The elementary theory of Abelian groups with m-chains of pure subgroups

by

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Abstract. An m-chain of pure subgroups is a chain $\hat{\mathfrak{A}} = [\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$ of Abelian groups such that $\mathfrak{A}_i$ is pure in $\mathfrak{A}_m$. These m-chains will be considered in an elementary language of group theory with additional predicates denoting the subgroups $\mathfrak{A}_i$. The well-known results of Szmielew (elimination of quantifiers, decidability) will be extended for the elementary theory of m-chains. These results are used to get the decidability of the $L(aa)$-theory of Abelian groups.

1. Introduction. An m-chain of pure subgroups of an Abelian group $\mathfrak{A}_m$ (short: m-chain) is a chain $\hat{\mathfrak{A}} = [\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$ of Abelian groups such that $\mathfrak{A}_i$ is pure in $\mathfrak{A}_m$. Let m be a fixed natural. We consider m-chains

$$\hat{\mathfrak{A}} = [\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m] \quad \text{and} \quad \hat{\mathfrak{B}} = [\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m].$$

By convention $\mathfrak{A}_0 = \{0\}$, $\mathfrak{B}_0 = \{0\}$. $f(x_1, \ldots, x_l) = \sum_{1 \leqslant i \leqslant l} k_i x_i$ is a k-term iff k_i are integers with $|k_i| \leqslant k$. To work in the group $\mathfrak{C}_m/\mathfrak{C}_{r-1}$ respectively $\mathfrak{C}_r/\mathfrak{C}_{r-1}$ we use the notations

$$(f(x_1, \ldots, x_l) = 0) \bmod \mathfrak{C}_{r-1} \quad \text{and} \quad (p^s | f(x_1, \ldots, x_l)) \bmod \mathfrak{C}_{r-1}.$$

DEFINITION 1.1. $\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$ is the following set of conditions:

(1) For every r with $0 < r \leqslant m$ and every k-term $f(x_1, \ldots, x_l)$

$$\hat{\mathfrak{A}} \models (f(a_1, \ldots, a_l) = 0) \bmod \mathfrak{A}_{r-1} \quad \text{iff} \quad \hat{\mathfrak{B}} \models (f(b_1, \ldots, b_l) = 0) \bmod \mathfrak{B}_{r-1}.$$

(2) For every r with $0 < r \leqslant m$, every $p^j \leqslant k$, and every k-term $f(x_1, \ldots, x_l)$

$$\hat{\mathfrak{A}} \models (p^j | f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1} \quad \text{iff} \quad \hat{\mathfrak{B}} \models (p^j | f(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}.$$

If $\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l$ are known we write Δ_k only. Remark that $\Delta_k \subseteq \Delta_h$ for $k \leqslant h$.

We define a recursive function $\pi(k)$ such that:

THEOREM 1.2. *Let $\hat{\mathfrak{A}} = [\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$ and $\hat{\mathfrak{B}} = [\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m]$ be two m-chains of pure subgroups such that $\mathfrak{A}_r/\mathfrak{A}_{r-1}$ and $\mathfrak{B}_r/\mathfrak{B}_{r-1}$ have the same Szmielew invariants. Then for every $a_{l+1} \in \mathfrak{A}_m$ $(b_{l+1} \in \mathfrak{B}_m)$ $\Delta_{\pi(k)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$ implies the existence of some $b_{l+1} \in \mathfrak{B}_m$ (resp. $a_{l+1} \in \mathfrak{A}_m$) such that*

$$\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1}).$$

This theorem extends the elimination procedure of Szmielew [12] for the elementary theory of Abelian groups. It follows the decidability of the elementary theory of m-chains with predicates denoting the pure subgroups.

Remark that the theory of Abelian groups with an additional predicate denoting a subgroup ([3], [9]+[10]) and the universal theory of lattices of pure subgroups of Abelian groups [11] are undecidable.

Furthermore Theorem 1.2 is used to extend the Szmielew results to the $L(aa)$-theory of Abelian groups. The decidability of the $L(aa)$-theory of Abelian groups is shown. This continues [2]. The $L(aa)$ results are obtained by Eklof and Mekler independently [6].

2. Preliminaries. Let p, q be primes, $h, i, j, k, l, m, n, r, s, t, u$ be naturals exept in terms $\Sigma k_i x_i$ where the k_i denote integers. $\varkappa$ and λ are used for cardinals. By a groups we understand always an additively written Abelian group. Let $k\mathfrak{A}$ be the subgroup of all ka with $a \in \mathfrak{A}$, and $[k]\mathfrak{A}$ be the subgroup of all a with $ka = 0$. $\{0\}$ is used to denote the trivial group (or subgroup) containing "0" only. If $a_1, \ldots, a_l \in \mathfrak{A}$ let $\{a_1, \ldots, a_l\}$ be the subgroup of $\mathfrak{A}$ generated by $a_1, \ldots, a_l$. $\oplus$ denotes the direct sum and $(\oplus\mathfrak{A})^\lambda$ the λ-fold direct sum of $\mathfrak{A}$. Let $\xi(p^n)$ be the cyclic group of order p^n, $\xi(p^\infty)$ the group of type p^∞, and ξ_p the additive subgroup of all rationals a/b where $p \nmid b$.

Define for every Abelian group $\mathfrak{A}$:

$$\zeta_1(p, n, \mathfrak{A}) \underset{\mathrm{Df}}{=} \sup\{\lambda\colon \text{there is a subgroup of } \mathfrak{A} \text{ isomorphic to } (\oplus\xi(p^n))^\lambda\}.$$

$$\zeta_2(p, n, \mathfrak{A}) \underset{\mathrm{Df}}{=} \zeta_1(p, n, \mathfrak{A}/p^n\mathfrak{A}).$$

$$\zeta_3(p, n, \mathfrak{A}) \underset{\mathrm{Df}}{=} \sup\{\lambda\colon \text{there is a direct summand of } \mathfrak{A} \text{ isomorphic to } (\oplus\xi(p^n))^\lambda\}.$$

$$\zeta_4(\mathfrak{A}) \underset{\mathrm{Df}}{=} \begin{cases} 0, & \text{if there is some } m \text{ with } \mathfrak{A} \models \forall x(mx = 0), \\ 1, & \text{otherwise.} \end{cases}$$

