Infinitary stationary logic and abelian groups

by

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Abstract. Necessary and sufficient conditions are given for abelian groups which are $L_{\infty\varkappa}$-free to be equivalent with respect to certain "filter logics" obtained by adding to $L_{\infty\varkappa}$ a second-order "almost all" quantifier. Examples and constructions are given of equivalent non-isomorphic groups.

0. Introduction. In this paper we investigate the properties of abelian groups which are expressible in various logics, $L^{\tilde{E}}_{\infty\varkappa}$ (aa), obtained by adding to $L_{\infty\varkappa}$ a second-order quantifier "aa$s$". The semantics of the quantifier depends on the choice of a stationary subset E of $\varkappa$; given E we interpret "aa$s$" as meaning — roughly — "for a set of small subsets s which, modulo a non-stationary set, contains E". (This is made precise in section 2.) In particular for $E = \varkappa$, "aa$s$" means "for a closed unbounded set of small sets s".

This paper may be considered as a sequel to [2]. In that paper the abelian groups are characterized which are $L_{\infty\varkappa}$-equivalent to a free group; these are called the strongly $\varkappa$-free groups (Theorem 1.1). Assuming V = L (or with no additional set theoretic hypothesis for $\varkappa = \omega_{n+1}$, $n \in \omega$) it may be shown that for most regular $\varkappa$ there are strongly $\varkappa$-free groups of cardinality $\varkappa$ which are not free (Theorem 1.5). In this paper we consider languages stronger than $L_{\infty\varkappa}$ which distinguish between strongly $\varkappa$-free groups. In particular we show (assuming V = L or for $\varkappa = \omega_{n+1}$) that there is a sentence of $L^{\tilde{E}}_{\infty\varkappa}$(aa) which picks out the free group among all abelian groups of cardinality $\varkappa$ if and only if E contains a closed unbounded set (Theorems 2.2 and 3.6).

In general, we give necessary and sufficient conditions for two strongly $\varkappa$-free groups of cardinality $\varkappa$ to satisfy the same sentences of $L^{\tilde{E}}_{\infty\varkappa}$(aa) (Theorem 3.4). (Of relevance here is an invariant of such groups (introduced in [6]): we associate to any $\varkappa$-free group A a stationary set $\Gamma(A)$ (or more precisely an equivalence class of stationary sets — see Section 1)). Moreover, we characterize algebraically the

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pairs of strongly $\varkappa$-free groups A and B which satisfy the same sentences of $L_{\infty\varkappa}^{\tilde{E}}(aa)$ for every E (Theorem 4.3) and we show — assuming V = L — how to construct non-trivial such pairs (Corollary 5.2) (1).

Throughout the paper we shall make use of the following conventions and terminology: (i) "group" always means "abelian group"; (ii) $|A|$ denotes the cardinality of a set or group; (iii) if A is a group and λ is a cardinal, $A^{(\lambda)}$ denotes the direct sum of λ copies of A; (iv) a cardinal is an initial ordinal and an ordinal is identified with the set of its predecessors; thus, $\aleph_\sigma = \omega_\sigma = \{\nu | \ \nu<\omega_\sigma\}$; (v) $\varkappa$ will always denote a regular uncountable cardinal.

1. Almost free groups. We begin by reviewing some terminology and some results dealing with groups which are "close" to being free (see [2] and [3]). A group A is said to be *$\varkappa$-free* if every subgroup of A of cardinality $<\varkappa$ is free. A subgroup B of a $\varkappa$-free group is said to be *$\varkappa$-pure* in A if A/B is $\varkappa$-free i.e., B is a direct summand of every extension C of B in A such that $|C|<\varkappa$. We say that A is *strongly $\varkappa$-free* if A is $\varkappa$-free and every subset of A of cardinality $<\varkappa$ is contained in a $\varkappa$-pure subgroup of A of cardinality $<\varkappa$. The property of being strongly $\varkappa$-free has model-theoretic significance:

1.1. THEOREM [2]. *A is strongly $\varkappa$-free if and only if A is $L_{\infty\varkappa}$-equivalent to a free group.* ■

1.2. DEFINITION. A *$\varkappa$-filtration* of A is an increasing chain $\Delta = \{A_\nu | \ \nu<\varkappa\}$ of subgroups of A satisfying for all $\nu<\varkappa$:

(i) $|A_\nu|<\varkappa$;

(ii) $A = \bigcup_{\nu<\varkappa} A_\nu$; and

(iii) if ν is a limit ordinal, $A_\nu = \bigcup_{\mu<\nu} A_\mu$.

1.3. LEMMA. (a) *A is $\varkappa$-free of cardinality $\varkappa$ if only if A has a $\varkappa$-filtration by free groups. In this case we can choose a $\varkappa$-filtration Δ such that*

(iv) *if A_ν is not $\varkappa$-pure in A, then $A_{\nu+1}/A_\nu$ is not free.*

(b) *A is strongly $\varkappa$-free if and only if A has a $\varkappa$-filtration Δ satisfying* (i)-(iv) *and*

(v) *$A_{\nu+1}$ is $\varkappa$-pure in A for all $\nu<\varkappa$.*

Proof. Given $A = \{a_\nu: \ \nu<\varkappa\}$ $\varkappa$-free (resp. strongly $\varkappa$-free) of cardinality $\varkappa$ we simply construct by transfinite induction a continuous increasing chain Δ so that $a_\nu \in A_{\nu+1}$. If A_ν is not $\varkappa$-pure in A, then by definition of $\varkappa$-purity we can choose $A_{\nu+1}$ so that $A_{\nu+1}/A_\nu$ is not free. If A is strongly $\varkappa$-free, then by definition we can choose $A_{\nu+1}$ so that $A_{\nu+1}$ is $\varkappa$-pure in A. ■

From now on, if A is a $\varkappa$-free group of cardinality $\varkappa$ and we write $A = \bigcup_{\nu<\varkappa} A_\nu$, we mean that $\Delta = \{A_\nu | \ \nu<\varkappa\}$ is a $\varkappa$-filtration of A satisfying (iv). Moreover, if A is strongly $\varkappa$-free, we assume that (v) holds.

