Hence, taking $\gamma = f(\{A\})$, we have from the definition of δ that $H^2(g(\gamma), \gamma) < \varepsilon$. Thus, by the triangle inequality, $H^2(k(\{A\}), \{A\}) < \varepsilon + \delta$. Hence

(2) k is within $\varepsilon+\delta$ of the identity of $F_1(C(S^1))$.

Finally, recall that

(3) $F_1(C(S^1))$ is naturally isometric to $C(S^1)$.

Since ε and δ may be chosen as small as we please, we see from (1), (2) and (3) that $\{S^1\}$ is a Z-set in $C(S^1)$. But it is well known that $C(S^1)$ is a 2-cell with S^1, as a point of $C(S^1)$, in its interior (see [13, (0.55)]). Thus $\{S^1\}$ cannot be a Z-set in $C(S^1)$ [8, VI 2, p. 75]. The contradiction proves that $\Gamma(C(S^1)) \not\approx Q$.

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A generalization of a theorem of Skala

by

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Abstract. Let $n \geqslant 1$, let (A, f) be some algebra of type $n+1$ satisfying

(a) $f(f(x_0, \ldots, x_n), y_1, \ldots, y_n) = f(x_0, f(x_1, y_1, \ldots, y_n), \ldots, f(x_n, y_1, \ldots, y_n))$ for any $x_0, \ldots, x_n, y_1, \ldots, y_n \in A$

and put

$$C := \{x \in A \mid f(x, x_1, \ldots, x_n) = x \text{ for any } x_1, \ldots, x_n \in A\},$$

$$S_i(M) := \{x \in A \mid f(x, x_1, \ldots, x_n) = x_i \text{ for any } x_1, \ldots, x_n \in M\} \quad (1 \leqslant i \leqslant n,\ M \subseteq A)$$

and

$$S(M) := S_1(M) \cup \ldots \cup S_n(M) \quad (M \subseteq A).$$

The following result of H. Skala (cf. [1]) is generalized:

THEOREM 1. *Let* $|C| \geqslant 3$ *and assume* $f(x_0, \ldots, x_n) \in \{x_0, \ldots, x_n\}$ *for any* $x_0, \ldots, x_n \in A$. T.f.a.e.:
(i) $a \in A \setminus C$.
(ii) $a \in S(C \cup \{a\})$.

In the following if $x \in A$ or if $x \subseteq A$ then $x(i)$ denotes the sequence $x, \ldots, x$ of length i $(1 \leqslant i \leqslant n)$.

LEMMA 1. *Let* $B \subseteq A$ *satisfying*

(b) $f(x, y, \ldots, y) = x$ *for any* $x, y \in B$

and let $a \in A$ *such that* (α) *and* (β):

(α) $f(a, x, \ldots, x) = x$ *for any* $x \in B$.

(β) $f(a, B(i-1), a, B, \ldots, B) \subseteq B \cup \{a\}$ *for any* $i = 1, \ldots, n$.

Further let $a_1, \ldots, a_n, b, b_1, \ldots, b_n \in B$ *and assume* $f(a, a_1, \ldots, a_n) = b$. *Finally, suppose* $b_i = b$ *whenever* $a_i = b$ $(1 \leqslant i \leqslant n)$. *Then* $f(a, b_1, \ldots, b_n) = b$.

Proof. We prove $c_i := f(a, b_1, \ldots, b_i, a_{i+1}, \ldots, a_n) = b$ for any $i = 0, \ldots, n$ by induction on i. $c_0 = b$ is our hypothesis. Now, let $0 < j \leqslant n$ and suppose $c_{j-1} = b$ to be already proved. If $a_j = b$ then $b_j = b = a_j$ whence $c_j = b$. If, otherwise, $a_j \neq b$ then $f(f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n), a_j, \ldots, a_j) = b$ by (a), (b) and (α) whence $f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n) = b$ by (β), (α) and (b) and therefore

$$c_j = f(f(a, b_1, \ldots, b_{j-1}, a, a_{j+1}, \ldots, a_n), b_j, \ldots, b_j) = f(b, b_j, \ldots, b_j) = b$$

by (a), (b) and (α).

THEOREM 2. *Let $B \subseteq A$ satisfying* (b), *suppose* $|B| \geqslant 3$ *and let* $a \in A$. T.f.a.e.:

(i) (α'), (β') *and* (γ) *hold:*

(α') $f(a, x(i-1), y, x, \ldots, x) \in \{x, y\}$ *for any* $x, y \in B$ *and for any* $i = 1, \ldots, n$.