Then the L-Szmielew invariants of $\mathfrak{A}$ are defined by

$$\zeta_i^L(p, n, \mathfrak{A}) = \zeta_i(p, n, \mathfrak{A}) \cap \omega \quad \text{for} \quad 1 \leqslant i \leqslant 3$$

and

$$\zeta_4^L(\mathfrak{A}) = \zeta_4(\mathfrak{A}).$$

They are described by the following Szmielew basic sentences:

$$\zeta_1^L(p, n, k) \underset{\mathrm{Df}}{=} \exists x_1 \ldots x_k\Big(\bigwedge_i (px_i = 0 \wedge p^{n-1}|x_i) \wedge \bigwedge_{\langle \ldots, k_i, \ldots\rangle \in S} \neg\big(\sum_{1 \leqslant i \leqslant k} k_i x_i = 0\big)\Big),$$

$$\zeta_2^L(p, n, k) \underset{\mathrm{Df}}{=} \exists x_1 \ldots x_k\Big(\bigwedge_i p^{n-1}|x_i \wedge \bigwedge_{\langle \ldots, k_i, \ldots\rangle \in S} \neg\big(p^n | \sum_{1 \leqslant i \leqslant k} k_i x_i\big)\Big),$$

$$\zeta_3^L(p, n, k) \underset{\mathrm{Df}}{=} \exists x_1 \ldots x_k\Big(\bigwedge_i (px_i = 0 \wedge p^{n-1}|x_i) \wedge \bigwedge_{\langle \ldots, k_i, \ldots\rangle \in S} \neg\big(p^n \nmid \sum_{1 \leqslant i \leqslant k} k_i x_i\big)\Big),$$

where S is the set of all $\langle k_1, \ldots, k_k\rangle \neq \langle 0, \ldots, 0\rangle$ with $0 \leqslant k_i < p$, $\zeta_4^L(m) \underset{\mathrm{Df}}{=} \forall x(mx = 0)$.

Remark that $\mathfrak{A}$ and $\mathfrak{B}$ have the same L-Szmielew invariants iff they fulfil the same Szmielew basic sentences. $\mathfrak{A} \models \zeta_i^L(p, n, k)$ iff $\zeta_i^L(p, n, \mathfrak{A}) \geqslant k$. Notice the following useful lemma:

LEMMA 2.1. (i) [12] *$\zeta_i(p, n, \mathfrak{A}) = \zeta_i(p, m, \mathfrak{A}) + \sum_{j=1}^{m-1} \zeta_3(p, j, \mathfrak{A})$ for $m > n$ and $1 \leqslant i \leqslant 2$.*

(ii) [7] *If $\mathfrak{B}$ is a pure subgroup of $\mathfrak{A}$ then $\zeta_i(p, n, \mathfrak{A}) = \zeta_i(p, n, \mathfrak{B}) + \zeta_i(p, n, \mathfrak{A}/\mathfrak{B})$ for $1 \leqslant i \leqslant 3$.*

Let K_m be the class of all m-chains. The elementary theory AG_m of K_m is formulated in the language L_m with a function symbol "+" for the group operation and unary predicates $P_i(1 \leqslant i < m)$ where P_i denotes the pure subgroup $\mathfrak{A}_i$ of $\mathfrak{A}_m$ in the m-chain $[\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$. Then K_m is EC_Δ in L_m. $L = L_1$ is the ordinary elementary language of group theory. Outside of L_m we use $P_0(x) \underset{\mathrm{Df}}{=} x = 0$ and $P_m(x) \underset{\mathrm{Df}}{=} x = x$.

Let L_m^* be the extension of L_m by the following definitions: "0" is a new constant uniquely determined by $\forall y(y + 0 = y)$.

$$x - z = y \underset{\mathrm{Df}}{\Leftrightarrow} y + z = x,$$

$$p^n|x \underset{\mathrm{Df}}{\Leftrightarrow} \exists y(p^n y = x),$$

$$(x = y) \bmod P_i \underset{\mathrm{Df}}{\Leftrightarrow} \exists z \in P_i(x = y + z),$$

$$(p^n|x) \bmod P_i \underset{\mathrm{Df}}{\Leftrightarrow} \exists y \exists z \in P_i(p^n y + z = x).$$

Let AG_m^* be the corresponding elementary theory. As usual $\equiv_{L_m}$ denotes elementary equivalence with respect to L_m. The L_m-basic sentences are defined to be the Szmielew basic sentences for the factor groups P_{j+1}/P_j $(0 \leqslant j < m)$ formulated in L_m.

3. Elimination of quantifiers. In this chapter we define $\pi(k)$ and prove Theorem 1.2.

LEMMA 3.1. *Let $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}$ be m-chains of pure subgroups, $a_1, \ldots, a_l \in \mathfrak{A}_m$, and $b_1, \ldots, b_l \in \mathfrak{B}_m$ such that $\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$. Furthermore assume $k\mathfrak{B}_m = \{0\}$, $k\mathfrak{A}_m = \{0\}$, and $\zeta_3^L(p, j, \mathfrak{A}_{i+1}/\mathfrak{A}_i) = \zeta_3^L(p, j, \mathfrak{B}_{i+1}/\mathfrak{B}_i)$ for $0 \leqslant i < m$ and $p^j|k$. Then for every $a_{l+1} \in \mathfrak{A}_m$ (resp. $b_{l+1} \in \mathfrak{B}_m$) there is some*

$$b_{l+1} \in \mathfrak{B}_m (a_{l+1} \in \mathfrak{A}_m) \quad \textit{such that} \quad \Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1}).$$

Proof. If $k\mathfrak{C} = \{0\}$ then $c = 0$ iff $p^s|c$ for every $p^s|k$. Therefore it is possible to assume w.l.o.g. $k = p^s$. The lemma will be proved for every $a_{l+1} \in \mathfrak{A}_r \setminus \mathfrak{A}_{r-1}$ and every $b_{l+1} \in \mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$ $(0 \leqslant r < m, \mathfrak{A}_{-1} = \mathfrak{B}_{-1} = \emptyset)$ by induction on r. The case $r = 0$ is trivial. Suppose that the assertion is proved for $r-1$. First let us consider the following case:

(*) There is some p^s-term $f(x_1, \ldots, x_l)$ such that $(a_{l+1} = f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$. Then w.l.o.g. $f(a_1, \ldots, a_l) = 0$ and the assertion follows from the induction hypothesis.

In the general case induction on h where $(p^{s-h}|a_{l+1}) \bmod \mathfrak{A}_{r-1}$ is used. The case $h = 0$ is clear by (*). Assume if $(p^{s-h}|a_{l+1}) \bmod \mathfrak{A}_{r-1}$ resp. $(p^{s-h}|b_{l+1}) \bmod \mathfrak{B}_{r-1}$ the assertion is proved.

