Absolute fixed point sets and AR-spaces

by

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Abstract. A subset A of a topological space X is a *fixed point set* of X if there is a map (continuous function) $f: X \to X$ such that $f(x) = x$ iff $x \in A$. A compactum X is an *absolute fixed point set* (AFS-*space*) if whenever X is embedded in a compactum Z, then X is a fixed point set of Z. In this paper it is shown that the class of AFS-spaces is properly contained in the class of compacta which are both contractible and locally contractible.

1. Introduction. The concept of an AFS-space was first defined in [2]. Clearly, every AR-space is an AFS-space. The results found in [2], [4] show that if X is an AFS-space, then X is a Peano continuum, and X is C^n if it is LC^n for $n = 1, 2, \ldots$ In [4] an example is given to show that there exists a contractible LC^∞ compactum which is not an AFS-space.

The purpose of this paper is to show that every AFS-space is contractible and locally contractible, and to give an example of a contractible and locally contractible compactum which is not an AFS-space.

2. Notation. The diameter of a set A will be denoted by $\delta(A)$. Hilbert space will be denoted by E^ω and given the metric ϱ defined in [1, p. 10]. Definitions for the concepts of LC^n, C^n, LC^∞, AR-space, and ANR-space may be found in [1]. We shall also make use of the concept of the *cap over a continuum* which we now define.

Let C be a continuum lying in E^ω and let A be a compact segment in E^ω. Consider the disjoint union $A \cup (C \times [0, \infty))$ of A and the product space $C \times [0, \infty)$. Let h be a one-to-one mapping from $A \cup (C \times [0, \infty))$ onto a continuum in E^ω such that the following properties are satisfied.

(1) $h(z) = z$ if $z \in A$.

(2) $h(z, 0) = z$ for all $z \in C$.

(3) $h|C \times [0, \infty)$ is a homeomorphism.

(4) $\lim_{t \to \infty} \delta(h(C \times \{t\})) = 0$.

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(5) For each $z \in C$, $h(\{z\} \times [0, \infty)) \cup A$ is homeomorphic to the closure of the curve in the plane E^2 whose equation is $y = \sin(\pi/x)$ for $0 < x \leqslant 1$.

The image of $A \cup (C \times [0, \infty))$ under h is defined to be the *cap CA of C and A*. We shall regard the cap CA as the set $A \cup (C \times [0, \infty))$ together with an assigned metric which makes the function h an isometry.

Essentially, the cap CA is obtained by taking a cone over C, removing the vertex, and then winding the resulting cylinder over C in a "$\sin(1/x)$" fashion in E^ω so as to converge to the limit interval A.

3. The results. The proof of the following theorem parallels the proof of Property 3 in [3, p. 165].

THEOREM 1. *Every locally contractible* AFS-*space is contractible.*

Proof. Suppose that X is a locally contractible AFS-space which is not contractible. Let A denote the closed interval $[-1, 1]$, and let Z denote the cap XA. We shall consider X to be identified with $X \times \{0\}$ in Z. It follows that $Z - A$ is locally contractible, and X is not contractible in Z.

There is a map $f\colon Z \to Z$ whose fixed point set is precisely X. Since A has the fixed point property, it follows that there is a point p in A such that $f(p) \in Z - A$. Let V be a neighborhood of $f(p)$ which is contractible in $Z - A$, and let $\beta\colon V \times I \to Z - A$ denote a homotopy which deforms V to a point. Since f is continuous at p, there is a neighborhood U of p in Z such that $f(U) \subset V$. Then, for some q in $[0, \infty)$, we have $X \times \{q\} \subset U$. Let $\alpha\colon X \times I \to Z - A$ denote a homotopy which deforms X onto $X \times \{q\}$. Define a homotopy $H\colon X \times I \to Z$ by

$$H(x, t) = \begin{cases} f(\alpha(x, 2t)), & (\text{if } 0 \leqslant t \leqslant \tfrac{1}{2}), \\ \beta(f(\alpha(x, 1)), 2t - 1), & (\text{if } \tfrac{1}{2} \leqslant t \leqslant 1). \end{cases}$$

It is easy to check that H is a homotopy which deforms X to a point in Z. This contradiction shows that X must be contractible.

COROLLARY. *A compactum X is an* AR-*space iff X is an* AFS-*space and X is an* ANR-*space.*

Proof. Suppose that X is an ANR-space and an AFS-space. Since every ANR-space is locally contractible, it follows from Theorem 1 that X is a contractible ANR-space, and hence an AR-space [1, p. 101]. This completes the proof since every AR-space is both an AFS-space and an ANR-space.