(β') $f(a, B \cup \{a\}, \ldots, B \cup \{a\}) \subseteq B \cup \{a\}$.

(γ) *There exist* $u, v, w \in B$, $u \neq v \neq w \neq u$, *such that* $f(a, u(i-1), v, w, \ldots, w) \in \{u, v, w\}$ *for any* i *with* $1<i<n$.

(ii) $a \in S(B \cup \{a\})$.

Proof. (i) is an immediate consequence of (ii). Therefore, suppose (i) holds. Then $f(a, u, \ldots, u, v) \neq w$ by (α') and (γ). Now put

$$k := \min\{i|\ 1 \leqslant i \leqslant n,\ f(a, u(i-1), v, w, \ldots, w) \neq w\}.$$

We will prove

(1) $$f(a, u(k-1), v, w, \ldots, w) = v.$$

If $k = 1$ then (1) follows from (γ), (α') and from the definition of k. Now suppose $k>1$. Then $f(a, u(k-2), v, w, \ldots, w) = w$ by definition of k whence

(2) $$f(a, u(k-1), w, \ldots, w) = w$$

by (γ) and Lemma 1. Now, $f(a, u(k-1), v, w, \ldots, w) = u$ would imply

$$f(a, u(k-1), w, \ldots, w) = u \neq w$$

by (γ) and Lemma 1 contradicting (2). Hence (1) follows from (γ), from the definition of k and from (α'). Now

(3) $$f(a, B(k-1), v, B, \ldots, B) = v$$

by (1), (γ) and Lemma 1. Let $c \in B$, $c \neq v$. Choose $d \in B$, $d \neq c, v$. Then

$$f(a, d(k-1), c, d, \ldots, d) = d$$

would imply $f(a, d(k-1), v, d, \ldots, d) = d \neq v$ by (γ) and Lemma 1 contradicting (3). Hence $f(a, d(k-1), c, d, \ldots, d) = c$ by (α') and thus $f(a, B(k-1), c, B, \ldots, B) = c$ by Lemma 1. Together with (3) this shows

(4) $$a \in S_k(B).$$

If there would exist $a_1, \ldots, a_n \in B \cup \{a\}$ with $f(a, a_1, \ldots, a_n) \neq a_k$ then choosing some $e \in B$, $e \neq f(a, a_1, \ldots, a_n), a_k$ we would obtain

$$f(a, f(a_1, e, \ldots, e), \ldots, f(a_n, e, \ldots, e)) = f(f(a, a_1, \ldots, a_n), e, \ldots, e) \neq f(a_k, e, \ldots, e)$$

by (a), (β'), (b) and (α') contradicting (4). Hence $a \in S_k(B \cup \{a\}) \subseteq S(B \cup \{a\})$ and (ii) holds.

Remark. Using the left ideal property of C, i.e. $f(A, C, \ldots, C) \subseteq C$, one easily verifies that Theorem 1 is an immediate consequence of Theorem 2. But Theorem 2 is more general than Theorem 1 as can be seen from the following example: Let M be some set, $|M| \geqslant 3$, put $A := \{f|\ f\colon M^n \to M\}$ and let σ be some equivalence relation on M^n satisfying

$$\prod_{X \in M^n/\sigma} |X \cap \operatorname{diag}(M^n)^2| \geqslant 3.$$

Further let $B \subseteq \{f \in A|\ \ker f = \sigma$ and $(fX, \ldots, fX) \in X$ for any $X \in M^n/\sigma\}$, $|B| \geqslant 3$. Finally, let $\bigcup\{fM^n|\ f \in B\} \subseteq L \subseteq M$, let $1 \leqslant j \leqslant n$ and let $a \in A$ such that $aM^n \subseteq L$ and $a(x_1, \ldots, x_n) = x_j$ for any $x_1, \ldots, x_n \in L$. Now consider the algebra (A, f) where f is the composition of functions, i.e.

$$(f(f_0, \ldots, f_n))(x_1, \ldots, x_n) := f_0(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$$

for any $f_0, \ldots, f_n \in A$ and for any $x_1, \ldots, x_n \in M$. Then Theorem 2 does apply to this case whereas Theorem 1 does not in general.

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