Now the situation $(p^{s-h-1}|a_{l+1}) \bmod \mathfrak{A}_{r-1}$ but $(p^{s-h}\nmid a_{l+1}) \bmod \mathfrak{A}_{r-1}$ will be considered. By induction on r w.l.o.g. $p^{s-h-1}|a_{l+1}$. By the induction about h w.l.o.g. $pa_{l+1} = a_{j_0}$ for some j_0 with $1 \leqslant j_0 \leqslant l$ because $(p^{s-h}|pa_{l+1}) \bmod \mathfrak{A}_{r-1}$. If $(p^{s-h}|a_{l+1} - f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$ for some p^s-term $f(x_1, \ldots, x_l)$ then by the induction on h and (*) there is some b_{l+1} with the desired properties. Therefore w.l.o.g. $(p^{s-h}\nmid a_{l+1} - f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$ for every p^s-term $f(x_1, \ldots, x_l)$.

Now it is sufficient to find some $b_{l+1} \in \mathfrak{B}_r$ such that $pb_{l+1} = b_{j_0}$, $p^{s-h-1}|b_{l+1}$, and $(p^{s-h}\nmid b_{l+1} - f(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}$ for every p^s-term $f(x_1, \ldots, x_l)$.

By Δ_k there is some $b' \in \mathfrak{B}_m$ with $p^{s-h-1}|b'$ and $pb' = b_{j_0}$. If

$$(p^{s-h}\nmid b' - f(b_1, \ldots, b_L)) \bmod \mathfrak{B}_{r-1}$$

for every p^s-term $f(x_1, \ldots, x_l)$ there is nothing to do. Otherwise assume

$$(p^{s-h}|b' - f_0(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}.$$

We can assume w.l.o.g. $(p^{s-h}|b') \bmod \mathfrak{B}_{r-1}$. Otherwise by induction on r w.l.o.g. $p^{s-h}|b' - f(b_1, \ldots, b_l)$. This would imply $p^{s-h-1}|f(b_1, \ldots, b_l)$ and therefore $p^{s-h-1}|a_{l+1} - f(a_1, \ldots, a_l)$. Then you could replace a_{l+1} by $a_{l+1} - f(a_1, \ldots, a_l)$ and b' by $b' - f(b_1, \ldots, b_l)$.

Now by induction on h there is some $a' \in \mathfrak{A}_r$ such that

$$\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, a', b_1, \ldots, b_l, b').$$

Then $pa_{l+1} = pa'$, $p^{s-h-1}|a'$ and $(p^{s-h}|a') \bmod \mathfrak{A}_{r-1}$. $pa_{l+1} = pa'$ implies $a_{l+1} = a' + c$ with $pc = 0$, $p^{s-h-1}|c$, and $(p^{s-h}\nmid c - f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$ for every p^s-term $f(x_1, \ldots, x_l)$. Therefore w.l.o.g. $pa_{l+1} = 0$, $p^{s-h-1}|a_{l+1}$, and $(p^{s-h}\nmid a_{l+1} - f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$ for every p^s-term $f(x_1, \ldots, x_l)$. Use $\bar{a}_i$, $\bar{c}_i$, $\bar{b}_j$ to denote elements of $\mathfrak{A}_r/\mathfrak{A}_{r-1}$ resp. $\mathfrak{B}_r/\mathfrak{B}_{r-1}$.

Let l_1 be the maximal natural ($\geqslant 0$) such that there are $\bar{c}_1, \ldots, \bar{c}_{l_1} \in [p]\{\bar{a}_1, \ldots, \bar{a}_l\}$ with $p\bar{c}_i = \bar{0}$, $p^{s-h-1}|\bar{c}_i$, and $p^{s-h}\nmid \sum_{1 \leqslant j \leqslant l_1} k_j \bar{c}_j$ for any nontrivial l_1-tupel $\langle k_1, \ldots, k_{l_1}\rangle$ with $0 \leqslant k_j < p$. The existence of a_{l+1} implies

$$\zeta_3(p, s-h, \mathfrak{B}_r/\mathfrak{B}_{r-1}) = \zeta_3(p, s-h, \mathfrak{A}_r/\mathfrak{A}_{r-1}) > l_1.$$

Then there is some $\overline{b_{l+1}} \in \mathfrak{B}_r/\mathfrak{B}_{r-1}$ with $p\bar{b}_{l+1} = \bar{0}$, $p^{s-h-1}|\overline{b_{l+1}}$, and $p^{s-h}\nmid \overline{b_{l+1}} - f(\bar{b}_1, \ldots, \bar{b}_l)$ for every p^s-term $f(x_1, \ldots, x_l)$. Let b^* be an element of $\mathfrak{B}_r$ such that $p^{s-h-1}b^*$ is in $\overline{b_{l+1}}$. Since $\mathfrak{B}_{r-1}$ is a pure subgroup there is some $d \in \mathfrak{B}_{r-1}$ such that $p^{s-h}b^* = p^{s-h}d$. Then $b_{l+1} = p^{s-h-1}(b^* - d)$ is an element of $\overline{b_{l+1}}$ with $pb_{l+1} = 0$ and p^{s-h-1}/b_{l+1}. Therefore b_{l+1} has the desired properties. ∎

DEFINITION 3.2. For every natural k let $k^\square$ be the *lowest common multiple of all j with $0 < j \leqslant k$.*

DEFINITION 3.3. Let $\pi(k)$ be $\pi(k, m)$ where $\pi(k, r)$ is defined for $0 \leqslant r \leqslant m$ by induction on r such that

$$\pi(k, r) \geqslant \pi(k, r-1), \quad \text{and} \quad \pi(k, 0) = 2kk^\square.$$

Now assume $r \geqslant 1$. For $u|k^{\square 2}$ define $\mu(k, r, u)$ by induction on $\sum_i s_i$ if $u = \prod_i p_i^{s_i}$:

$$\mu(k, r, 1) = \pi(2k^2k^{\square 2}, r-1).$$

For every prime p $\mu'(k, r, p) = \mu(\mu(2k, r, 1), r, 1)$. Since $\mu'(k, r, p)$ does not depend on p and $\mu(k, r, u)$ depends on $\sum_i s_i$ only it is possible to define:

$$\mu(k, r, u) = \mu(\mu'(k, r, p), r, u/p) \quad \text{for} \quad p|u.$$

Then $\pi(k, r) = \mu(k, r, k^{\square 2})$. Remark $\mu(k, r, u) \geqslant \mu(k, r, v)$ if $v|u$.

