z, i.e. $U=\langle U,e,i\rangle$ is an internal interpretation of Z. There is a function-like formula φ which to each x assigns the constant function with value x, i.e. the following holds:

$$Z\vdash\forall x\exists!y\varphi,$$
$$Z\vdash\forall x,y\{\varphi\rightarrow U(y)\,\&\,\forall n[y(n)=x]\}.$$

Thus φ defines an elementary embedding in the obvious sense. Indeed, from Lemma 3.3 we obtain the following schema:

$$Z\vdash\varphi(x_1,y_1)\,\&\ldots\&\,\varphi(x_k,y_k)\rightarrow[p^U(y_1,\ldots,y_k)\equiv p(x_1,\ldots,x_k)],$$
$$\text{for all } p(x_1,\ldots,x_k).$$

Let c_x be a constant function with value x and d the diagonal, $d(n)=n$. We immediately see that $Z\vdash\forall n[c_n<^U d]$, which proves that $U=\langle U,e,i\rangle$ is an ω-nonstandard interpretation.

Note that the nonstandard interpretation of equality in U can be replaced by the identity. Indeed, since V = L is assumed in Z, we may choose, in a definable way, one element from each i-equivalence class.

Now consider again the system of second order arithmetic. Let A be A_2 plus the consistent axiom $(\mathrm{V}=\mathrm{L})^T$ (every tree is constructible). Interpret A in A in the following way: first, interpret A in Z in a natural way, then Z in Z under U, then Z in A under T.

More exactly, let φ and ψ be formulas of Z defining addition and multiplication of numbers, respectively. We fix the following formulae of A:

$$S_1^*(x)\quad\text{is}\quad(x\in\omega)^{UT},$$
$$S_2^*(x)\quad\text{is}\quad(x\subseteq\omega)^{UT},$$

$$x\in^* y\quad\text{is}\quad S_1^*(x)\,\&\,S_2^*(y)\,\&\,(x\in y)^{UT},$$
$$(x+y=z)^*\quad\text{is}\quad\varphi(x,y,z)^{UT},$$
$$(x\cdot y=z)^*\quad\text{is}\quad\psi(x,y,z)^{UT},$$
$$(x=y)^*\quad\text{is}\quad(x=y)^{UT}.$$

Let $\langle *\rangle$ denote the sequence of the above formulas and let p^* be the translation of p under $\langle *\rangle$. The main result of this section is

THEOREM 3.5 *The sequence of formulas $\langle *\rangle$ is an ω-nonstandard interpretation of A in A.*

Proof. That $\langle *\rangle$ is an interpretation follows from Lemma 3.3 and the results cited in Section 2. To see that it is ω-nonstandard we define in A trees t_n and t such that t_n encodes the constant function with value n and t encodes the diagonal. Then $A\vdash\forall n[t_n<^{UT}t]$ is easily checked by again using 3.3 and the lemma of Section 4 of Zbierski [11].

Section 4. ω-standard but β-nonstandard interpretations. In this section we shall deal with interpretations of class (2), see Section 0.

We fix the numbers $n\geqslant 2$ and $1\leqslant k<n$. We shall define a k-standard and $(k+1)$-non-standard interpretation I_k of A_n in a consistent extension denoted by A_n^β, where A_n^β is A_n plus the axiom: "there is a code for a countable β-model of A_n". Since the numbers k, n are fixed throughout this section, we put $A=A_n$ and $A^\beta=A_n^\beta$. From the additional axiom it follows in A^β that there is a code M_0 for the smallest β-model of A_n. It is well known that the smallest β-model is uniquely determined and can be characterized in different ways, e.g. in terms of a ramified analytical hierarchy, see Gandy [1], Marek and Mostowski [3], Vetulani [10] or constructible sets, Zbierski [11]. We fix formulas

$$\varphi_1,\ldots,\varphi_n \tag{4.1}$$

defining the types $S_1,\ldots,S_n$ of the smallest β-model M (i.e. $S_i^M=\{x\colon\varphi_i(x)\}$). These formulas involve a parameter M_0 (a code for M) but, for different codes, are equivalent in A^β. A similar remark applies to all the formulas constructed in the sequel.

