that $\mathfrak{A}$ not $\models A$ and $\mathfrak{B}$ not $\models B$. Consider then the ω-model $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$. Since it is an ω-model of HAS we have that

$$(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}} \models (A \vee B)\,.$$

However using Lemma 5.7 we see that it is not possible to find a bar B in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ such that for all nodes $k \in B$ either A or B is satisfied at k. But then $(A \vee B)$ is not true in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$.

Remark. It should be clear that similar methods could be applied to obtain other common closure properties of intuitionistic systems.

References

[1] D. van Dalen, *An interpretation of Intuitionistic Analysis*, Preprint No. 14, University of Utrecht, 1975.
[2] A. Heyting, *Intuitionism. An introduction*, North Holland, Amsterdam 1956.
[3] E. G. K. López-Escobar, *Infinite rules in finite systems*, Article in the Proceedings of the III Latin American Symposium in Mathematical Logic, North Holand, Amsterdam 1976.
[4] P. S. Novikov, *On the consistency of certain logical calculus*, Math. Sbornik 12 (54) No. 2 (1943), pp. 231–261.
[5] D. Prawitz, *Comments on Gentzen-type procedures and the classical notion of truth*, An article in the Proceedings of the International Summer Institute and Logic Colloquium, Kiel 1974, Lecture Notes in Math. 500, Springer Verlag, 1975.
[6] J. Shoenfield, *On a restricted* ω-rule, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 7 (1959), pp. 405–407.
[7] C. Smorynski, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Chap. V, Lecture Notes in Math. 344, Springer Verlag, 1973.
[8] M. Takahashi, *A theorem on the second-order arithmetic with the* ω*-rule*, J. Math. Soc. Japan 22 (1970), pp. 15–24.
[9] A. Troelstra, *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Lecture Notes in Math. 344, Springer Verlag, 1973.

Accepté par la Rédaction le 15. 1. 1979

Set-valued mappings on metric spaces

by

Brian Fisher (Leicester)

Abstract. In this paper we consider a mapping F of a complete metric space (X, d) into the class $B(X)$ of nonempty, bounded subsets of X. For A in $B(X)$ we define $FA = \bigcup_{a\in A} Fa$ and for A, B in $B(X)$ we define $\delta(A, B) = \sup\{d(a, b)\colon a\in A, b\in B\}$. It is proved that if F maps $B(X)$ into $B(X)$ and satisfies the inequality

$$\delta(Fx, Fy)\leqslant c.\max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}$$

for all x, y in X, where $0\leqslant c<1$, then there exists a unique point z in X such that $z\in Fz$ and further $Fz = \{z\}$.

In a paper by Kaulgud and Pai, see [3], they consider mappings F of a metric space (X, d) into either $b(X)$, the class of nonempty, closed and bounded subsets of X, or $\mathrm{Cpt}(X)$, the class of nonempty, compact subsets of X, or 2^X, the class of nonempty, closed subsets of X. The classes $b(X)$ and $\mathrm{Cpt}(X)$ are given the Hausdorff metric D induced by the metric d. With F satisfying various conditions, they prove a number of fixed point theorems for F, a fixed point being defined as a point z in X for which z is in the set Fz. For example, they prove the following theorem in which $d(x, A)$ with x in X and A in $\mathrm{Cpt}(X)$ is defined by

$$d(x, A) = \inf\{d(x, A)\colon a\in A\}\,.$$

THEOREM 1. *Let F be a mapping of a complete metric space (X, d) into* $\mathrm{Cpt}(X)$ *satisfying the inequality*

$$D(Fx, Fy)\leqslant a_1 d(x, Fx)+a_2 d(y, Fy)+a_3 d(x, Fy)+a_4 d(y, Fx)+a_5 d(x, y)$$

for all x, y in X, where $a_1, \ldots, a_5\geqslant 0$ and $a_1+\ldots+a_5<1$. Then F has a fixed point in X.