THEOREM 1.2. *Let $\hat{\mathfrak{A}} = [\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$ and $\hat{\mathfrak{B}} = [\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m]$ be two m-chains of pure subgroups such that $\mathfrak{A}_r/\mathfrak{A}_{r-1}$ and $\mathfrak{B}_r/\mathfrak{B}_{r-1}$ have the same Szmielew invariants. Then for every $a_{l+1} \in \mathfrak{A}_m$ ($b_{l+1} \in \mathfrak{B}_m$) $\Delta_{\pi(k)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$ implies the existence of some $b_{l+1} \in \mathfrak{B}_m$ (resp. $a_{l+1} \in \mathfrak{A}_m$) such that*

$$\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1}).$$

Proof. Let a_{l+1} be an element of $\mathfrak{A}_r \setminus \mathfrak{A}_{r-1}$ (resp. $b_{l+1} \in \mathfrak{B}_r \setminus \mathfrak{B}_{r-1}$) for some r with $0 \leqslant r \leqslant m$ ($\mathfrak{A}_{-1} = \emptyset = \mathfrak{B}_{-1}$). By induction on r it will be proved that

$$\Delta_{\pi(k,r)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$$

implies the existence of some $b_{l+1} \in \mathfrak{B}_m$ with $\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1})$. By definition it is $\pi(k, r) \leqslant \pi(k)$. If $r = 0$ there is nothing to do. Assume for $r-1$ that the assertion is true.

Case 1. $(k^{\square 2}a_{l+1} \neq f(a_1, \ldots, a_l)) \bmod \mathfrak{A}_{r-1}$ for every $kk^{\square 2}$-term $f(x_1, \ldots, x_l)$. This condition implies $(g(a_1, \ldots, a_l) \neq ua_{l+1}) \bmod \mathfrak{A}_{r-1}$ for every k-term $g(x_1, \ldots, x_l)$ and $0 < |u| \leqslant k$. Let $\mathfrak{C}_1$ be a pure subgroup of $\mathfrak{C}_2$. Then $\mathfrak{C}_1/h\mathfrak{C}_1 \hookrightarrow \mathfrak{C}_1 + h\mathfrak{C}_2/h\mathfrak{C}_2$ is a pure subgroup of $\mathfrak{C}_2/h\mathfrak{C}_2$ under the natural embedding. Therefore it is possible to consider the m-chains $\hat{\mathfrak{A}}/h\hat{\mathfrak{A}} = [\tilde{\mathfrak{A}}_1 \subseteq \ldots \subseteq \tilde{\mathfrak{A}}_m]$ and $\hat{\mathfrak{B}}/h\hat{\mathfrak{B}} = [\tilde{\mathfrak{B}}_1 \subseteq \ldots \subseteq \tilde{\mathfrak{B}}_m]$ where $\tilde{\mathfrak{A}}_r \hookrightarrow \mathfrak{A}_r/h\mathfrak{A}_r$ and $\tilde{\mathfrak{B}}_r \hookrightarrow \mathfrak{B}_r/h\mathfrak{B}_r$. Let h be $k^\square$ and use $\tilde{a}_i$ and $\tilde{b}_j$ to denote the images of a_i in $\hat{\mathfrak{A}}/h\hat{\mathfrak{A}}$ resp. of b_j in $\hat{\mathfrak{B}}/h\hat{\mathfrak{B}}$. Then $k^\square(\hat{\mathfrak{A}}/k^\square\hat{\mathfrak{A}}) = 0$, $k^\square(\hat{\mathfrak{B}}/k^\square\hat{\mathfrak{B}}) = 0$, and $\tilde{\mathfrak{A}}_r/\tilde{\mathfrak{A}}_{r-1}$ and $\tilde{\mathfrak{B}}_r/\tilde{\mathfrak{B}}_{r-1}$ have the same Szmielew invariants. $\Delta_{k^\square}(\hat{\mathfrak{A}}/k^\square\hat{\mathfrak{A}}, \hat{\mathfrak{B}}/k^\square\hat{\mathfrak{B}}, \tilde{a}_1, \ldots, \tilde{a}_l, \tilde{b}_1, \ldots, \tilde{b}_l)$ is fulfilled. Then

Since $\pi(k, r) \geqslant k^\square$ $\Delta_{k^\square}(\hat{\mathfrak{A}}/k^\square\hat{\mathfrak{A}}, \hat{\mathfrak{B}}/k^\square\hat{\mathfrak{B}}, \tilde{a}_1, \ldots, \tilde{a}_l, \tilde{b}_1, \ldots, \tilde{b}_l)$ is fulfilled. Then by Lemma 3.1 there is some $\tilde{b}_{l+1}$ such that

$$\Delta_{k^\square}(\hat{\mathfrak{A}}/k^\square\hat{\mathfrak{A}}, \hat{\mathfrak{B}}/k^\square\hat{\mathfrak{B}}, \tilde{a}_1, \ldots, \tilde{a}_{l+1}, \tilde{b}_1, \ldots, \tilde{b}_{l+1}).$$

Let b^* be an element of $\tilde{b}_{l+1}$ in $\mathfrak{B}_r$. Then $\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_l, b^*)$ (2) is fulfilled. Let $\theta(k^{\Box^2}(\mathfrak{A}_r/\mathfrak{A}_{r-1}))$ be the number of elements $f(\bar{a}_1, \ldots, \bar{a}_l)$ in $k^{\Box^2}(\mathfrak{A}_r/\mathfrak{A}_{r-1})$ where $f(x_1, \ldots, x_l)$ is a $kk^{\Box}$-term.

By assumption $\mathrm{card}(k^{\Box^2}(\mathfrak{A}_r/\mathfrak{A}_{r-1}))>\theta$. Since $\mathfrak{A}_r/\mathfrak{A}_{r-1}$ and $\mathfrak{B}_r/\mathfrak{B}_{r-1}$ have the same Szmielew invariants $\mathrm{card}(k^{\Box^2}(\mathfrak{A}_r/\mathfrak{A}_{r-1})) = \mathrm{card}(k^{\Box^2}(\mathfrak{B}_r/\mathfrak{B}_{r-1}))<\omega$ or $\mathrm{card}(k^{\Box^2}(\mathfrak{B}_r/\mathfrak{B}_{r-1}))\geqslant\omega$. Furthermore $\theta(k^{\Box^2}(\mathfrak{A}_r/\mathfrak{A}_{r-1})) = \theta(k^{\Box^2}(\mathfrak{B}_r/\mathfrak{B}_{r-1}))$ by $\Delta_{\pi(k)}$ and $\pi(k, r)\geqslant 2kk^{\Box}$. Therefore there is some $k^{\Box^2}b' \in \mathfrak{B}_r$ such that

$$(k^{\Box^2}b' \neq f(b_1, \ldots, b_l)-k^{\Box}b^*)\bmod \mathfrak{B}_{r-1} \quad \text{for every } kk^{\Box}\text{-term } f(x_1, \ldots, x_l).$$

If $b_{l+1} = b^*+k^{\Box}b'$ then $(b_{l+1} = b^*)\bmod(k^{\Box}\hat{\mathfrak{B}})$ and $(g(b_1, \ldots, b_l) \neq ub_{l+1})\bmod\mathfrak{B}_{r-1}$ for every k-term $g(x_1, \ldots, x_l)$ and $0<|u|\leqslant k$. It follows

$$\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1}).$$

Case 2. There is some u with $u|k^{\Box^2}$ such that

(1) $(f(a_1, \ldots, a_l) = ua_{l+1})\bmod\mathfrak{A}_{r-1}$ for some $kk^{\Box^2}$-term $f(x_1, \ldots, x_l)$.