The proof of existence of the interpretation I_k is obtained by formalization in A^β the Mostowski–Suzuki construction [6]. First, we shall fix some notation. We shall use formal languages $L, L^*, L^\#$, where L is the language of the model M, i.e. L contains the constants c_m^i, $1\leqslant i\leqslant n$; $m\in\omega$; L^* in an extension of L by a new constant α and $L^\#$ is an extension of L^* by additional constants e_m. We assume that all three languages are — via Gödel's procedure — defined in A^β. Thus, formulae are some natural numbers. We use the same notation in L (resp. $L^*, L^\#$) as in the metalanguage, e.g. $\neg p$ (for p in L) denotes the negation of p, the phrase "p is $\exists x[S_1(x)\,\&\,q]$" means that p is a formula of L having the indicated form, etc. The sets of sentences of L, L^* and $L^\#$ are denoted by Sn, Sn^* and $Sn^\#$, respectively.

We assume that the languages $L, L^*, L^\#$ are defined in A^β in such a way that $Sn \subseteq Sn^* \subseteq Sn^\#$. We fix a standard interpretation of constants of L in the smallest model M (depending on the code M_0) and use the symbol $M_0 \models p[a_1 \ldots a_m]$, for p in L, as satisfaction formula defined in A^β. Under this interpretation the c_n^i's denote the elements of S_i^M. We distinguish also constants denoted by b_n from among the c_m^k's denoting codes for consecutive natural numbers in the type S_k. Finally, let p_n be a definable enumeration of Sn such that p_0 is $\mathrm{Bord}(d)$.

We now state the following

LEMMA 4.2. *There are formulas Φ_0, Φ_1 of A^β such that $A^\beta \vdash \exists! Z \Phi_0(Z)$; $A^\beta \vdash \Phi_0(Z) \to Z \subseteq Sn^*$; $A^\beta \vdash \forall n \exists! u \Phi_1$; $A^\beta \vdash \Phi_1(n, u) \to S_1(n) \,\&\, S_1(u)$, i.e. Φ_0 defines a set of sentences of L^*, Φ_1 defines a sequence u_n. In addition, Z and u_n have (provably in A^β) the following properties:*

(i) *Z is consistent and complete,*

(ii) *if $M_0 \models p$, $p \in Sn$, then $p \in Z$,*

(iii) *if $p \in Z$ and p has the form $\exists x[S_i(x) \,\&\, q]$, $1 \leqslant i \leqslant k$, then $q\,(c_m^i/x)$ is in Z, for a certain m (i.e. Z is S_i-closed for all $1 \leqslant i \leqslant k$).*

(iv) *the sentences $b_{u_{n+1}} <_d b_{u_n}$ are in Z for all n*

Proof. Let W be the set of all well-orderings of the model M in S_{k+1}^M. Of course, W is definable in A^β. Let $\Psi_n(s, r, x_0, \ldots, x_n)$ be the formula of A^β describing the following: (s, r are in W) & ($x_0 > \ldots > x_n$-strictly decrease in r) & (s can be embedded into r below x_n). We shall define in A^β a set Y such that all sections $Y^{(n)}$ are finite subsets of Sn^*. Then Z is defined as the union $\bigcup_n Y^{(n)}$. The sets $Y^{(n)}$ and numbers u_n are defined inductively according to the following conditions:

(R1) $Y^{(0)}$ consists of $\mathrm{Bord}(d)$ and $S_{k+1}(d)$; u_0 is 0;

(R2) for $n > 0$, $p_n \in Y^{(n)}$ or $\neg p_n \in Y^{(n)}$;
if $p_n \in Sn$ and $M_0 \models p_n$, then $p_n \in Y^{(n)}$;
if $p_n \in Y^{(n)}$ and p_n has the form $\exists x[S_i(x) \,\&\, q]$, $1 \leqslant i \leqslant k$, then $q(c_m^i/x) \in Y^{(n)}$, for some m; the sentence $b_{u_n} <_d b_{u_{n-1}}$ is in $Y^{(n)}$;

(R3) for $n > 0$, the following holds:
$\forall s \in W \exists r \in W\{M_0 \models \psi_n[s, r, b_u, \ldots, b_{u_n}]$ and $\langle M_0, r\rangle \models \bigwedge Y^{(n)}\}$.

In (R3) the symbol $\langle M_0, r\rangle$ denotes a code for the model M expanded by an $r \in W$.