A *closed unbounded set* (or *cub*) in $\varkappa$ is a subset C of $\varkappa$ such that $\sup C = \varkappa$ and $\sup X \in C \cup \{\varkappa\}$ whenever $X \subseteq C$. Let $I(\varkappa)$ be the set of all subsets of $\varkappa$ disjoint from a cub; I forms an ideal in $\mathfrak{P}(\varkappa)$, the Boolean algebra of all subsets of $\varkappa$. Let $D(\varkappa)$ denote the Boolean algebra $\mathfrak{P}(\varkappa)/I$. If $E \in \mathfrak{P}(\varkappa)$ we let $\tilde{E}$ denote its image in $D(\varkappa)$: thus $\tilde{E}_1 = \tilde{E}_2$ if and only if $E_1 \cap C = E_2 \cap C$ for some cub C in $\varkappa$. The greatest element of $D(\varkappa)$ is $1 = \tilde{\varkappa} = \tilde{C}$ for any cub; the least element is $0 = \tilde{\emptyset}$. A subset E of $\varkappa$ is called *stationary* if and only if $\tilde{E} \neq 0$ i.e., for all cubs C, $E \cap C \neq \emptyset$.

We are going to define a map $\Gamma_\varkappa$ from the set of all $\varkappa$-free groups of cardinality $\varkappa$ to $D(\varkappa)$. Given A $\varkappa$-free of cardinality $\varkappa$ and a $\varkappa$-filtration $\Delta = \{A_\nu | \ \nu<\varkappa\}$ of A, let

$$E = \{\nu<\varkappa | \ A_{\nu+1}/A_\nu \text{ is not free}\} = \{\nu<\varkappa | \ A_\nu \text{ is not } \varkappa\text{-pure in } A\}$$

and let $\Gamma_\varkappa(A) = \tilde{E} \in D(\varkappa)$.

1.4. LEMMA. (1) *$\Gamma_\varkappa$ is well-defined*; (2) *$\Gamma_\varkappa(A) = 0$ if and only if A is free*; *and* (3) *if $\Gamma_\varkappa(A) \neq 1$, then A is strongly $\varkappa$-free.*

Proof. See [6], Theorem 2.5, p. 259. ■

We conclude this section with a summary of some results about the existence of $\varkappa$-free groups.

1.5. THEOREM. (1) (Shelah [22]) *If λ is a singular cardinal and A is λ-free of cardinality λ, then A is free.*

(2) (Shelah, Mekler, et al) *If λ is weakly compact and A is λ-free of cardinality λ, then A is free.*

(3) (Shelah) *It is consistent with* ZFC *that every $2^{\aleph_0}$-free group is free (assuming the consistency of the existence of a supercompact cardinality).*

(4) (Gregory [11]) *Assuming* V = L, *there exists for every regular non-weakly compact $\varkappa$ a strongly $\varkappa$-free group of cardinality $\varkappa$ which is not free.*

(5) (Eklof [3]) *If there is a $\varkappa$-free group of cardinality $\varkappa$ which is not free, then there is a strongly $\varkappa^+$-free group of cardinality $\varkappa^+$ which is not free.*

(6) (Mekler [19]) *If there is a strongly $\varkappa$-free group of cardinality $\varkappa$ which is not free, then for every $\tilde{E} \subseteq \Gamma_\varkappa(A)$, there exists an A with $\Gamma_\varkappa(A) = \tilde{E}$.*

(7) (Mekler [20]) *For every $n \in \omega$ and every $\tilde{E} \in D(\omega_{n+1})$, there exists A with $\Gamma_{\omega_{n+1}}(A) = \tilde{E}$.* ■

(8) (Mekler [20]) *Assuming* V = L, *for successor cardinals $\varkappa$ and $\tilde{E} \subseteq D(\varkappa)$, there exists A such that $\Gamma(A) = \tilde{E}$ if and only if $\tilde{E} \subseteq 1 - \tilde{W}$ where $W = \{\nu | \ \mathrm{cf}(\nu)$ is weakly compact$\}$.*

For additional results on Γ, see [20]. Recently Shelah has proved that GCH implies not every $\aleph_{\omega+1}$-free group (of cardinality $\aleph_{\omega+1}$) is free.

Part (6) is weaker than the result claimed in Theorem 2.7 (2) of [6]. This is because the hypothesis that the group A be strongly $\varkappa$-free is needed for the con-

(1) In another paper we study the $L_{\omega\omega}(aa)$ theory of abelian groups and prove, among other things, that it is decidable (Ann. Math. Logic 17 (1979), pp. 227–270).

struction used in the proof of Lemma 1.3, p. 326 of [19]. (For the same reason Theorem 1.1 of [19] should read "if there exists a *strongly* $\varkappa$-free non-free group of cardinality $\varkappa$, then there exist $2^\varkappa$ strongly $\varkappa$-free groups of cardinality $\varkappa$". We do not know if Theorem 1.1 of [19] is true as stated, although we suspect it is.)