Remember $\pi(k,r) = \pi(k, r, k^{\Box^2})$ and $\mu(k, r, u)\geqslant\mu(k, r, v)$ if $v|u$. In (1) let u be $\prod_i p_i^{s_i}$, then $p_i\leqslant k$. For $u|k^{\Box^2}$ it is sufficient to show the following by induction on $\sum_i s_i$ (short: u-induction):

(2) Let $a_{l+1}\in\mathfrak{A}_m$ (resp. $b_{l+1}\in\mathfrak{B}_m$) be an element fulfilling (1). Then

$$\Delta_{\mu(k,r,u)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$$

implies the existence of some $b_{l+1}\in\mathfrak{B}_m$ $(a_{l+1}\in\mathfrak{A}_m)$ such that

$$\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1}).$$

The case $u = 1$ follows from the induction hypothesis about r because $\mu(k, r, 1)\geqslant\pi(2k^2k^{\Box^2}, r-1)$. Now assume $u>1$ and that for every $v|u$ with $v<u$ the assertion (2) is true. Let p be a prime with $p|u$. Then $p\leqslant k$ and $k>1$. Since $\mu(k, r, u)\geqslant\mu(\mu'(k, r, p), r, u/p)$ by u-induction we can confine us to the case

(1′) $pa_{l+1} = a_{j_0}$ for some j_0 with $1\leqslant j_0\leqslant l$ and

$$\Delta_{\mu'(k,r,p)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l).$$

Consider some equation

(3) $(u_1a_{l+1} = f_1(a_1, \ldots, a_l))\bmod\mathfrak{A}_{r-1}$ with $p\nmid u_1$, $0<u_1\leqslant k$, and $f_1(x_1, \ldots, x_l)$ k-term.

Then $1 = u_1s_1+ps_2$ with $|s_i|\leqslant k$ and therefore

$$(a_{l+1} = s_1f_1(a_1, \ldots, a_l)+s_2a_{j_0})\bmod\mathfrak{A}_{r-1}.$$

That means $(a_{l+1} = g(a_1, \ldots, a_l))\bmod\mathfrak{A}_{r-1}$ where $g(x_1, \ldots, x_l)$ is some $2k^2$-term. Using u-induction and $\mu'(k, r, p)\geqslant\mu(k, r, 1)$ (2) is proved.

Therefore it is possible to confine us to the case that (3) is not fulfilled for any k-term $f_1(x_1, \ldots, x_l)$ and any u_1 with $0<u_1\leqslant k$ and $p\nmid u_1$. As shown above $(b_{l+1}\neq g(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every $2k^2$-term $g(x_1, \ldots, x_l)$ implies $(u_1b_{l+1}\neq f_1(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every k-term $f_1(x_1, \ldots, x_l)$ and every $1\leqslant u_1\leqslant k$ with $p\nmid u_1$. Therefore it is sufficient to get some $b_{l+1}\in\mathfrak{B}_r$ with $pb_{l+1} = b_{j_0}$, $(b_{l+1}\neq g(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every $2k^2$-term $g(x_1, \ldots, x_l)$, and fulfilling Δ_k (2). To ensure $\Delta_k(2)$ we consider $\hat{\mathfrak{A}}/k^{\Box^2}\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}/k^{\Box^2}\hat{\mathfrak{B}}$ as above. Let $\tilde{\tilde{a}}_i$ and $\tilde{\tilde{b}}_j$ be the images of a_i in $\hat{\mathfrak{A}}/k^{\Box^2}\hat{\mathfrak{A}}$ resp. of b_j in $\hat{\mathfrak{B}}/k^{\Box^2}\hat{\mathfrak{B}}$. By Lemma 3.1 there is some $\tilde{\tilde{b}}_{l+1}$ such that

(4) $$\Delta_{\check{k}^{\Box^2}}(\hat{\mathfrak{A}}/k^{\Box^2}\hat{\mathfrak{A}}, \hat{\mathfrak{B}}/k^{\Box^2}\hat{\mathfrak{B}}, \tilde{\tilde{a}}_1, \ldots, \tilde{\tilde{a}}_{l+1}, \tilde{\tilde{b}}_1, \ldots, \tilde{\tilde{b}}_{l+1}).$$

$(\mu'(k, r, p)\geqslant k^{\Box^2})$. If $b'\in\mathfrak{B}_r$ is some element of $\tilde{\tilde{b}}_{l+1}$ then $pb' = b_{j_0}+k^{\Box^2}d$ for some $d\in\mathfrak{B}_r$. Therefore $b^* = b'-(k^{\Box^2}/p)d\in\mathfrak{B}_r$ has the properties $(b^* = b')\bmod(k^{\Box}\hat{\mathfrak{B}})$ and $pb^* = b_{j_0}$. Then (4) implies

(5) $\Delta_{k^{\Box}}(\hat{\mathfrak{A}}/k^{\Box}\hat{\mathfrak{A}}, \hat{\mathfrak{B}}/k^{\Box}\hat{\mathfrak{B}}, \tilde{a}_1, \ldots, \tilde{a}_{l+1}, \tilde{b}_1, \ldots, \tilde{b}_l, \tilde{b}^*)$ where $\tilde{a}_i$ and $\tilde{b}_j$ are the corresponding images of a_i resp. b_j.

If now $(b^*\neq g(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every $2k^2$-term $g(x_1, \ldots, x_l)$ then $b_{l+1} = b^*$ has the desired properties. Otherwise assume $(b^* = g_0(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ where $g_0(x_1, \ldots, x_l)$ is a $2k^2$-term. Since $\mu'(k, r, p)\geqslant\mu(\mu(2k, r, 1), r, 1)$ it is possible to apply u-induction to b^*. You get some a^* such that

(6) $$\Delta_{\mu(2k,r,1)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, a^*, b_1, \ldots, b_l, b^*).$$

Then $pa_{l+1} = pa^* = a_{j_0}$ and therefore $a_{l+1} = a^*+c$ with $pc = 0$. $c\in\mathfrak{A}_r$. By (5) $(\tilde{b}^* = \widetilde{g_0(b_1, \ldots, b_l)})\bmod\tilde{\mathfrak{B}}_{r-1}$ implies $(\tilde{a}_{l+1} = \widetilde{g_0(a_1, \ldots, a_l)})\bmod\tilde{\mathfrak{A}}_{r-1}$. Since by (6) $(\tilde{a}^* = \widetilde{g_0(a_1, \ldots, a_l)})\bmod\tilde{\mathfrak{A}}_{r-1}$ it follows $(\tilde{a}^* = \tilde{a}_{l+1})\bmod\tilde{\mathfrak{A}}_{r-1}$. Therefore $(k^{\Box}|c)\bmod\mathfrak{A}_{r-1}$.