We remark that Proposition 1 in [4] shows that if X is an AFS-space which is LC^n, then X is C^n for $n = 0, 1, 2, \ldots$ It follows that (see [1, p. 122]), for the case of finite dimensional AFS-spaces, Theorem 1 is a consequence of Proposition 1 of [4].

THEOREM 2. *Every* AFS-*space is locally contractible.*

Proof. Let X be an AFS-space and suppose that X is not locally contractible at a point p. Then there is a neighborhood U of p in X which contains a decreasing sequence $V_1, V_2, \ldots$ of compact neighborhoods of p such that $\lim_{i \to \infty} \delta(V_i) = 0$ and no V_i is contractible in U. Let I denote the closed unit interval $[0, 1]$, and let $A_i = [1/2i, 1/(2i-1)]$ for $i = 1, 2, \ldots$ Consider a sequence of disjoint continua $Y_1, Y_2, \ldots$ with $\lim_{i \to \infty} \delta(Y_i) = 0$ obtained by first taking the disjoint union $\bigcup_{i=1}^{\infty} V_i$ and then letting $Y_i = \operatorname{cap} V_i A_i$ for $i = 1, 2, \ldots$ Let $Y = I \cup \bigcup_{i=1}^{\infty} Y_i$. Form the compactum Z_1 obtained by taking the disjoint union $X \cup Y$ and then identifying V_i in X with $V_i \times \{0\}$ in Y for $i = 1, 2, \ldots$, and identifying the point p in X with the point 0 in I.

Now suppose $f_1\colon Z_1 \to Z_1$ is a map whose fixed point set is precisely X. First we show that there are infinitely many sets of the form $f_1(A_i)$ such that $f_1(A_i) \cap (0, 1] = \emptyset$. To see this, suppose $f_1(A_i) \cap (0, 1] \neq \emptyset$ for all but finitely many of the sets $f_1(A_i)$, $i = 1, 2, \ldots$ Let W be a neighborhood of p in Z_1 such that $W \cap X = U$. Since f_1 is continuous at p, there is a neighborhood V of p in Z_1 such that $V \subset W$ and $f_1(V) \subset W$. Now let k be a positive integer such that $Y_k \subset V$ and $f_1(A_k) \cap (0, 1] \neq \emptyset$.

Suppose there is a point a in A_k and a neighborhood M of a in W such that $f_1(M) \subset I$. Then, for some $t_1 > 0$, $f_1(V_k \times \{t_1\}) \subset I$. Since f_1 is the identity on $V_k = V_k \times \{0\}$ and V_k can be deformed onto $V_k \times \{t_1\}$ in Y_k, it follows that V_k is contractible in $f_1(Y_k)$. Let $r\colon f_1(Y_k) \to f_1(Y_k) \cap X$ be a retraction defined by

$$(*) \qquad r(y) = \begin{cases} p & \text{if } y \in f_1(Y_k) \cap I, \\ y & \text{if } y \in f_1(Y_k) \cap X, \\ (x, 0) & \text{if } y = (x, t) \in f_1(Y_k) \cap \left(\bigcup_{i=1}^{\infty} Y_i\right). \end{cases}$$

It then follows that V_k is contractible in $r(f_1(Y_k)) \subset W \cap X = U$. This contradiction shows that no neighborhood M of a point a in A_k has the property that $f_1(M) \subset I$.

Now let N be a positive integer such that if $i > N$, then $Y_i \subset V$ and $f_1(A_i) \cap (0, 1] \neq \emptyset$. It follows from the above argument that if $i > N$, then $f_1(A_i) = A_j$ for some j. Moreover, since 0 is the only point in I which remains fixed under f_1, it is easy to show that if x, $f_1(x) \in I$, then $f_1(x) < x$. Consequently, if $i > N$, we have $f_1(A_i) = A_j$ for some $j > i$.

Define $\sigma(0) = N+1$ and, if m is a positive integer, let $\sigma(m)$ denote the unique positive integer satisfying the equation $f_1(A_{\sigma(m-1)}) = A_{\sigma(m)}$. Since $f_1(A_{N+1}) = A_{\sigma(1)}$, it follows that, for some $t_2 > 0$, $f_1(V_{N+1} \times \{t_2\}) \subset Y_{\sigma(1)} \cap (Z - (0, 1])$. Then, as in $(*)$, we may define a retraction

$$s\colon f_1(V_{N+1} \times [0, t_2]) \to f_1(V_{N+1} \times [0, t_2]) \cap X.$$

Since $V_{N+1} = V_{N+1} \times \{0\}$ is homotopic to $V_{N+1} \times \{t_2\}$ in $V_{N+1} \times [0, t_2]$, it follows that $C_1 = V_{N+1}$ is homotopic to $C_2 = sf_1(V_{N+1} \times \{t_2\})$ under a homotopy H_1 whose image, $\operatorname{Im} H_1$, lies in U. This argument can be repeated to obtain a sequence of sets $C_1, C_2, \ldots, C_i, \ldots$ in X and a corresponding sequence $H_1, H_2, \ldots, H_i, \ldots$ of homotopies with values in U such that the following properties are satisfied.