For the purposes of some constructions used in later sections to exhibit some examples we shall outline some of the ideas involved in the proof of part (7), which will appear in a forthcoming paper [20]. (The simple proof claimed in Corollary 2.8 of [6] does not work, because of the error mentioned above.) The case $n = 1$ is quite elementary and well-known and does not require the following machinery. Thus the reader who so desires can skip the following and in the later examples consider only the case $n = 1$.

1.6. Definition (Hill [12]). Define by induction on $n \in \omega$ a class $\mathfrak{F}_n$ of torsion-free groups of cardinality $\leqslant \omega_n$. Let $\mathfrak{F}_0$ = the class of all countable torsion-free groups. For $n \geqslant 0$, $\mathfrak{F}_{n+1}$ = the class of all groups A which have an ω_{n+1}-filtration $\{A_\nu |\ \nu < \omega_{n+1}\}$ such that for all $\nu < \omega_{n+1}$, $A_{\nu+1}/A_\nu$ belongs to $\mathfrak{F}_n$.

1.7. Theorem (Hill-Mekler). *Let E be a stationary subset of $\aleph_{n+1}$ consisting only of limit ordinals. Let Φ be any function from E into $\mathfrak{F}_n$. Then there is a strongly ω_{n+1}-free group A of cardinality ω_{n+1} with an ω_{n+1}-filtration $\Delta = \{A_\nu |\ \nu < \omega_{n+1}\}$ such that: A_ν is ω_{n+1}-pure in A if $\nu \notin E$; and $A_{\nu+1}/A_\nu \cong \Phi(\nu)$ if $\nu \in E$.* ■

2. Stationary logic. We now introduce a class of languages, stronger than $L_{\infty\varkappa}$, using which we can differentiate between the strongly $\varkappa$-free groups, which are all $L_{\infty\varkappa}$-equivalent. We fix a regular cardinal $\varkappa$ and an element $\tilde{E}$ of $D(\varkappa)$. Let $\mathfrak{F}^{\tilde{E}} = \{S \in \mathscr{P}(\varkappa)\colon \tilde{E} \subseteq \tilde{S}\}$, a filter in $\mathscr{P}(\varkappa)$, the power set of $\varkappa$.

First we define the syntax of our language $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ — which is the same for all $\tilde{E}$. The non-logical symbols of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ are those of the language of abelian groups (a binary function symbol + and a constant symbol 0) plus a countable number of unary predicate symbols $s_0, s_1, s_2, \ldots$ Then $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ is the smallest class of formulas containing the atomic formulas and closed under the formation rules of $L_{\infty\varkappa}$ (negation, conjunction and disjunction over arbitrary sets of formulas; quantification over sets of variables of cardinality $<\varkappa$) plus the additional rule:

If φ is a formula of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$, then so is $\mathrm{aa}\, s_i \varphi$. Let stat $s\varphi$ be an abbreviation for $\neg \mathrm{aa}\, s \neg \varphi$. (See [1] and [16], where $L_{\omega\omega}(\mathrm{aa})$ and $L_{\omega_1\omega}(\mathrm{aa})$ are studied in general and in detail. The idea of studying such languages was first suggested in [23].)

We shall define a semantics for $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ only for groups A, of cardinality $\varkappa$. Fix a $\varkappa$-filtration $\{A_\nu |\ \nu < \varkappa\}$ of A. If $\varphi(v_1, \ldots, v_n; s_1, \ldots, s_m)$ is a formula of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ — whose free first and second order variables are displayed — then for any $a_1, \ldots, a_n \in A$ and $\nu_1, \ldots, \nu_m \in \varkappa$ we define

$$A \models_{\tilde{E}} \varphi[a_1, \ldots, a_n; \nu_1, \ldots, \nu_m]$$

by induction on formulas. If φ is $s_i(v_j)$, then $A \models_{\tilde{E}} \varphi[a, \nu]$ if and only if $a \in A_\nu$. For other atomic formulas and for the cases of negation, conjunction and quanti-

fication of first order variables the definition is the usual one. If φ is of the form $\mathrm{aa}\, s_0 \psi$, then we define

$$A \models_{\tilde{E}} \varphi[a_1, \ldots, a_n; \nu_1, \ldots, \nu_m]$$

if and only if $\{\nu_0 < \varkappa |\ A \models_{\tilde{E}} \psi[a_1, \ldots, a_n; \nu_0, \nu_1, \ldots, \nu_m]\} \in \mathfrak{F}^{\tilde{E}}$. (Compare [1], § 8.3.) It is easy to check that for sentences θ of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$, the definition of $A \models_{\tilde{E}} \theta$ is independent of the choice of the $\varkappa$-filtration, since any two $\varkappa$-filtrations of A agree on a cub. Notice that $L^0_{\infty\varkappa}(\mathrm{aa})$ has the same strength as $L_{\infty\varkappa}$.