If $(c = g'(a_1, \ldots, a_l))\bmod\mathfrak{A}_{r-1}$ for some $4k^2$-term $g'(x_1, \ldots, x_l)$ then $(a_{l+1} = a^*+g'(a_1, \ldots, a_l) = g''(a_1, \ldots, a_l, a^*))\bmod\mathfrak{A}_{r-1}$ where $g''(x_1, \ldots, x_l, x)$ is a $4k^2$-term. By u-induction the assertion follows from (6). Otherwise let s be the maximal natural such that $p^s\leqslant k$. $\bar{a}$, $\bar{b}$ are used to denote the image of a in $\mathfrak{A}_r/\mathfrak{A}_{r-1}$ resp. of b in $\mathfrak{B}_r/\mathfrak{B}_{r-1}$. Then $\bar{c}\in[p](p^s(\mathfrak{A}_r/\mathfrak{A}_{r-1}))$ with $\bar{c}\neq g(\bar{a}_1, \ldots, \bar{a}_l)$ for every $4k^2$-term $g(x_1, \ldots, x_l)$. Since $\mu'(k, r, p)\geqslant p4k^2$ by $\Delta_{\mu'(k,r,p)}(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_l, b_1, \ldots, b_l)$ the number θ of elements $g(\bar{x}_1, \ldots, \bar{x}_l)$ where $g(x_1, \ldots, x_l)$ is a $4k^2$-term is the same in $[p](p^s(\mathfrak{A}_r/\mathfrak{A}_{r-1}))$ and $[p](p^s(\mathfrak{B}_r/\mathfrak{B}_{r-1}))$. Then there is some $\bar{d}\in[p](p^s(\mathfrak{B}_r/\mathfrak{B}_{r-1}))$ with $\bar{d}\neq g(\bar{b}_1, \ldots, \bar{b}_l)$ for every $4k^2$-term $g(x_1, \ldots, x_l)$. Otherwise you would get $\theta = p^{\zeta_1(p,s+1,\mathfrak{B}_r/\mathfrak{B}_{r-1})}$. But since $\zeta_1(p, s+1, \mathfrak{A}_r/\mathfrak{A}_{r-1}) = \zeta_1(p, s+1, \mathfrak{B}_r/\mathfrak{B}_{r-1})$ this contradicts the existence of c.

Now let $d'\in\mathfrak{B}_r$ be some element with $p^s\bar{d}' = \bar{d}$. Since $\mathfrak{B}_{r-1}$ is a pure subgroup there is some $e'\in\mathfrak{B}_{r-1}$ with $p^{s+1}e' = p^{s+1}d'\in\mathfrak{B}_{r-1}$. Then $d = p^s(d'-e')$ has the following properties: $pd = 0$, $p^s|d$, and $(d\neq g(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every $4k^2$-term $g(x_1, \ldots, x_l)$ since $(d = p^sd')\bmod\mathfrak{B}_{r-1}$. Define $b_{l+1} = b^*+d$. Since $k^{\Box}|d$ by (5) $\Delta_k(\hat{\mathfrak{A}}, \hat{\mathfrak{B}}, a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1})$ (2) is fulfilled. Furthermore $pb_{l+1} = b_{j_0}$, and $(b_{l+1}\neq g(b_1, \ldots, b_l))\bmod\mathfrak{B}_{r-1}$ for every $2k^2$-term $g(x_1, \ldots, x_l)$.

To prove this assume $(b_{l+1} = g_1(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}$ for some $2k^2$-term $g_1(x_1, \ldots, x_l)$. Since $(b^* = g_0(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}$ then

$$(d = g_1(b_1, \ldots, b_l) - g_0(b_1, \ldots, b_l)) \bmod \mathfrak{B}_{r-1}$$

where $(g_1 - g_0)(x_1, \ldots, x_l)$ is a $4k^2$-term. This contradicts the construction. ■

COROLLARY 3.4. *Let* $\mathfrak{A}$ *be* $[\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$, $\mathfrak{B}$ *be* $[\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m]$, *and* $\mathfrak{A}_0 = \{0\} = \mathfrak{B}_0$. *If* $\mathfrak{A}_{i+1}/\mathfrak{A}_i$ *and* $\mathfrak{B}_{i+1}/\mathfrak{B}_i$ *have the same Szmielew invariants* $(0 \leqslant i < m)$ *then* $\mathfrak{A} \equiv_{L_m} \mathfrak{B}$.

COROLLARY 3.5. *Every sentence of* L_m *is equivalent relative to* AG_m *to a Bolean combination of* L_m*-basic sentences.*

COROLLARY 3.6. *For every sentence* φ *of* L_m *fulfilled in some* m*-chain there are* m *direct sums* $\mathfrak{C}_i$ *of finitely many groups* $\mathfrak{Z}(p^n)$, $\mathfrak{Z}(p^\infty)$, *and* $\mathfrak{Z}_p$ *such that* φ *is true in* $[\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m]$ *where* $\mathfrak{A}_i = \bigoplus_{1 \leqslant j \leqslant i} \mathfrak{C}_i$.

COROLLARY 3.7. AG_m *is decidable.*

Furthermore AG_m admits elimination of quantifiers in the following sence:

COROLLARY 3.8. *Every formula of* L_m^* *is equivalent relative to* AG_m^* *to a Boolean combination of* L_m*-basic sentences and atomic formulas of* L_m^*.

4. The $L(aa)$-theory of Abelian groups. The basic investigations of the logic $L(aa)$ are done by Barwise, Kaufmann, and Makkai in [1]. They proved Completeness and Compactness. As well known in $L(ad)$ the generalized quantifier Q_1 "There exist uncountably many" is definable. To define $L(aa)$ — Ehrenfeucht games $\Gamma_n^{aa}(\mathfrak{C}_1, \mathfrak{C}_2)$ let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be arbitrary relational structures. As in the elementary case $\Gamma_n^{aa}(\mathfrak{C}_1, \mathfrak{C}_2)$ is a game for two players I and II about n rounds:

Player I begins every round and decides at first whether he plays an element — round or an cub — round. Consider the mth round:

Element round. Player I takes an element $c_i^m \in \mathfrak{C}_i$. Then II has to choose an element $c_{3-i}^m \in \mathfrak{C}_{3-i}$.

