Ideals on the real line and Ulam's problem

by

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Abstract. We prove that if $\{S_n: n \in \omega\}$ is a family of 2^ω-complete fields of sets such that $S_n \subset P(2^\omega)$, $\{x\} \in S_n$ for $x \in 2^\omega$ and $S_n \neq P(2^\omega)$ then $\bigcup_n S_n \neq P(2^\omega)$.

The proof uses techniques due to A. Taylor and E. Grzegorek and the following key lemma: if $I \subset P(2^\omega)$ is a 2^ω-complete ideal then $|P(2^\omega)/I| > 2^\omega$.

0. Introduction. We shall be concerned with ideals and measures on the set of reals. An ideal will always mean a non-trivial ideal containing singletons. By a measure we shall understand a σ-additive function $m: F \to \langle 0, 1\rangle$ such that $F \subset P(X)$, $m(\{x\}) = 0$ for $x \in X$ and if $m(a) = 0$ and $b \subset a$ then $b \in F$ and $m(b) = 0$. We also require $m(X) = 1$.

In Section 1 we discuss the cardinality of $P(2^\omega)/I$ where I is an ω_1-complete ideal. It is proved that $|P(2^\omega)/I| > 2^\omega$ when I is 2^ω-complete. If $2^\varkappa = 2^\omega$ for some uncountable $\varkappa$, then we show an ω_1-complete ideal J such that $|P(2^\omega)/J| = 2^\omega$. Some corollaries are also proved.

In Section 2 we prove the main result, mentioned in the abstract.

In Section 3 we discuss the so-called representation problem and give a counter-example to the converse of a theorem from [6].

1. The cardinality of $P(2^\omega)/I$.

Proposition 1.1. *Let $\varkappa \leqslant 2^\omega$ and let $I \subset P(\varkappa)$ be an ω_1-complete ideal. Then $|P(\varkappa)/I| \geqslant 2^\omega$.*

Proof. Straightforward. ■

Theorem 1.2. (1). *Let $I \subset P(2^\omega)$ be a 2^ω-complete ideal. Then $|P(2^\omega)/I| > 2^\omega$.*

Proof. We consider two cases.

1. I is not 2^ω-saturated. There are 2^ω disjoint sets outside I. All sums belong to different classes, so in this case $|P(2^\omega)/I| = 2^{2^\omega}$.

2. I is 2^ω-saturated. In this case we shall use the technique of generic ultra-powers. See [3] for the details.

We view the universe as the ground model and denote it by M. We consider

(1) After completing this paper the author learned that Theorem 1.2 was already known (for 2^ω-complete ω_1-saturated ideals) to K. Kunen.

the Boolean extension given by the algebra $B = P(2^\omega)/I$. B is complete since I is 2^ω-complete 2^ω-saturated. We shall prove that B has cardinality greater than $(2^\omega)^M = \varkappa$.

In the generic extension $M[G]$ we consider an ultrapower modulo G consisting of equivalence classes of functions belonging to M with domain $\varkappa$. Since G is M-complete, the usual fundamental theorem holds.

I is $\varkappa$-saturated, and hence this ultrapower is well founded (cf. [3]). By Mostowski's contraction theorem we can identify it with its transitive collapse $N \subset M[G]$. We have the canonical elementary embedding $j: M \to N$. Denote $j(\varkappa) = \lambda$.

Since $M \models |2^\omega| = \varkappa$, we have $N \models |2^\omega| = \lambda$. We have $(2^\omega)^N \subset (2^\omega)^{M[G]}$, and hence $M[G] \models |2^\omega| \geqslant |\lambda|$. B is $\varkappa$-saturated; so cardinals $\geqslant \varkappa$ in M and in $M[G]$ are the same.

Consider now an arbitrary function $f: \varkappa \to \varkappa$ belonging to M. The transitive collapse of the class of this function is an element of λ. By a well-known combinatorial theorem there are at least $\varkappa^+$ almost disjoint functions $f: \varkappa \to \varkappa$. ($f$ and g are almost disjoint iff $|\{\alpha < \varkappa: f(\alpha) = g(\alpha)\}| < \varkappa$). Each such function determines a different class and a different element of N. Hence we have shown a subset of λ of cardinality $\varkappa^+$. $M[G] \models |2^\omega| \geqslant \varkappa^+$; in other words, $(2^\omega)^{M[G]} \geqslant \varkappa^+$.