(1) $C_i \subset V_{\sigma(i-1)} \times \{0\}$ for $i = 1, 2, \ldots$

(2) $H_i\colon C_i \times I \to U$ with $H_i(C_i \times \{0\}) = C_i$ and $H_i(C_i \times \{1\}) = C_{i+1}$ for $i = 1, 2, \ldots$

(3) $\lim_{i\to\infty} \delta(H_i(C_i \times I)) = 0$.

It is possible to construct a homotypy $H\colon C_1 \times I \to U$ such that, for $i = 1, 2, \ldots$ $H(C_1 \times \{(i-1)/i\}) = C_i$ and $H(C_1 \times \{1\}) = p$. This contradicts the fact that $C_1' = V_{N+1}$ is not contractible in U. Consequently, for infinitely many of the integers $i = 1, 2, \ldots, f_1(A_i) \cap (0, 1] = \emptyset$.

Let $Q = \bigcup \{Y_i | \ f_1(A_i) \cap (0, 1] = \emptyset\}$. Then, as in (*), we may define a retraction $r_1\colon f_1(X \cup Q) \to X$. Let

$$Q_1 = \bigcup \{Y_i | \ Y_i \subset Q \text{ and } r_1 f_1(Y_i) \subset U\}.$$

Henceforth, we shall only consider the subsequence $Y_{i_1}, Y_{i_2}, \ldots$ of $\{Y_i\}_{i=1}^{\infty}$ such that $Q_1 = \bigcup_{j=1}^{\infty} Y_{i_j}$. To facilitate notation we shall relabel this subsequence $Y_1, Y_2, \ldots$ Since $r_1 f_1(Y_i) \subset U$, it follows that $\delta(r_1 f_1(A_i)) \neq 0$ for $i = 1, 2, \ldots$ Otherwise, some V_j would be contractible in U which is not possible. Thus, we may assume that $\delta(r_1 f_1(A_i)) = \lambda_i$ where $\lambda_1, \lambda_2, \ldots$ is a sequence of positive numbers which converges to 0.

Let a_i denote the midpoint of A_i, and let $0 = t_{i,0} < t_{i,1} < \ldots < t_{i,j} < \ldots$ be a sequence of numbers in $[0, \infty)$ such that $\lim_{j\to\infty} V_i \times \{t_{i,j}\} = a_i$. Set $b_i = r_1 f_1(a_i)$, $B_{i,j} = r_1 f_1(V_i \times \{t_{i,j}\})$, and let $Y_{i,j} = \operatorname{cap}(B_{i,j} A_{i,j})$ denote a null sequence of caps such that $\lim_{j\to\infty} Y_{i,j} = b_i$. Let $\{I_i\}_{i=1}^{\infty}$ be a null sequence of arcs such that $I_i \cap X = \{b_i\}$ and, for all $j = 1, 2, \ldots$, $A_{i,j} \subset I_i$. Form the compactum $Z_2 = X \cup \bigcup_{i,j=1}^{\infty} (Y_{i,j} \cup I_i)$.

Since X is an AFS-space, there is a map $f_2\colon Z_2 \to Z_2$ whose fixed point set is precisely X. Then, using arguments similar to previous arguments, there exists a set Q_2 which is the union of infinitely many sets of the form $Y_{i,j}$, and there exists a retraction $r_2\colon f_2(X \cup Q_2) \to X$ such that $r_2 f_2(Y_{i,j}) \subset U$ for each $Y_{i,j} \subset Q_2$. The process can be continued so that in the nth stage we obtain a retraction $r_n\colon f_n(X \cup Q_n) \to X$ such that $r_n f_n(Y_{i_1, i_2, \ldots, i_n}) \subset U$ for each of the $Y_{i_1, i_2, \ldots, i_n}$ lying in Q_n. In particular, there is a sequence $V_{j_1} = B_{j_1,0}, B_{j_1,j_2}, B_{j_1,j_2,j_3}, \ldots, B_{j_1,j_2,\ldots,j_n}, \ldots$ of sets lying in U, a point $q \in U$, and a sequence of homotopies $\{F_n\}_{n=1}^{\infty}$ with values in U such that the following properties are satisfied.

(1) $\lim_{n\to\infty} B_{j_1,j_2,\ldots,j_n} = \{q\}$.