2.1. Example. (1) Let δ be the sentence

$$\mathrm{aa}\, s\, \mathrm{aa}\, s' (s \subseteq s' \to s'/s \text{ is not free})$$

which is a sentence of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ since the clause "s'/s is not free" can be replaced by a formula of $L_{\infty\varkappa}$ (namely, the disjunction of the descriptions of all non-free groups of cardinality $<\varkappa$). Then

$$A \models_{\tilde{E}} \delta \quad \text{if and only if} \quad \Gamma(A) \supseteq \tilde{E}.$$

(2) Let θ be the sentence

$$\mathrm{aa}\, s\, \mathrm{aa}\, s' (s \subseteq s' \to s'/s \text{ is free}).$$

Then $A \models_{\tilde{E}} \theta$ if and only if $\Gamma(A) \subseteq 1 - \tilde{E}$. In the special case of $\tilde{E} = 1$ we obtain, by Lemma 1.4:

2.2. Theorem. *There is a sentence θ of $L^1_{\infty\varkappa}(\mathrm{aa})$ such that for any group A of cardinality $\varkappa$, $A \models_1 \theta$ if and only if A is free.* ■

We shall see later (3.6 and 3.7) that these examples are best possible. In particular, for any $\tilde{E} \neq 1$, the property of being free is not expressible in $L^{\tilde{E}}_{\infty\omega_n}(\mathrm{aa})$.

2.3. Example. (See [5].) Let C be any group of cardinality $<\varkappa$. Let ψ_C be the following sentence of $L^{\tilde{E}}_{\infty\omega_\varkappa}(\mathrm{aa})$

$$\mathrm{aa}\, s\, \mathrm{aa}\, s' [s \subseteq s' \to \mathrm{Ext}(s'/s, C) = 0]$$

Then, assuming V = L, for any group A of cardinality $\varkappa$, $A \models_1 \psi_C$ if and only if $\mathrm{Ext}(A, C) = 0$. Moreover in general there is no sentence of $L_{\infty\varkappa}$ which expresses the property of A that $\mathrm{Ext}(A, C) = 0$.

2.4. Remark. In the case $\tilde{E} = 1$ we can extend the definition of the semantics of $L^1_{\infty\varkappa}(\mathrm{aa})$ to groups of arbitrary cardinality by analogy with the definition of the semantics of $L(\mathrm{aa})$ (cf. [1]). Thus $A \models \mathrm{aa}\, s\varphi$ if and only if $\{S < A\colon |S| < \varkappa \text{ and } A \models \varphi[S]\}$ a closed unbounded subset of $\mathscr{P}_\varkappa(A)$, the set of subsets of A of cardinality $<\varkappa$. Theorems 3.2 and 3.4 extend in a natural way to this setting.

3. $\tilde{E}$-equivalence of groups. Let us say that two groups of cardinality $\varkappa$ are *$\tilde{E}$-equivalent* if they satisfy the same sentences of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$. We shall generalize the idea of Examples 2.1 (1) and (2) in order to give necessary and sufficient conditions for two strongly $\varkappa$-free groups of cardinality $\varkappa$ to be $\tilde{E}$-equivalent. We begin with a back-and-forth criterion for $\tilde{E}$-equivalence, due to Makowsky.

3.1. DEFINITION. Let A and B be groups of cardinality $\varkappa$. Fix $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$, $B = \bigcup_{\nu<\varkappa} B_\nu$ of the groups. A *partial isomorphism* from A to B is a pair (f, σ) consisting of:

(i) an isomorphism $f\colon A_\varrho \to B_\tau$ for some $\varrho, \tau < \varkappa$,

(ii) a bijection $\sigma\colon X \to Y$, where X, Y are finite subsets of $\varkappa$,

satisfying: for every $\mu \in X$, if $\mu \leqslant \varrho$, then $f(A_\mu) = B_{\sigma(\mu)}$.

We write $(f, \sigma) \subseteq (f', \sigma')$ if $f \subseteq f'$ and $\sigma \subseteq \sigma'$.

3.2. THEOREM (Makowsky [17]). *If A and B are groups of cardinality $\varkappa$, A is $\tilde{E}$-equivalent to B if and only if there is a set I of partial isomorphisms from A to B satisfying*

(1) *For every subset Z of A (resp. B) of cardinality $<\varkappa$ and every $(f, \sigma) \in I$ there is $(f', \sigma') \in I$ such that $(f, \sigma) \subseteq (f', \sigma')$ and $Z \subseteq \operatorname{dom} f'$ (resp. $Z \subseteq \operatorname{codom} f'$).*

(2) *For every $S \in \mathfrak{F}^{\tilde{E}}$ and every $(f, \sigma) \in I$ there exists $S' \in \mathfrak{F}^{\tilde{E}}$ such that for every $\mu' \in S'$ there exists $\mu \in S$ such that $(f, \sigma \cup \{(\mu, \mu')\}) \in I$.*

(2′) *For every $S' \in \mathfrak{F}^{\tilde{E}}$ and every $(f, \sigma) \in I$ there exists $S \in \mathfrak{F}^{\tilde{E}}$ such that for every $\mu \in S$ there exists $\mu' \in S'$ such that $(f, \sigma \cup \{(\mu, \mu')\}) \in I$.* ∎

(We shall need only the sufficiency of the conditions, which is proved by an induction on formulas in the usual way.)

3.3. DEFINITION. Define an equivalence relation $\sim$ on groups of cardinality $<\varkappa$ by $G \sim H$ if and only if there is a free group F of cardinality $<\varkappa$ such that $G \oplus F \cong H \oplus F$.