On the other hand, $(2^\omega)^{M[G]} \leqslant (|B|^\omega)^M$. If $|B| \leqslant \varkappa$, then $(2^\omega)^{M[G]} \leqslant (\varkappa^\omega)^M$. But in view of $M \models 2^\omega = \varkappa$ we have $(\varkappa^\omega)^M = \varkappa$, which gives a contradiction. Hence $|P(2^\omega)/I| > 2^\omega$. ∎

The following remark is due to E. Grzegorek:

PROPOSITION 1.3. If $2^\varkappa = 2^\omega$ for a certain uncountable $\varkappa$, then there is an ω_1-complete ideal $I \subset P(2^\omega)$ such that $|P(2^\omega)/I| = 2^\omega$.

Proof. We take $A \subset 2^\omega$, $|A| = \omega_1$ and an ω_1-complete ideal $J \subset P(A)$. Let

$$I = \{X \subset 2^\omega: X \cap A \in J\}.$$

I is an ω_1-complete ideal. There are at most $2^{|A|}$ equivalence classes, and hence the conclusion holds. ∎

We show some applications of the above results.

THEOREM 1.4. *The following are equivalent*:

(i) Con(ZFC + *there exist cardinals* $\varkappa < \lambda$ *such that* $\varkappa$ *carries a* $\varkappa$*-complete* ω_1*-saturated ideal and* λ *carries a* λ*-complete* ω_1*-saturated ideal* $+ 2^\omega \leqslant \lambda$);

(ii) Con(ZFC+MA+2^ω *carries a* 2^ω*-complete* ω_1*-saturated ideal* I *and an* ω_1*-complete* ω_1*-saturated ideal* J *such that* $|P(2^\omega)/I| \neq |P(2^\omega)/J| = 2^\omega$).

Proof. (i) → (ii). We want to get a generic extension of the ground model M s.t. $M[G] \models \text{MA} + 2^\omega = \lambda$. The theorem of Solovay and Tennenbaum cannot be used directly since we do not know whether $\forall \xi < \lambda [2^\xi \leqslant \lambda]$.

Hence we first add λ generic reals by a ccc forcing. In the extension $M[H_1] \models 2^\omega = \lambda$ carries a 2^ω-complete ω_1-saturated ideal and $\varkappa < 2^\omega$ carries a $\varkappa$-complete ω_1-saturated ideal (see [8]). By a theorem of Prikry (see [3]), the condition of Solovay and Tennenbaum is satisfied in this model. We can get a ccc extension $M[H_1][H_2]$ where $\text{MA} + 2^\omega = \lambda$ holds. This is the desired model $M[G]$. The proof of Proposition 1.3. gives the ideal J.

(ii) → (i). The model $M[G] \models 2^\omega = \lambda$ carries a λ-complete ω_1-saturated ideal. It cannot be the first such cardinal, because it carries also the ideal J (which is not λ-complete). We take as $\varkappa$ the first cardinal carrying an ω_1-complete ω_1-saturated ideal. ∎

Similarly (adding Solovay reals) we can prove:

THEOREM 1.5. *The following are equivalent*:

(i) Con (ZFC + *there exist real-valued measurable cardinals* $\varkappa < \lambda$ *such that* $2^\omega \leqslant \lambda$);

(ii) Con (ZFC+2^ω *carries a* 2^ω*-additive measure* m *and a measure* n *such that* $|P(2^\omega)/I_m| \neq |P(2^\omega)/I_n| = 2^\omega$).

2. Ulam's problem on sets of measures. The following question was raised by S. Ulam [cf. [1], [10]): Let $\varkappa$ be less than the first measurable cardinal. What is the minimal number λ of two valued measures μ_α: $\alpha < \lambda$ s.t. every $X \subset \varkappa$ is measurable with respect to at least one of them?

Erdös and Alaoglu have proved that if there is no ω_1-complete ω_1-saturated ideal $I \subset P(\varkappa)$, then $\lambda > \omega$ (see [1]).