In the following we consider a mapping F of a metric space (X, d) into $B(X)$, the class of all nonempty, bounded subsets of X. We define the function $\delta(A, B)$ with A, B in $B(X)$ by

$$\delta(A, B) = \sup\{d(a, b)\colon a\in A, b\in B\}\,.$$

If the set A consists of a single point a we write

$$\delta(A, B) = \delta(a, B)$$

and if B also consists of single point b we write

$$\delta(A, B) = \delta(a, b) = d(a, b).$$

It follows easily from the definition of δ that

$$\delta(A, B) = \delta(B, A) \geqslant 0 \quad \text{and} \quad \delta(A, B) \leqslant \delta(A, C) + \delta(C, B)$$

for A, B, C in $B(X)$.

Further, if A is any nonempty subset of X, we define the set FA by

$$FA = \bigcup_{a \in A} Fa.$$

We now prove the following theorem

THEOREM 2. *Let F be a mapping of a complete metric space (X, d) into $B(X)$ satisfying the inequality*

(1) $$\delta(Fx, Fy) \leqslant c\max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$. If F also maps $B(X)$ into itself, that is $FA \in B(X)$ for $A \in B(X)$, then F has a unique fixed point z in X and further, $Fz = \{z\}$.

Proof. It follows from inequality (1) that if A, B are any sets in $B(X)$ then

$$\delta(Fa, Fb) \leqslant c\max\{\delta(a, Fa), \delta(b, Fb), \delta(a, Fb), \delta(b, Fa), d(a, b)\}$$

for all a in A and b in B and so on taking the supremum over a in A and b in B of both sides of this inequality we have

(2) $$\delta(FA, FB) \leqslant c\max\{\delta(A, FA), \delta(B, FB), \delta(A, FB), \delta(B, FA), \delta(A, B)\}$$

for all A, B in $B(X)$, both sides being finite since we are supposing that F maps $B(X)$ into itself.

Now let x be an arbitrary point in X and define the set $F^n x$ inductively by

$$F^n x = F(F^{n-1}x)$$

for $n = 2, 3, \ldots$ Let us suppose that the sequence $\{\delta(F^n x, Fx): n = 1, 2, \ldots\}$ is unbounded. Then there exists some $n > 1$ such that

(3) $$\begin{aligned}\delta(F^n x, Fx) &> \frac{c}{1-c}\,\delta(x, Fx)\\ &\geqslant \max\{\delta(F^r x, Fx): r = 1, 2, \ldots, n-1\}.\end{aligned}$$

Note that $n > 1$, since if $n = 1$ we would have

$$\begin{aligned}(1-c)\delta(Fx, Fx) &> c\delta(x, Fx)\\ &\geqslant c[\delta(x, Fx) - \delta(Fx, Fx)]\end{aligned}$$

which implies that $\delta(Fx, Fx) > c\delta(x, Fx)$ where as inequality (1) implies that $\delta(Fx, Fx) \leqslant c\delta(x, Fx)$. It follows from inequalities (3) that

$$\begin{aligned}(1-c)\delta(F^n x, Fx) &> c\delta(x, Fx)\\ &\geqslant c[\delta(x, F^r x) - \delta(F^r x, Fx)]\\ &\geqslant c[\delta(x, F^r x) - \delta(F^n x, Fx)]\end{aligned}$$

for $r = 1, 2, \ldots, n$ and so

(4) $$\delta(F^n x, Fx) > c\max\{\delta(x, F^r x): r = 1, 2, \ldots, n\}.$$

We will now prove that

(5) $$\delta(F^n x, Fx) > c\max\{\delta(F^r x, F^s x): r, s = 0, 1, 2, \ldots, n\}$$

where $F^0 x = x$ and so

$$\delta(F^0 x, F^0 x) = d(x, x) = 0.$$

For if not

$$\begin{aligned}\delta(F^n x, Fx) &\leqslant c\max\{\delta(F^r x, F^s x): r, s = 0, 1, 2, \ldots, n\}\\ &\leqslant c\max\{\delta(F^r x, F^s x): r, s = 1, 2, \ldots, n\}\end{aligned}$$