3.4. THEOREM. *Let A and B be strongly $\varkappa$-free groups of cardinality $\varkappa = \lambda^+$; let $\tilde{E} \in D(\varkappa)$. Fix $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$, $B = \bigcup_{\nu<\varkappa} B_\nu$ of the groups. The following are equivalent.*

(1) *A is $\tilde{E}$-equivalent to B.*

(2) *For every class $\mathfrak{H}$ of groups of cardinality $<\varkappa$ which is closed under $\sim$ and $\cong$, $\{\nu \in E \mid A_{\nu+1}/A_\nu \in \mathfrak{H}\}$ is stationary in $\varkappa$ $\Leftrightarrow$ $\{\nu \in E \mid B_{\nu+1}/B_\nu \in \mathfrak{H}\}$ is stationary in $\varkappa$.*

(3) *For every class $\mathfrak{H}$ as in (2),*

$$\{\nu<\varkappa \mid A_{\nu+1}/A_\nu \in \mathfrak{H}\} \in \mathfrak{F}^{\tilde{E}} \Leftrightarrow \{\nu<\varkappa \mid B_{\nu+1}/B_\nu \in \mathfrak{H}\} \in \mathfrak{F}^{\tilde{E}}.$$

Proof. (2) $\Leftrightarrow$ (3). This follows from the fact that $\{\nu \in E \mid A_{\nu+1}/A_\nu \in \mathfrak{H}\}$ is not stationary in $\varkappa$ if and only if $\{\nu<\varkappa \mid A_{\nu+1}/A_\nu \in \bar{\mathfrak{H}}\} \in \mathfrak{F}^{\tilde{E}}$, where $\bar{\mathfrak{H}}$ denotes the complement of $\mathfrak{H}$ in the class of all groups of cardinality $<\varkappa$.

(1) $\Rightarrow$ (3). Suppose $\mathfrak{H}$ is as in (2) and suppose

$$Y = \{\nu<\varkappa \mid A_{\nu+1}/A_\nu \in \mathfrak{H}\} \in \mathfrak{F}^{\tilde{E}};$$

say C is a cub such that $E \cap C \subseteq Y \cap C$. Let φ be the sentence

$$\mathrm{aa}\, s\, \mathrm{aa}\, s' (s \subseteq s' \to s'/s \in \mathfrak{H}).$$

This is a sentence of $L^{\tilde{E}}_{\infty\varkappa}(\mathrm{aa})$ because the clause "$s'/s \in \mathfrak{H}$" is expressible in $L_{\infty\varkappa}$ (by using a large disjunction; cf. Example 2.1 (1)). But $A \models_{\tilde{E}} \varphi$ if and only if $\{\nu<\varkappa \mid A_{\nu+1}/A_\nu \in \mathfrak{H}\} \in \mathfrak{H}^{\tilde{E}}$. (To see this use property 1.3(v) of a $\varkappa$-filtration and the fact that $\mathfrak{H}$ is closed under $\sim$; we have $A_{\nu+1}/A_\nu \in \mathfrak{H}$ if and only if $A_\mu/A_\nu \in \mathfrak{H}$ for all $\mu \geqslant \nu+1$, because $A_\mu/A_\nu \cong A_{\nu+1}/A_\nu \oplus A_\mu/A_{\nu+1}$ and $A_\mu/A_{\nu+1}$ is free.) Since the same holds for B, and A and B are $\tilde{E}$-equivalent, we are done.

(3) $\Rightarrow$ (1). We shall prove this implication using Theorem 3.2. Since $\varkappa = \lambda^+$ is a successor cardinal we may assume that the $\varkappa$-filtrations are chosen so that (∗) for every $\nu<\varkappa$, $A_{\nu+1}/A_\nu \cong A_{\nu+1}/A_\nu \oplus Z^{(\lambda)}$, so that $A_{\nu+1}/A_\nu \cong A_\mu/A_\nu$ for every $\mu \geqslant \nu+1$, and similarly for B. Let I consist of all partial isomorphisms (f, σ) — see Definition 3.1 — where $f\colon A_{\varrho+1} \to B_{\tau+1}$ for some $\varrho, \tau \in \varkappa$, and for all $\mu \in \operatorname{dom}\sigma$, $A_{\mu+1}/A_\mu \cong B_{\sigma(\mu)+1}/B_{\sigma(\mu)}$.

Let us first verify 3.2(2) (and by symmetry 3.2(2′)). Given $(f, \sigma) \in I$, where $f\colon A_{\varrho+1} \to B_{\tau+1}$, and given $S \in \mathfrak{F}^{E}$ let C be a cub such that $E \cap C \subseteq S \cap C$. Let $\mathfrak{H}$ be the closure of

$$\{A_{\mu+1}/A_\mu \mid \mu \in E \cap C,\ \mu > \varrho+1\}$$

under isomorphism and $\sim$. By hypothesis, $\{\mu<\varkappa \mid B_{\mu+1}/B_\mu \in \mathfrak{H}\}$ belongs to $\mathfrak{F}^{\tilde{E}}$. So let

$$S' = \{\mu \in E \mid \mu > \tau+1 \text{ and } B_{\mu+1}/B_\mu \in \mathfrak{H}\}$$

which also belongs to $\mathfrak{F}^{\tilde{E}}$. By definition of S', for every $\mu' \in S'$ there is a $\mu > \varrho+1$ such that

$$B_{u+1}/B_\mu \oplus F \cong A_{\mu+1}/A_\mu \oplus F$$

for some free group F of rank $\leqslant \lambda$. But then by assumption (∗),

$$B_{\mu+1}/B_\mu \cong B_{\mu+1}/B_\mu \oplus F \cong A_{\mu+1}/A_\mu \oplus F \cong A_{\mu+1}/A_\mu.$$

Hence $(f, \sigma \cup \{(\mu, \mu')\})$ belongs to I.