(2) F_1 is a homotopy which deforms V_{j_1} onto B_{j_1,j_2} and, for $n > 1$, F_n deforms $B_{j_1,j_2,\ldots,j_n}$ onto $B_{j_1,j_2,\ldots,j_n,j_{n+1}}$.

(3) $\lim_{n\to\infty} \delta(\operatorname{Im} F_n) = 0$.

It is possible to construct a homotopy F with values in U which deforms V_{j_1} to the point q. This contradiction shows that X must be locally contractible as required.

Theorems 1 and 2 show that every AFS-space is contractible and locally contractible. Thus every finite dimensional AFS-space is an AR-space [1, p. 122]. Since every AR-space is an AFS-space, we have the following theorem.

THEOREM 3. *A finite dimensional compactum X is an* AFS-*space iff X is an* AR-*space.*

The following result was pointed out to the author by E. D. Tymchatyn. Since a retract of an AR-space is an AR-space [1, p. 101], the finite dimensional case follows from Theorem 3.

PROPOSITION. *Every retract of an* AFS-*space is an* AFS-*space.*

Proof. Let X be a retract of an AFS-space Y, and let $e\colon X \to Z$ be an embedding of X into a compactum Z. Let W be the identification space obtained by taking the disjoint union of Y and Z, and then identifying each $x \in X$ with $e(x) \in Z$. We shall regard X, Y, Z as subspaces of the compactum W. Since Y is an AFS-space, there is a mapping $g\colon W \to W$ whose fixed point set is precisely Y. Let $r\colon Y \to X$ be a retraction of Y onto X. Define a function $f\colon Z \to Z$ by

$$f(z) = \begin{cases} g(z) & \text{if } g(z) \in Z, \\ rg(z) & \text{if } g(z) \in Y - X. \end{cases}$$

It is easy to check that f is a mapping whose fixed point set is precisely X.

An example of a contractible LC^{∞} compactum which is not an AFS-space is given in [4]. We now use similar techniques to those found in [4] to show that a well-known example due to Borsuk [1, p. 126] is in fact an example of a contractible and locally contractible compactum which is not an AFS-space.

EXAMPLE. Consider the following subsets of the Hilbert cube Q^{∞} (for notation see [1, p. 10]):

$$X_0 = \{x = \{x_i\} | \ x_1 = 0\},$$

$$B_k = \{x = \{x_i\} | \ 1/(k+1) \leqslant x_1 \leqslant 1/k \text{ and } x_i = 0 \text{ for } i > k\} \text{ for } k = 1, 2, \ldots$$

The boundary $\operatorname{Bd} B_k$ of B_k is a $(k-1)$-sphere which we shall denote by X_k for $k = 1, 2, \ldots$ Let $X = X_0 \cup \bigcup_{k=1}^{\infty} X_k$. Then, if Y denotes the cone over X with vertex p, Y is a contractible and locally contractible compactum which is not an ANR-space [1, p. 126]. We now show that Y is not an AFS-space by constructing a compactum Z containing Y such that Y is not a fixed point set of Z.

Let $C_k = \operatorname{cap} X_k A_k$, $k = 1, 2, \ldots$, denote a sequence of caps in Hilbert space E^{∞} such that the following properties are satisfied.

(1) $C_k \cap Y = X_k \times \{0\} = X_k$ for $k = 1, 2, \ldots$

(2) $C_i \cap C_j = X_i \cap X_j$ for $i, j = 1, 2, \ldots, i \neq j$.

(3) $\lim_{k\to\infty} C_k = X_0$.

Define $Z = Y \cup \bigcup_{k=1}^{\infty} C_k$.

Suppose that $f\colon Z \to Z$ is a mapping whose fixed point set is precisely Y. Since $\lim_{k\to\infty} C_k = X_0$ and $f(X_0) = X_0$, there is a positive integer j such that $f(C_j) \subset Z - \{p\}$. Furthermore, since no point in A_j remains fixed under f, it follows that $f(A_j) \subset Z - (A_j \cup \{p\})$. Let r denote a retraction which maps $Z - (A_j \cup \{p\})$ onto X_j. Then $rf \mid C_j$ is a retraction from C_j onto $X_j = X_j \times \{0\}$. It then follows from the proof of Theorem 1 that X_j is contractible. This contradiction shows that Y is not a fixed point set of Z.

We remark that the above example together with Theorems 1 and 2 show that the class of AFS-spaces is properly contained in the class of contractible and locally contractible compacta. In view of this, it seems appropriate to pose the following question.

QUESTION. *Does the class of* AFS-*spaces coincide with the class of* AR-*spaces*?

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