on using inequality (4). We can now apply inequality (2) indefinitely to these terms, since whenever a term of the form $\delta(x, F^r x)$ appears, it can be omitted because of inequality (4). This means that

$$\delta(F^n x, Fx) \leqslant c^k \max\{\delta(F^r x, F^s x): r, s = 1, 2, \ldots, n\}$$

for $k = 1, 2, \ldots$ and on letting k tend to infinity it follows that $\delta(F^n x, Fx) = 0$, giving a contradiction. Inequality (5) is thus proved. However, on using inequality (2), we now have

$$\begin{aligned}\delta(F^n x, Fx) &\leqslant c\max\{\delta(F^{n-1}x, F^n x), \delta(x, Fx), \delta(F^{n-1}, Fx), \delta(x, F^n x), \delta(F^{n-1}x, x)\}\\ &\leqslant c\max\{\delta(F^r x, F^s x): r, s = 0, 1, 2, \ldots, n\}\end{aligned}$$

which is impossible because of inequality (4). This contradiction implies that the sequence $\{\delta(F^n x, Fx): n = 1, 2, \ldots\}$ is in fact bounded.

Thus since

$$\delta(F^r x, F^s x) \leqslant \delta(F^r x, Fx) + \delta(Fx, F^s x),$$

it follows that

$$M = \sup\{\delta(F^r x, F^s x): r, s = 0, 1, 2, \ldots\}$$

is finite. Now, for arbitrary $\varepsilon > 0$, choose N such that $c^N M < \varepsilon$. It follows that for $m, n \geqslant N$, inequality (2) can be applied N times to the term $\delta(F^m x, F^n x)$ and so

(6) $$\delta(F^m x, F^n x) \leqslant c^N M < \varepsilon.$$

Choosing a point x_n in $F^n x$ for $n = 1, 2, \ldots$, we have

$$d(x_m, x_n) \leqslant \delta(F^m x, F^n x) < \varepsilon$$

for $m, n \geqslant N$. The sequence $\{x_n: n = 1, 2, \ldots\}$ is therefore a Cauchy sequence in the complete metric space X and so has a limit z in X. Further,

$$\begin{aligned}\delta(z, F^n x) &\leqslant d(z, x_m) + \delta(x_m, F^n x) \\ &\leqslant d(z, x_m) + \delta(F^m x, F^n x) \\ &< d(z, x_m) + \varepsilon\end{aligned}$$

for $m, n \geqslant N$. Letting m tend to infinity we have

$$(7) \qquad \delta(z, F^n x) \leqslant \varepsilon,$$

for $n \geqslant N$ and so

$$(8) \qquad \begin{aligned}\delta(F^n x, Fz) &\leqslant \delta(F^n x, z) + \delta(z, Fz) \\ &\leqslant \delta(z, Fz) + \varepsilon\end{aligned}$$

for $n \geqslant N$. On using inequalities (2), (6), (7) and (8), we have for $n > N$

$$\begin{aligned}\delta(F^n x, Fz) &\leqslant c\max\{\delta(F^{n-1}x, F^n x), \delta(z, Fz), \delta(F^{n-1}x, Fz), \delta(z, F^n x), \delta(F^{n-1}x, z)\} \\ &\leqslant c\max\{\varepsilon, \delta(z, Fz), \delta(z, Fz) + \varepsilon, \varepsilon, \varepsilon\} \\ &= c[\delta(z, Fz) + \varepsilon].\end{aligned}$$

It follows that since x_n is in $F^n x$, we have

$$\delta(x_n, Fz) \leqslant \delta(F^n x, Fz) \leqslant c[\delta(z, Fz) + \varepsilon]$$

for $n > N$ and on letting n tend to infinity we have

$$\delta(z, Fz) \leqslant c\delta(z, Fz)$$

since ε was arbitrary. It follows that $\delta(z, Fz) = 0$ and so $Fz = \{z\}$.