In the verification of 3.2(1) we use the following result of J. Erdös (see [8], p. 196).

3.5. LEMMA. *If $H \subseteq F$, $H' \subseteq F'$ are free groups of infinite rank such that F/H is isomorphic to F'/H' and torsion-free and such that $|H| = |H'|$, then there is an isomorphism $\varphi\colon F \to F'$ such that $\varphi(H) = H'$.* ∎

Now suppose we are given $Z \subseteq A$ of cardinality $<\varkappa$. (The case of $Z \subseteq B$ is entirely symmetrical.) Chose ϱ' such that $Z \subseteq A_{\varrho'+1}$. Given $(f, \sigma) \in I$ where $f\colon A_{\varrho+1} \to B_{\tau+1}$ let $\mu_1 \leqslant \ldots \leqslant \mu_r$ be all the elements of the domain of σ which are $>\varrho+1$ and $\leqslant \varrho'+1$. By increasing ϱ' if necessary we may assume that the rank of $A_{\varrho'+1}/A_{\mu_r+1}$ is λ. We must show that we can extend f to $f'\colon A_{\varrho'+1} \to B_{\tau'+1}$ for some $\tau' \in \varkappa$ such that for $i = 1, \ldots, r$, $f'(A_{\mu_i}) = B_{\sigma(\mu_i)}$. First we show how to extend f to $g_1\colon A_{\mu_1+1} \to B_{\sigma(\mu_1)+1}$. By property 1.3(v) we have:

$$A_{\mu_1+1} = A_{\varrho+1} \oplus \bar{A}; \quad B_{\sigma(\mu_1)+1} = B_{\tau+1} \oplus \bar{B}$$

for some $\bar{A}$, $\bar{B}$. Hence

$$A_{\mu_1} = A_{\varrho+1}\oplus(\bar{A}\cap A_{\mu_1});\qquad B_{\sigma(\mu_1)} = B_{\tau+1}\oplus(\bar{B}\cap B_{\sigma(\mu_1)}).$$

Therefore

$$\bar{A}/(\bar{A}\cap A_{\mu_1}) \cong A_{\mu_1+1}/A_{\mu_1} \cong B_{\sigma(\mu_1)+1}/B_{\sigma(\mu_1)} \cong \bar{B}/(\bar{B}\cap B_{\sigma(\mu_1)}).$$

Hence by Lemma 3.5 there is an isomorphism $\varphi\colon \bar{A}\to\bar{B}$ such that $\varphi(\bar{A}\cap A_{\mu_1}) = \bar{B}\cap B_{\sigma(\mu_1)})$. Then define g_1 to be f on $A_{\varrho+1}$ and φ on A. It should now be clear that in a finite number of steps like that above we can extend f to $g_r\colon A_{\mu_r+1}\to B_{\sigma(\mu_r)+1}$. We can then easily extend g_r to the desired f' since for sufficiently large τ' we have

$$A_{\varrho'+1}/A_{\mu_r+1} \cong Z^{(\lambda)} \cong B_{\tau'+1}/B_{\sigma(\mu_r)+1}. \ \blacksquare$$

If $\varkappa$ is a weakly inaccessible cardinal and A and B are strongly $\varkappa$-free groups of cardinality $\varkappa$, then for any E in $D(\varkappa)$, A is $\tilde{E}$-equivalent to B if and only if A and B have $\varkappa$-filtrations, $A = \bigcup_{\nu<\varkappa} A_\nu$, $B = \bigcup_{\nu<\varkappa} B_\nu$ such that for every $\nu\in E$, $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$. In this case we say A is *E-quotient-equivalent* to B (cf. section 4).

The proof of sufficiency is like that of the implication (3)⇒(1) in 3.4. As for necessity, let $\{\varkappa_\nu\colon \nu<\varkappa\}$ enumerate the cardinals less than $\varkappa$ in increasing order. Filter $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ so that for all ν, $A_{\nu+1}$ and $B_{\nu+1}$ are $\varkappa$-pure, and $\varkappa_\nu = |A_\nu| = |B_\nu|$, and

$$A_{\nu+1}/A_\nu \oplus \pi^{\varkappa^{\nu+1}} \cong A_{\nu+1}/A_\nu$$

(and similarly for $B_{\nu+1}/B_\nu$). For each group G, let $G(A) = \{\varkappa_\nu\colon A_{\nu+1}/A_\nu \sim G\}$. Then

$$A \models_{\tilde{E}} \mathrm{aa}\,s\,\mathrm{aa}\,s' \wedge \big(s'/s\sim G \Leftrightarrow |s|\in G(A)\big)$$

where the conjunction is over all groups G of cardinality $<\varkappa$. Let C be a cub such that for all $\nu\in C\cap E$,

$$B \models \mathrm{aa}\,s' \wedge \big(s'/B_\nu\sim G \Leftrightarrow B_\nu\in G(A)\big).$$

Hence for all $\nu\in C\cap E$, $B_{\nu+1}/B_\nu\sim A_{\nu+1}/A_\nu$, since $|B_\nu| = \varkappa_\nu = |A_\nu|$. But then it is easy to see, using the properties of the filtration, that $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$.