The case we shall be concerned with in this section is $\varkappa = 2^\omega$. What is the situation if 2^ω is large, e.g. real-valued measurable? In this case the theorem of Erdös and Alaoglu gives no information. The main result of this section is

THEOREM 2.1. *For every countable family of* 2^ω*-additive two-valued measures* μ_n *defined on* $S_n \subset P(2^\omega)$ *there is an* $X \subset 2^\omega$ *non-measurable with respect to any of them.*

A Boolean algebra B is said to be *separable* iff it contains a countable dense set (in the forcing sense). An ideal $I \subset P(X)$ is said to be *separable* iff $P(X)/I$ is separable.

Theorem 1.2 gives:

THEOREM 2.2. *If* $I \subset P(2^\omega)$ *is a* 2^ω*-complete ideal, then* I *is not separable.*

Proof. Assume I is separable. It is clear that I is ω_1-saturated. Hence $P(2^\omega)/I$ is a complete algebra. I cannot have atoms, since it would yield a two-valued measure on $P(2^\omega)$. Hence $P(2^\omega)/I$ is isomorphic to Bor/I_k where Bor denotes the family of Borel subsets of 2^ω and I_k the ideal of meager sets. The cardinality of this algebra is 2^ω. This contradicts Theorem 1.2. ∎

The proof of a theorem in Taylor [10] gives, in view of Theorem 2.2, the following

THEOREM 2.3. *For every countable family of* 2^ω*-complete ideals* $I_n \subset P(2^\omega)$ *there exists a family* $\{a_k: k \in \omega\} \subset P(2^\omega)$ *of pairwise disjoint sets s.t.* $a_k \notin \bigcup_{n\in\omega} I_n$ *for all* $k \in \omega$.

The question whether the above follows from Theorem 2.2 was suggested to us by E. Grzegorek.

Before proving Theorem 2.3 we shall finish the proof of Theorem 2.1.

Let $\{\mu_n: n \in \omega\}$ denote any family of two-valued 2^ω-additive measures and $I_n = \{X \subset 2^\omega: \mu_n(X) = 0\}$. I_n are 2^ω-complete ideals. By Theorem 2.3 there is a pairwise disjoint family $\{a_k: k \in \omega\}$ outside $\bigcup_{n\in\omega} I_n$. We claim that all these sets are non-measurable with respect to any measure μ_n. $\mu_n(a_k) = 0$ is impossible since $a_k \notin \bigcup_{n\in\omega} I_n$ and $\mu_n(a_k) = 1$ is impossible since in this case $\mu_n(a_m) = 0$ for $m \neq k$. ■

It remains to prove Theorem 2.3.

Proof (essentially due to Taylor [10]):

An ideal $I \subset P(X)$ is said to be *nowhere ω_1-saturated* iff $I|A = \{B \subset X: B \cap A \in I\}$ is not ω_1-saturated for any $A \in P(X) - I$.

LEMMA 1. *Let $Q \subset P(P(2^\omega))$ be a countable set of nowhere ω_1-saturated ω_1-complete ideals. Then there exists a pairwise disjoint family $\{a_\xi: \xi < \omega_1\} \subset P(2^\omega) - \bigcup Q$.*

Proof. The argument of Erdös and Alaoglu works (cf. [1] and [10]). ■

LEMMA 2. *If Q is a countable set of 2^ω-complete ω_1-saturated ideals $I_n \subset P(2^\omega)$, then there exists a pairwise disjoint family $\{a_n: n \in \omega\} \subset P(2^\omega) - \bigcup Q$.*