Now we introduce formulas of A^β decribing this inductive process. The initial condition (R1) is described by the formula $P(X, u)$:

$$\forall p[p \in X \equiv p \in Sn \,\&\, p \text{ is } \mathrm{Bord}(d) \text{ or } p \text{ is } S_{k+1}(d) \,\&\, u = 0].$$

To describe the inductive step from $Y^{(n)}$ to $Y^{(n+1)}$ and from u_n to u_{n+1} we fix the formula $\varphi(n, Y, q, l)$:

$$\forall s \in W \exists r \in W\{M_0 \models \Psi_n[s, r, (l)_0, \ldots, (l)_n] \text{ and } \langle M_0, r\rangle \models \bigwedge Y^{(n)} \text{ and } \langle M_0, r\rangle \models q\}.$$

Clearly, all these notions are formalizable in A^β. In addition, one easily checks that the following holds:

$$A^\beta \vdash \varphi(n, Y, q, l) \vee \varphi(n, Y, \neg q, l),$$
(4.3)
$$A^\beta \vdash \varphi(n, Y, q, l) \to \exists l' \varphi(n+1, Y, q, l * l'),$$
$$A^\beta \vdash M_0 \models S_i[a] \to \exists m M_0 \models (x = c_m^i)[a],$$

$((l))_j$ is the jth term of the sequence number l; $l * l'$ is the number of $\langle (l)_0, \ldots, (l)_n, l'\rangle$). Using (4.3) we see that the formula $H(n, Y, g)$ describing the inductive step can be defined thus:

$$\forall p\{p \in Y^{(n+1)} \equiv p \in Sn^* \,\&\, (p \in Y^{(n)}) \vee$$
$$\vee (p \text{ is } p_{n+1} \text{ and } M_0 \models p \text{ or } \varphi(n, Y, p_{n+2}, \bar{g}(n))) \vee$$
$$\vee (p \text{ is } \neg p_{n+1} \text{ and } \neg\varphi(n, Y, p_{n+1}, \bar{g}(n))) \vee$$
$$\vee (p \text{ is } bg(n+1) <_d bg(n) \text{ and } g(n+1) = \min_l (\varphi(n+1, Y, p_{n+1}, \bar{g}(n) * l)$$
$$\text{or } \neg\varphi(n, Y, p_{n+1}, \bar{g}(n)) \text{ and}$$
$$g(n+1) = \min_l (\varphi(n+1, Y, \neg p_{n+1}, \bar{g}(n) * l))) \vee$$
$$\vee \exists q, i \leqslant k, m, x[(x \text{ — is a variable of } L) \text{ and } (p_{n+1} \text{ is } \exists x[S_i(x) \,\&\, q]$$
$$\text{and } \varphi(n+1, Y, p_{n+1}, \bar{g}(n+1)) \text{ and } (p \text{ is } q(c_m^i/x))$$
$$\text{and } \forall m' < m \neg\varphi(n+1, Y, q(c_{m'}^i/x), \bar{g}(n+1)))]\}.$$

Now let $\Gamma(n, X, u)$ be the formula

$$\exists Y, g \forall j < n[P(Y^{(0)}, g(0)) \,\&\, H(j, Y, g) \,\&\, X = Y^{(n)} \,\&\, u = g(n)].$$

Using (4.3) and induction, we have:

$$A^\beta \vdash \forall n \exists! X, u \Gamma(n, X, u),$$
(4.4)
$$A^\beta \vdash \forall n, X, u \exists l \forall p[\Gamma(n, X, u) \,\&\, p \in X \to p < l].$$

Clearly, the set X and number u defined by $\Gamma(n, X, u)$ are the $Y^{(n)}$ and u_n as required in (R2), (R3). Thus the formulae

$$\Phi_0\colon\ \forall p\{p \in Z \equiv p \in Sn^* \,\&\, \exists n, X, u[\Gamma(n, X, u) \,\&\, p \in X]\},$$
$$\Phi_1\colon\ \exists X \Gamma(n, X, u)$$

define the required set Z and sequence u_n, which finishes the proof.

We have to show yet that the set Z has provably in A^β a k-standard and $(k+1)$-nonstandard model definable in A^β. The definition of this model then determines the required interpretation I_k of A in A^β. The proof of existence of such a model is obtained by formalization of the Henkin–Orey construction.

Let q_n be a definable enumeration of the set $Sn^{\#}$ and let $Cn(Z, p)$ be the formula of A^{β} expressing: "$p \in Sn^{\#}$ and p is consistent with Z".