The following corollary should be compared with Example 2.1(2) and Theorem 2.2.

3.6. COROLLARY. *Let* $\tilde{E}\in D(\varkappa)$

(1) *If A and B are strongly $\varkappa$-free groups of cardinality $\varkappa$ such that $\Gamma(A)\subseteq 1-\tilde{E}$ and $\Gamma(B)\subseteq 1-\tilde{E}$, then A is $\tilde{E}$-equivalent to B.*

(2) *Hence, for every $n\in\omega$ and every $\tilde{E}\in D(\omega_{n+1})$ with $\tilde{E}\neq 1$, there is a non-free group A such that A is $\tilde{E}$-equivalent to a free group.*

Proof. (1) This is an immediate consequence of 3.4(2) since $\{\nu\in E\,|\,A_{\nu+1}|A_\nu\in\mathscr{K}\}$ is stationary in $\varkappa \Leftrightarrow \mathscr{K}$ contains a free group $\Leftrightarrow \{\nu\in E\,|\, B_{\nu+1}/B_\nu\in\mathscr{K}\}$ is stationary in $\varkappa$.

(2) By Theorem 1.5(7) there is a group A such that $\Gamma(A) = 1-\tilde{E}$, and by Lemma 1.4(2), A is not free if $\tilde{E}\neq 1$. ■

The following corollary should be compared with Example 2.1(1).

3.7. COROLLARY. *Let* $\tilde{E}\in D(\varkappa)$

(1) *If A and B are strongly $\varkappa$-free groups with $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ such that for all $\nu\in E$, $A_{\nu+1}/A_\nu\sim B_{\nu+1}/B_\nu$ then A is $\tilde{E}$-equivalent to B.*

(2) *Hence, for every $\tilde{E}\in D(\omega_{n+1})$ and every $\tilde{E}_1, \tilde{E}_2\in D(\omega_{n+1})$ such that $\tilde{E}\subseteq\tilde{E}_i$ $(i = 1, 2)$ there exist strongly $\varkappa$-free groups A and B such that $\Gamma_\varkappa(A) = \tilde{E}_1$, $\Gamma_\varkappa(B) = \tilde{E}_2$ and A is $\tilde{E}$-equivalent to B.*

Proof. (1) is an immediate consequence of 3.4(2).

(2) Let H be an $\mathfrak{F}_n$-group which is not free (see Definition 1.6). By Theorem 1.7 there exist groups $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ such that $\Gamma(A) = \tilde{E}_1$, $\Gamma(B) = \tilde{E}_2$ and for every $\nu\in\omega_{n+1}$,

$$A_{\nu+1}/A_\nu \cong \begin{cases} H & \text{if } \nu\in E_1, \\ \text{free} & \text{if } \nu\notin E_1\end{cases}$$

and

$$B_{\nu+1}/B_\nu \cong \begin{cases} H & \text{if } \nu\in E_2, \\ \text{free} & \text{if } \nu\notin E_2.\end{cases}$$

By part (1), A and B are $\tilde{E}$-equivalent. ■

The following should be compared with Example 2.3. Recall that a group G is *cotorsion* if $\mathrm{Ext}(A, G) = 0$ for all torsion-free groups A or, equivalently, if $\mathrm{Ext}(Q, G) = 0$ (see [9], § 54).

3.8. COROLLARY. (V = L) *Let G be a countable group which is not cotorsion. Let $\tilde{E}\in D(\omega_{n+1})$ such that $\tilde{E}\neq 1$. There is no sentence ψ of $L^{\tilde{E}}_{\infty\omega_{n+1}}(\mathrm{aa})$ such that for every torsion-free group A of cardinality ω_{n+1}, $A\models\psi$ if and only if* $\mathrm{Ext}(A, G) = 0$.

Proof by induction on $n\in\omega$. For $n = 1$ we construct $A^{(1)} = \bigcup_{\nu<\omega_1} A_\nu$ such that $A^{(1)}$ is strongly ω_1-free of cardinality ω_1, $\Gamma(A^{(1)}) = 1-\tilde{E}$ and for $\nu\in\omega_1-E$, $A_{\nu+1}/A_\nu\cong Q$. Then by Corollary 3.6(1) $A^{(1)}$ is $\tilde{E}$-equivalent to a free group but by Theorem 2.1 of [5], $\mathrm{Ext}(A^{(1)}, G)\neq 0$. Suppose that we have constructed an $\mathfrak{F}_n$-group $A^{(n)}$ such that $\mathrm{Ext}(A^{(n)}, G)\neq 0$. By Theorem 1.7 there is a group $A^{(n+1)} = \bigcup_{\nu<\omega_{n+1}} A_\nu$ which is strongly ω_{n+1}-free of cardinality ω_{n+1}, such that $\Gamma(A^{(n+1)}) = 1-\tilde{E}$ and for $\nu\in\omega_{n+1}-E$, $A_{\nu+1}/A_\nu\cong A^{(n)}$. Then by Corollary 3.6(1), $A^{(n+1)}$ is $\tilde{E}$-equivalent to a free group, but by Theorem 2.1 of [5],

$$\mathrm{Ext}(A^{(n+1)}, G)\neq 0. \ \blacksquare$$

4. Quotient-equivalent groups. We are going to characterize the pairs of strongly $\varkappa$-free groups of cardinality $\varkappa$ which are $\tilde{E}$-equivalent for every $\tilde{E} \in D(\varkappa)$. The following is the algebraic condition which turns out to be what we want.