Cub-round. Player I takes an element X_i^m of the cub-filter of $\mathfrak{C}_i$, and then II takes an element X_{3-i}^m of the cub filter of $\mathfrak{C}_{3-i}$. Now player I takes an element C_{3-i}^m of X_{3-i}^m and at last II has to choose an C_i^m of X_i^m. Player II wins $\Gamma_n^{aa}(\mathfrak{C}_1, \mathfrak{C}_2)$ iff $\chi(c_1^m) = c_2^m$ is a partiell isomorphism with respect to the relations of the underlying language and the predicates P^j denoting the choosen C_1^j in $\mathfrak{C}_1$ and C_2^j in $\mathfrak{C}_2$.

THEOREM 4.1 [8]. *If player* II *has a winning strategy for every* n *in* $\Gamma_n^{aa}(\mathfrak{C}_1, \mathfrak{C}_2)$ *then* $\mathfrak{C}_1 \equiv_{aa} \mathfrak{C}_2$.

Let $\mathrm{AG}(aa)$ be the $L(aa)$-theory of Abelian groups. If you want to describe $\zeta_i(p, n, \mathfrak{A})$ with help of the elementary language L $\zeta_i(p, n, \mathfrak{A}) \geqslant k$ (for $1 \leqslant i \leqslant 3$) is expressible only. In the case of $L(aa)$ it is further possible to define $\zeta_i(p, n, \mathfrak{A}) \geqslant \omega_1$ and to refine $\zeta_4(\mathfrak{A})$. The $L(aa)$-invariants are the following:

$$\zeta_i^{L(aa)}(p, n, \mathfrak{A}) = \zeta_i(p, n, \mathfrak{A}) \cap \omega_1 \quad \text{for} \quad 1 \leqslant i \leqslant 3,$$
$$\zeta_4^{L(aa)}(\mathfrak{A}) = \zeta_4(\mathfrak{A}), \text{ and}$$
$$\zeta_5^{L(aa)}(\mathfrak{A}) = \begin{cases} 0, & \mathfrak{A} \models \neg Q_1 x(m|x) \text{ for some } m, \\ 1, & \text{otherwise.} \end{cases}$$

In [2] is proved the essential part of

LEMMA 4.2. *The following are equivalent*:

(i) $\mathfrak{A} \models \neg Q_1(m|x)$.

(ii) *There are subgroups* $\mathfrak{C}$ *and* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{C}$, $\mathrm{card}(\mathfrak{B}) < \omega_1$, *and* $m\mathfrak{C} = 0$.

(iii) *There is some countable subgroup* $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that* $m(\mathfrak{A}/\mathfrak{B}) = 0$.

To formulate the $L(aa)$-invariants we define the following $L(aa)$-basic sentences:

$$\zeta_i^{L(aa)}(p, n, k) = \zeta_i^L(p, n, k) \quad \text{for } 1 \leqslant i \leqslant 3,$$
$$\zeta_4^{L(aa)}(m) = \zeta_4^L(m),$$
$$\zeta_1^{L(aa)}(p, n, \omega_1) \underset{\mathrm{Df}}{=} aas\exists x(x \notin s \wedge px \in s \wedge (p^{n-1}|x) \bmod s).$$

(In $L(aa)$ this is equivalent to $Q_1 x(px = 0 \wedge p^{n-1}|x)$.)

$$\zeta_2^{L(aa)}(p, n, \omega_1) \underset{\mathrm{Df}}{=} aas\exists x((p^{n-1}|x) \bmod s \wedge (p^n \nmid x) \bmod s),$$
$$\zeta_3^{L(aa)}(p, n, \omega_1) \underset{\mathrm{Df}}{=} aas\exists x(px \in s \wedge (p^{n-1}|x) \bmod s \wedge (p^n \nmid x) \bmod s),$$
$$\zeta_5^{L(aa)}(m) \underset{\mathrm{Df}}{=} aas\forall x(mx \in s).$$

(By Lemma 4.2 this equivalent to $\neg Q_1 x(m|x)$.)

LEMMA 4.3. (i) *For* $1 \leqslant i \leqslant 3$ $\mathfrak{A} \models \zeta_i^{L(aa)}(p, n, \omega_1)$ *iff* $\zeta_i^{L(aa)}(p, n, \mathfrak{A}) \geqslant \omega_1$.

(ii) $\mathfrak{A} \models \zeta_5^{L(aa)}(m)$ *for some* m *iff* $\zeta_5^{L(aa)}(\mathfrak{A}) = 0$.

(iii) $\mathfrak{A}$ *and* $\mathfrak{B}$ *have the same* $L(aa)$*-invariants iff they fulfil the same* $L(aa)$*-basic sentences.*

THEOREM 4.4. *Two Abelian groups are* $L(aa)$*-equivalent iff they fulfil the same* $L(aa)$*-basic sentences.*

COROLLARY 4.5. *Every sentence of* $L(aa)$ *is equivalent relative to* $\mathrm{AG}(aa)$ *to a Boolean combination of* $L(ad)$*-basic sentences.*

COROLLARY 4.6. *Every* $L(aa)$*-sentence fulfilled in some Abelian group is true in a direct sum of finitely many groups* $(\oplus \mathfrak{Z}(p^n))_{\lambda_1}$, $(\oplus \mathfrak{Z}(p^\infty))^{\lambda_2}$, $(\oplus \mathfrak{Z}_p)^{\lambda_3}$ *where* λ_i *is finite or* ω_1.

COROLLARY 4.7. $\mathrm{AG}(aa)$ *is decidable.*

THEOREM 4.8. *For every* $L(aa)$*-formula* $\varphi(\vec{x})$ *with free element variables only there is a Boolean combination of* $L(aa)$*-basic sentences and atomic formulas of* L^* *equivalent to* $\varphi(\vec{x})$ *relative to* $\mathrm{AG}^*(aa)$.

To prove Theorems 4.4 and 4.8 let $\mathfrak{A}$ and $\mathfrak{B}$ be two Abelian groups with the same $L(aa)$-invariants (or equivalently that fulfil the same $L(aa)$-basic sentences). Furthermore let $\vec{a} = \langle a_1, \ldots, a_{l_1} \rangle$, and $\vec{b} = \langle b_1, \ldots, b_{l_1} \rangle$ be l_1-tupels of elements of $\mathfrak{A}$ resp. of $\mathfrak{B}$ that fulfil the same open formulas $\varphi(x_1, \ldots, x_{l_1})$ of L^*.