Proof. Let $Q = \{I_n: n \in \omega\}$ and $I = \bigcap_{n\in\omega} I_n$. I is ω_1-saturated. For each $n \in \omega$ let $\mathscr{A}_n$ be a maximal collection of sets in $I_n - I$ that are almost disjoint (mod$I$). Then $|\mathscr{A}_n| < \omega_1$ so $A_n = \bigcup \mathscr{A}_n \in I_n$. Hence $B_n = 2^\omega - A_n \notin I$ and $I_n = I|B_n$. Since $\{B_n: n > 0\}$ is not a dense set for $I|B_n$ (because this ideal is 2^ω-complete), we can choose $C_0 \subset B_0$ s.t. $C_0 \notin I$ and for each $n > 0$ $B_n^0 = B_n - C_0 \notin I$. If C_j has been defined and for each $n > j$ we have $B_n^j \notin I$, then we can choose $C_{j+1} \subset B_{j+1}^j$ s.t. $C_{j+1} \notin I$ and for each $n > j+1$ $B_n^{j+1} = B_n^j - C_{j+1} \notin I$. (Otherwise $I|B_{j+1}^j$ which is 2^ω-complete would be separable). This yields a pairwise disjoint family $\{C_n: n \in \omega\}$ s.t. $C_n \in P(B_n) - I$ for each $n \in \omega$. Since $C_n \notin I_n$ for each n, we can find a pairwise disjoint partition $\{C_n^m: m \in \omega\}$ of C_n s.t. $C_n^m \notin I_n$. (Otherwise we would get a two-valued measure on $P(2^\omega)$). Now $\{a_m: m \in \omega\}$ defined by $a_m = \bigcup_{m\in\omega} C_n^m$ is the desired family.

Let us now return to the proof of Theorem 2.5. We have our given set $Q = \{I_n: n \in \omega\}$ of 2^ω-complete ideals. We define:

$$Q_0 = \{I \in Q: I \text{ is nowhere } \omega_1\text{-saturated}\},$$

$$Q_1 = \{I \in Q: \text{there is an } A_I \in P(2^\omega) - I \text{ s.t. } I|A_I \text{ is } \omega_1\text{-saturated}\},$$

$$Q_2 = \{I|A_I: I \in Q_1\}.$$

By Lemma 1 we get a pairwise disjoint family $\{Y_\alpha: \alpha < \omega_1\} \subset P(2^\omega) - \bigcup Q_0$. For any $J \in Q_2$ at most one Y_α can satisfy $2^\omega - Y_\alpha \in J$. Hence there is a $\gamma < \omega_1$ s.t. for any $J \in Q_2$ $2^\omega - Y_\gamma \notin J$. Let $A = Y_\gamma$, $B = 2^\omega - Y_\gamma$. Then $A \notin \bigcup Q_0$, $B \notin \bigcup Q_2$ and $A \cap B = 0$.

Applying Lemma 2 to $\{J|B: J \in Q_2\}$ yields a pairwise disjoint partition $\{B_n: n \in \omega\}$ of B s.t. for each $n \in \omega$ $B_n \in P(2^\omega) - \bigcup Q_2 \subset P(2^\omega) - \bigcup Q_1$. Similarly, if we apply Lemma 1 to $\{I|A: I \in Q_0\}$, we obtain a pairwise disjoint partition $\{A_n: n \in \omega\}$ of A s.t., for each $n \in \omega$, $A_n \in P(2^\omega) - \bigcup Q_0$. The family $\{A_n \cup B_n: n \in \omega\}$ satisfies the conclusion of Theorem 2.3. ■

Now the proof of Theorem 2.1. is complete.

Theorem 2.1 together with a result of Grzegorek (see [2], Theorem 3) gives the following

THEOREM 2.6. *Let $\{S_n: n \in \omega\}$ be a family of 2^ω-complete fields of sets s.t. $S_n \subset P(2^\omega)$, $\{x\} \in S_n$ for $x \in 2^\omega$ and $S_n \neq P(2^\omega)$. Then $\bigcup_{n\in\omega} S_n \neq P(2^\omega)$.*

3. The representation problem. The following will be referred to as the representation problem. Let $F \subset P(X)$ be a σ-algebra and $I \subset F$ an ideal. Does there exist a function f: $F \to F$ such that:

$$f(x) \equiv x(\text{mod} I), \quad \text{if } x \equiv y(\text{mod} I) \text{ then } f(x) = f(y),\ f(\emptyset) = \emptyset,\ f(X) = (X),$$
$$f(a \cup b) = f(a) \cup f(b), \quad f(a \cap b) = f(a) \cap f(b).$$

Such a function f is called the *solution of the problem* (F, I).

The following theorem is proved in [6].