LEMMA 4.5. *There is a function f, definable in A^{β}, on $Sn^{\#}$ whose values are constants c_m^i, with $1 \leqslant i \leqslant k$, such that the following holds:*

$$A^{\beta} \vdash \forall p, n[Cn(Z, p) \rightarrow Cn(Z, p \,\&\, \bigwedge_{1 \leqslant i \leqslant k} \neg S_i(e_n) \vee e_n = f(p))] .$$

Proof. To each $p \in Sn^{\#}$ we assign (a code for) the sequence $\langle e_{j_1}, \ldots, e_{j_m} \rangle$ of all constants not in L^* occurring in p. Then we fix variables $x_{j_1}, \ldots, x_{j_m}$ which do not occur in p. The variable corresponding to e_n is denoted by x. Let $\hat{p}$ be the formula obtained from p by substituting $x_{j_1}, \ldots, x_{j_m}$ for $e_{j_1}, \ldots, e_{j_m}$ and quantifying all variables but x. The resulting function $g(p) = \hat{p}$ is recursive and we have:

$$A^{\beta} \vdash Cn(Z, p) \equiv Cn(Z, \hat{p}) .$$

If $\neg Cn(Z, \hat{p} \,\&\, S_i(x))$ for all $1 \leqslant i \leqslant k$, then $A^{\beta} \vdash Cn(Z, \hat{p} \,\&\, \bigwedge_{1 \leqslant i \leqslant k} \neg S_i(x))$, by predicate calculus. If $Cn(Z, \hat{p} \,\&\, S_i(x))$ holds for a certain $1 \leqslant i \leqslant k$, then also

$$Cn(Z, \exists x [S_i(x) \,\&\, \hat{p}])$$

holds and $\exists x [S_i(x) \,\&\, \hat{p}]$ is in Z, since Z is complete. By Lemma 4.2 $\hat{p}(c_m^i/x)$ is in Z for a certain m, from which we infer $Cn(Z, p \,\&\, (e_n = c_m^i))$.

Thus we put $f(p) = c_m^i$, with the smallest such m. The function f is clearly definable from Z and hence definable in A^{β}, which completes the proof.

LEMMA 4.6. *There is a formula Φ_2 defining in A^{β} a set $Z^{\#} \subseteq Sn^{\#}$ having (provably in A^{β}) the following properties: $Z^{\#}$ is consistent and complete, and contains Z; if $\exists xq$ is in $Z^{\#}$, then $q(e_m/x) \in Z^{\#}$, for a certain m; for all $1 \leqslant i \leqslant k$ if $S_i(e_m) \in Z^{\#}$, then $e_m = c_n^i$ is in $Z^{\#}$, for some n.*

Proof. We define $Z^{\#}$ as a union $\bigcup_n X^{(n)}$, where all $X^{(n)}$'s are finite sets $\subseteq Sn^{\#}$ and are defined inductively in A^{β} according to the following conditions: at the step n we add to $\bigcup_{j<n} X^{(j)}$ the sentence

$$\neg \bigwedge_{1 \leqslant i \leqslant k} S_i(e_n) \vee e_n = f(\bigwedge X^{(n-1)}) \tag{4.7}$$

where f is the function from Lemma 4.5.

We add also the sentence q_n or $\neg q_n$ depending on, which is consistent with Z, $X^{(n-1)}$ and the already added sentence (4.7), with preference of q_n. If a sentence of the form $\exists xp$ has been added, then we add also $p(e_m/x)$, where e_m is the smallest constant not yet used. Thus, the induction is similar to that in the proof of Lemma 4.2 and it suffices to describe the inductive step $I(n, X)$. Let us denote by $X_*^{(n)}$ the set

$$X^{(n)} \cup \{\bigwedge_{1 \leqslant i \leqslant k} \neg S_i(e_{n+1}) \vee e_{n+1} = f(\bigwedge X^{(n)})\} .$$

The formula $I(n, X)$ can be defined thus:

$$\begin{aligned}\forall q \{q \in X^{(n-1)} \equiv q \in X^{(n)} \vee (q \text{ is } \bigwedge_{1 \leqslant i \leqslant k} S_i(e_{n+1}) \vee e_{n+1} \\ = f(\bigwedge X^{(n)})) \vee (q \text{ is } q_{n+1} \text{ and } Cn(Z \cup \bigwedge X_*^{(n)}, q_n)) \vee \\ \vee (q \text{ is } \neg q_{n+1} \text{ and } \neg Cn(Z \cup \bigwedge X_*^{(n)}, q_{n+1})) \vee \\ \vee \exists x, p, m[(x \text{ is a variable}) \text{ and } (q_{n+1} \text{ is } \exists xp) \\ \text{and } Cn(Z \cup X_*^{(n)}, q_{n+1}) \text{ and } (q \text{ is } p(e_m/x)) \\ \text{and } (m = \min_{m'}\{e_{m'} \text{ does not occur in } \bigwedge X_*^{(n)}\})]\} .\end{aligned}$$

Now, by induction, we obtain $A^{\beta} \vdash \exists! X \forall n I(n, X)$ and we put Φ_2 as

$$\forall q \{q \in Z^{\#} \equiv \exists n, X \forall m [I(m, X) \,\&\, q \in X^{(n)}]\} .$$

Clearly, the set $Z^{\#}$ has the required properties and the proof is finished.