Now suppose F has a second fixed point w in X, so that w is in Fw. Then on using inequality (1) we have

$$\delta(Fw, Fw) \leqslant c\delta(w, Fw) \leqslant c\delta(Fw, Fw).$$

It follows that $\delta(Fw, Fw) = 0$ and so Fw only contains the single point w. Then

$$\begin{aligned}d(z, w) &= \delta(Fz, Fw) \\ &\leqslant c\max\{\delta(z, Fz), \delta(w, Fw), \delta(z, Fw), \delta(w, Fz), d(z, w)\} \\ &= cd(z, w).\end{aligned}$$

The uniqueness of z now follows. This completes the proof of the theorem.

We now note that although the mapping F in the theorem has a unique fixed point z it is possible for the point z to be contained in other sets Fx. To see this let x be the closed interval $[0, 1]$ with the usual metric. Define the function F by putting

$$Fx = \begin{cases}\{0\} & \text{for } x = 0, \\ [0, \tfrac{1}{2}x] & \text{for } x \neq 0.\end{cases}$$

Inequality (1) is satisfied with $c = \frac{1}{2}$ and the fixed point 0 is contained in every set in the range of F.

We finally prove two corollaries to Theorem 2. First of all we have

COROLLARY 1. *Let F be a mapping of a complete metric space (X, d) into $B(X)$ satisfying the inequality*

$$\delta(Fx, Fy) \leqslant a_1\delta(x, Fx) + a_2\delta(y, Fy) + a_3\delta(x, Fy) + a_4\delta(y, Fx) + a_5 d(x, y)$$

for all x, y in X, where $a_1, \ldots, a_5 \geqslant 0$ and $a_1 + \ldots + a_5 < 1$. If F also maps $B(X)$ into itself, then F has a unique fixed point z in X and further, $Fz = \{z\}$.

Proof. We have

$$\begin{aligned}\delta(Fx, Fy) &\leqslant a_1\delta(x, Fx) + a_2\delta(y, Fy) + a_3\delta(x, Fy) + a_4\delta(y, Fx) + a_5 d(x, y) \\ &\leqslant c\max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}\end{aligned}$$

where $c = a_1 + \ldots + a_5$. The result follows immediately from the theorem.

COROLLARY 2. *Let T be a mapping of a complete metric space (X, d) into itself satisfying the inequality*

$$d(Tx, Ty) \leqslant c\max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$. Then T has a unique fixed point z.

Proof. Define a mapping F of X into $B(X)$ by putting $Fx = \{Tx\}$ for all x in X. It follows that F satisfies inequality (1). Further, by noting the condition that F maps bounded sets into bounded sets was only used to prove inequality (2) and because we required the sets $\{F^n x: n = 1, 2, \ldots\}$ to be bounded, this condition is not needed in this corollary since $F^n x$ is now a set always consisting of a single point. In such a case inequality (1) can always be used instead of inequality (2) throughout the proof of the corollary. Thus, there exists a unique point z in X with $Fz = \{z\} = \{Tz\}$. The result now follows.

The result of this corollary was given in [2].

It should also be noted that it follows from Theorem 1 that the condition M be F-orbitally complete in Theorem 3 of Ćirić [1] can be replaced by the condition M be complete and it follows from Corollary 2 that the condition M be T-orbitally complete in Theorem 1 of [1] can also be replaced by the condition M be complete.

References

[1] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. 45 (1974), pp. 267–273.

[2] B. Fisher, *Quasi-contractions on metric spaces*, Proc. Amer. Math. Soc. 75 (1979), pp. 321–325.

[3] N. N. Kaulgud and D. V. Pai, *Fixed point theorems for set-valued mappings*, Nieuw Arch. Wisk. 23 (1975), pp. 49–66.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY
Leicester, England

Accepté par la Rédaction le 22. 1. 1979