Let L' be the group language with a ternary relation "+". Using Theorem 4.1 it will be shown that $(\mathfrak{A}, \vec{a}) \equiv_{L'(aa)} (\mathfrak{B}, \vec{b})$. But then $(\mathfrak{A}, \vec{a}) \equiv_{L^*(aa)} (\mathfrak{B}, \vec{b})$, and Theorems 4.4 and 4.8 are proved.

Consider $\Gamma_n^{aa}((\mathfrak{A}, \vec{a}), (\mathfrak{B}, \vec{b}))$. Using Theorem 1.2 player II can play in such a way that after l_2 element-rounds and m cub-rounds he has the following situation Σ:

Let l be l_1+l_2, $a_{l_1+1}, \ldots, a_l \in \mathfrak{A}$, $b_{l_1+1}, \ldots, b_l \in \mathfrak{B}$ be the choosen elements, and $\mathfrak{A}_1, \ldots, \mathfrak{A}_m \subseteq \mathfrak{A}$, $\mathfrak{B}_1, \ldots, \mathfrak{B}_m \subseteq \mathfrak{B}$ be the choosen subsets in the cub-rounds.

1) $\mathfrak{A}_j$ and $\mathfrak{B}_j$ are countable subgroups of $\mathfrak{A}$ resp. $\mathfrak{B}$, and $\mathfrak{A}_1 \leqslant \ldots \leqslant \mathfrak{A}_m \leqslant \mathfrak{A}$, $\mathfrak{B}_1 \leqslant \ldots \leqslant \mathfrak{B}_m \leqslant \mathfrak{B}$.

2) If $\zeta_5(\mathfrak{A}) = \zeta_5(\mathfrak{B}) = 0$ then $\zeta_4(\mathfrak{A}/\mathfrak{A}_1) = \zeta_4(\mathfrak{B}/\mathfrak{B}_1) = 0$. Otherwise

$$\zeta_4(\mathfrak{A}_{j+1}/\mathfrak{A}_j) = \zeta_4(\mathfrak{B}_{j+1}/\mathfrak{B}_j) = 1$$

and

$$\zeta_4(\mathfrak{A}/\mathfrak{A}_m) = \zeta_4(\mathfrak{B}/\mathfrak{B}_m) = 1 .$$

If $\zeta_i(p, n, \mathfrak{A}) = \zeta_i(p, n, \mathfrak{B}) < \omega_1$ then

$$\zeta_i(p, n, \mathfrak{A}_{j+1}/\mathfrak{A}_j) = \zeta_i(p, n, \mathfrak{B}_{j+1}/\mathfrak{B}_j) = 0$$

for $1 \leqslant j < m$ and

$$\zeta_i(p, n, \mathfrak{A}/\mathfrak{A}_m) = \zeta_i(p, n, \mathfrak{B}/\mathfrak{B}_m) = 0 .$$

Otherwise

$$\zeta_i(p, n, \mathfrak{A}_{j+1}/\mathfrak{A}_j) = \zeta_i(p, n, \mathfrak{B}_{j+1}/\mathfrak{B}_j) = \omega$$

and

$$\zeta_i(p, n, \mathfrak{A}/\mathfrak{A}_m) = \zeta_i(p, n, \mathfrak{B}/\mathfrak{B}_m) = \omega_1 .$$

3) $\Delta_{\pi^{n-l_2}(1)}([\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m \subseteq \mathfrak{A}], [\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m \subseteq \mathfrak{B}], a_1, \ldots, a_l, b_1, \ldots, b_l)$.

To finish the proof it will be shown that II can ensure Σ in the next round. First assume it is an element-round. By Σ 1) $[\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m \subseteq \mathfrak{A}]$ and $[\mathfrak{B}_1 \subseteq \ldots \mathfrak{B}_m \subseteq \mathfrak{B}]$ are $(m+1)$-chains of pure subgroups where $\mathfrak{B}_1$ and $\mathfrak{A}_1$ have the same Szmielew invariants. By Σ 2) furthermore $\mathfrak{A}_{j+1}/\mathfrak{A}_j$ and $\mathfrak{B}_{j+1}/\mathfrak{B}_j$ for $1 \leqslant j < m$ and $\mathfrak{A}/\mathfrak{A}_m$ and $\mathfrak{B}/\mathfrak{B}_m$ have the same Szmielew invariants. If player I chooses w.l.o.g. $a_{l+1} \in \mathfrak{A}$ then Σ 3) implies by Theorem 1.2 the existence of some b_{l+1} such that

$$\Delta_{\pi^{n-l_1-1}(1)}([\mathfrak{A}_1 \subseteq \ldots \subseteq \mathfrak{A}_m \subseteq \mathfrak{A}], [\mathfrak{B}_1 \subseteq \ldots \subseteq \mathfrak{B}_m \subseteq \mathfrak{B}], a_1, \ldots, a_{l+1}, b_1, \ldots, b_{l+1})$$

$\Gamma_n^{aa}((\mathfrak{A}, \vec{a}), (\mathfrak{B}, \vec{b}))$ is in the situation Σ again. Now assume w.l.o.g. player I has choosen X in the cub filter of $\mathfrak{A}$. Then II chooses the following element Y of the cub filter of $\mathfrak{B}$:

1. Case $m = 0$. Then $Y = \{\mathfrak{B}_1 \colon b_1, \ldots, b_l \in \mathfrak{B}_1, \mathfrak{B}_1$ is a countable elementary subgroup of $\mathfrak{B}$, $\zeta_i(p, n, \mathfrak{B}/\mathfrak{B}_1) = 0$ if $\zeta_i(p, n, \mathfrak{B}) < \omega_1$, and $\zeta_4(\mathfrak{B}/\mathfrak{B}_1) = 0$ if $\zeta_5(\mathfrak{B}) = 0$.

2. Case $m > 0$. Then $Y = \{\mathfrak{B}_{m+1} \colon \mathfrak{B}_m \subseteq \mathfrak{B}_{m+1}, b_1, \ldots, b_l \in \mathfrak{B}_{m+1}, \mathfrak{B}_{m+1}$ is a countable elementary subgroup of $\mathfrak{B}$, and $\zeta_i(p, n, \mathfrak{B}_{m+1}/\mathfrak{B}_m) = \omega$ if $\zeta_i(p, n, \mathfrak{B}) \geqslant \omega_1\}$.

After I has choosen some $\mathfrak{B}_{m+1}$ of Y II can find some elementary subgroup $\mathfrak{A}_{m+1}$ of $\mathfrak{A}$ in X with the properties described in the definition of Y. It follows that $\Gamma_n^{aa}((\mathfrak{A}, \vec{a}), (\mathfrak{B}, \vec{b}))$ is again in the situation Σ. ■

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Accepté par la Rédaction le 1. 2. 1979