Now, it is easy to define a model $M^{\#}$ of $Z^{\#}$. The constants e_n, e_m are equivalent if the sentence $e_n = e_m$ is in $Z^{\#}$.

Choosing one element from each equivalence class, we may assume that $e_n \neq e_n \in Z^{\#}$, for $n \neq m$.

Formulae $\psi_1^{\#}, \ldots, \psi_n^{\#}$, where $\psi_i^{\#}(m)$ is $S_i(e_m) \in Z^{\#}$, define the types $S_1, \ldots, S_n$ of $M^{\#}$. Similarly, we define $+^{\#}, \cdot^{\#}, \in^{\#}$. Hence there is a definable code $M_0^{\#}$ for $M^{\#}$ and by induction we show

$$A^{\beta} \vdash M_0^{\#} \models p \equiv p \in Z^{\#} .$$

Since $Z^{\#}$ is S_1-closed, there is a definable isomorphism h_1 of $S_1^{M^{\#}}$ onto natural numbers. We extend h_1 to all higher levels as described in Section 1.

We obtain the formulas $\psi_1, \ldots, \psi_n$ defining the ranges of $h_1, \ldots, h_n$, respectively. Since $Z^{\#}$ is S_i-closed, for $1 \leqslant i \leqslant k$ we infer that $A^{\beta} \vdash \psi_i \equiv \varphi_i$, for $1 \leqslant i \leqslant k$, where φ_i's are as in (4.1). We put $I_k = \langle \psi_1, \ldots, \psi_n \rangle$, i.e. we interpret $S_1, \ldots, S_n$ as $\psi_1, \ldots, \psi_n$, respectively, and addition, multiplication and membership are standard.

THEOREM 4.8. *I_k is a k-standard and a $(k+1)$-nonstandard interpretation of A_n in A_n^{β}.*

Proof. From the definition of I_k it follows that we interpret the objects of types $\leqslant k$ as elements of the minimal β-model. Hence I_k is clearly k-standard. That it is $(k+1)$-nonstandard follows from Lemma 4.2.

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Extensions normales de demi-groupes inverses

par

Mario Petrich (Montpellier)

Abstract. The concept of a normal extension of an inverse semigroup is formulated in analogy to Schreier group extensions, and a general extension problem is posed. As a means of a study of these extensions the normal hull is introduced. Certain properties of this hull concerning congruences are established. The problem posed is then solved in two special cases. The paper is concluded by the construction of the normal hull of the Reilly semigroup and of its centralizer of idempotents.

1. Introduction et sommaire. Les demi-groupes inverse représentent un des domaines de recherche les plus fructueux dans la théorie des demi-groupes. La grande variété des résultats concernant la structure des demi-groupes inverses fait qu'il est nécessaire d'avoir une théorie systématique qui couvrirait le plus grand nombre possible de résultats déjà existants. Une approche susceptible d'être utile dans cette direction est basée sur la notion d'extension normale d'un demi-groupe inverse. Cela donne un autre point de vue concernant les congruences sur les demi-groupes inverses, et représente une généralisation de la théorie de Schreier des extensions des groupes.

Nous rappelons quelques définitions et un résultat concernant les congruences sur un demi-groupe inverse dans le paragraphe 2. Dans le paragraphe 3, nous introduisons la notion d'extension normale d'un demi-groupe inverse par un autre et formulons un problème général. Le paragraphe 4 contient la construction de l'enveloppe normale d'un demi-groupe inverse. Les résultats principaux se trouvent dans le paragraphe 5: le premier concerne une propriété intéressante des congruences sur l'enveloppe normale, le deuxième et le troisième donnent des constructions des extensions normales dans deux cas particuliers. Le paragraphe 6 contient des constructions de l'enveloppe normale d'un demi-groupe de Reilly et du centralisateur de ses idempotents.

2. Rappel. Soit S un demi-groupe. Si $a, b \in S$ sont tels que $a = aba$ et $b = bab$, alors b est un *inverse* de a. Un *demi-groupe inverse* est un demi-groupe dont tout élément possède un seul inverse (l'inverse de a sera noté a^{-1}). Le demi-treillis des idempotents de S sera noté E_S.

Une relation d'équivalence ϱ sur S régulière à droite et à gauche est une *congruence*; on définit le *demi-groupe quotient* S/ϱ de façon naturelle.