THEOREM 3.1. *Let $F \subset P(X)$ be a σ-algebra and $I \subset F$ an ideal. If I is $|F/I|$-complete, then (F, I) has a solution.*

Using the results of Section 1, we shall produce a counterexample to the converse of Theorem 3.1. In order to do it we need a classical result due to von Neumann and Maharam (see [4]):

THEOREM 3.2. *Let $F \subset P(X)$ be a σ-algebra and m a measure on F. Then (F, I_m) has a solution.*

We come back to Theorem 3.1. If nothing is required about the σ-algebra F, then it is easy to give a counterexample to the converse of this theorem in ZFC. Actually there exist σ-algebras F with a measure m defined on F s.t. $F \subset P(2^\omega)$ and $|F/I_m| > 2^\omega$. I_m is not $|F/I_m|$-complete and (F, I_m) has a solution by Theorem 3.2.

A more interesting problem arises if we want F to be $P(2^\omega)$.

THEOREM 3.3. *No ω_1-complete ideal $I \subset P(2^\omega)$ is $|P(2^\omega)/I|$-complete.*

Proof. Let $I \subset P(2^\omega)$ be an ω_1-complete ideal. If I is not 2^ω-complete, then I is not $|P(2^\omega)/I|$-complete in view of Proposition 1.1. If I is 2^ω-complete, then it is not $|P(2^\omega)/I|$-complete in view of Theorems 1.2. ■

THEOREM 3.4. *Let m be a σ-additive measure on $P(2^\omega)$ and I the ideal of null sets. Then I is not $|P(2^\omega)/I|$-complete but $(P(2^\omega), I)$ has a solution.*

Proof. Apply Theorems 3.2 and 3.3. ■

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Sur une propriété des fonctions de deux variables

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Résumé. Dans l'article [1] O'Malley a démontré que toute fonction $f\colon [0, 1] \to R$ étant surpassement continue est de première classe de Baire et a la propriété de Darboux. Dans cet article on introduit les trois différentes définitions de la continuité surpasse des fonctions réelles de deux variables et on examine si cettes propriétés impliquent la première classe de Baire et la propriété de Darboux définie par Mišik dans l'article [2].

Dans l'article [1] O'Malley a introduit la définition suivante:

(1) Let f be a measurable real-valued function defined on $[0, 1]$. Let (a, b) be any open interval and $E = \{x\colon f(x) \in (a, b)\}$. Then f is preponderantly continuous if, for every $x \in E$, we can find a $\delta = \delta(x, (a, b)) > 0$ such that $m(E \cap I)/m(I) > \frac{1}{2}$ for all interval I containing x with $0 < m(I) < \delta$. (Here m denotes Lebesgue measure.)

et a démontré le théorème suivant:

THEOREM. *If f is preponderantly continuous on* $[0, 1]$ (*according to* (1)), *then f is Baire* 1, *Darboux.*

Dans cet article j'établis des théorèmes analogues concernant les fonctions de deux variables.

DÉFINITION 1. Désignons par R l'espace des nombres réels et par R^2 l'espace produit $R \times R$. On dit qu'une fonction mesurable (au sens de Lebesgue) $f\colon R^2 \to R$ a la propriété:

(P_1) lorsqu'il existe pour tout point $A \in R^2$ et pour tout intervalle ouvert (a, b) contenant $f(A)$ un nombre positif $\delta = \delta(A, (a, b))$ tel que

$$m_2(S(A, r) \cap f^{-1}((a, b)))/m_2(S(A, r)) > \tfrac{1}{2}$$

pour tout $0 < r < \delta$, où m_2 désigne la mesure de Lebesgue dans l'espace R^2 et $S(A, r) = \{X \in R^2\colon \varrho(A, X) < r\}$ et ϱ désigne la distance euclidienne dans R^2;

(P_2) lorsqu'il existe pour tout point $A \in R^2$ et pour tout intervalle $(a, b) \ni f(A)$ un nombre positif $\delta = \delta(A, (a, b))$ tel que

$$\liminf_{r \to 0} m_2(S(A, r) \cap f^{-1}((a, b)))/m_2(S(A, r)) > \tfrac{1}{2};$$