4.1. DEFINITION (cf. [6]). If A and B are strongly $\varkappa$-free groups of cardinality $\varkappa$, we say that they are *quotient-equivalent* if there are $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$, $B = \bigcup_{\nu<\varkappa} B_\nu$ such that for every $\nu \in \varkappa$, $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$. Equivalently, A and B are quotient-equivalent if for any $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ of the groups, there is a cub C such that for $\nu \in C$, $A_{\nu+1}/A_\nu \sim B_{\nu+1}/B_\nu$ (cf. Definition 3.3). Note that if A is quotient-equivalent to B, then $\Gamma(A) = \Gamma(B)$.

4.2. DEFINITION. Let $\mathscr{B}$ be a Boolean algebra. Let us say that a subset Σ of $\mathscr{B}$ *separates points* if for any two different elements $c_0 \neq c_1$ of $\mathscr{B}$ there is $a \in \Sigma$ such that either (I) $c_0 \cap a = 0 \langle\neq\rangle c_1 \cap a = 0$ (i.e., $c_0 \cap a = 0$ and $c_1 \cap a \neq 0$ or vice versa) or (II) $c_0' \cap a = 0 \langle\neq\rangle c_1' \cap a = 0$, (where c' is the complement of c).

Obviously a dense subset of $\mathscr{B}$ separates points. But for any non-trivial $\mathscr{B}$ there are non-dense subsets which separate points. Indeed, if $b \in \mathscr{B} - \{0, 1\}$, then $\Sigma = \{a \in \mathscr{B} \mid a \cap b \neq 0\}$ is not dense but separates points.

4.3. THEOREM. *Let A and B be strongly $\varkappa$-free groups of cardinality $\varkappa$. The following are equivalent.*

(1) *A is quotient-equivalent to B;*

(2) *A is $\tilde{E}$-equivalent to B for every $\tilde{E}$ in $D(\varkappa)$;*

(3) *For some subset Σ of $D(\varkappa)$ which separates points, A is $\tilde{E}$-equivalent to B for every $\tilde{E}$ in Σ.*

Before proving the theorem, we present a set-theoretic result which we shall need. We are grateful to S. Shelah for supplying us with the following proof.

4.4. LEMMA (Shelah). *Let $\varkappa$ be a regular cardinal, E a stationary subset of $\varkappa$ and f and g functions from E into $\varkappa$ such that for every $\nu \in E$, $f(\nu) \neq g(\nu)$. Then there is a stationary set $E' \subseteq E$ such that*

$$\{f(\nu) \mid \nu \in E'\} \cap \{g(\nu) \mid \nu \in E'\} = \emptyset .$$

Proof. There is a stationary set $E_1 \subseteq E$ such that for all $\mu, \nu \in E_1$, $f(\mu) < \mu \Leftrightarrow f(\nu) < \nu$ and $g(\mu) < \mu \Leftrightarrow g(\nu) < \nu$ (because if we write E as the disjoint union of 4 subsets, then one of them is stationary). There is a cub C such that for every $\eta \in C$ and every $\nu < \eta$ we have $f(\nu) < \eta$ and $g(\nu) < \eta$. By Fodor's Theorem there is a stationary set $E_2 \subseteq E_1$ such that:

(i) if $f(\nu) < \nu$ for all $\nu \in E_1$, then $f|E_2$ is constant; and

(ii) if $g(\nu) < \nu$ for all $\nu \in E_1$, then $g|E_2$ is constant.

Finally, let $E' = E_2 \cap C$. We claim that if $\nu, \eta \in E'$, then $f(\nu) \neq g(\eta)$. Suppose false. Say $f(\nu) = g(\eta) = \alpha$. Then certainly $\nu \neq \eta$ since $E' \subseteq E$.

Case 1. $\alpha < \max\{\nu, \eta\}$. Say $\eta = \max\{\nu, \eta\}$. Then $g(\eta) = \alpha < \eta$ so by choice of E_1 and E_2, $g|E_2$ is constant. Thus $g(\eta) = g(\nu)$, but this contradicts the fact that $g(\nu) \neq f(\nu)$.

Case 2. $\alpha \geqslant \max\{\nu, \eta\}$. Say $\eta = \max\{\nu, \eta\}$, so $\nu < \eta$. Then since $\eta \in C$, $f(\nu) < \eta \leqslant \alpha$, a contradiction. ∎

Proof of 4.3. (1) ⇒ (2) follows immediately from Theorem 3.4. (2) ⇒ (3) is trivial. Thus it remains only to prove (3) ⇒ (1). Suppose $A = \bigcup_{\nu<\varkappa} A_\nu$ is not quotient-equivalent to $B = \bigcup_{\nu<\varkappa} B_\nu$ i.e., $E = \{\nu > \varkappa \mid A_{\nu+1}/A_\nu \nsim B_{\nu+1}/B_\nu\}$ is stationary in $\varkappa$. Fix a one-one correspondence between $\varkappa$ and the $\sim$-equivalence classes of groups of the form $A_{\nu+1}/A_\nu$ or $B_{\nu+1}/B_\nu$. Define functions $f, g \colon E \to \varkappa$ such that for all $\nu \in E$, $f(\nu)$ (resp. $g(\nu)$) equals the element of $\varkappa$ corresponding to the $\sim$-equivalence class of $A_{\nu+1}/A_\nu$ (resp. $B_{\nu+1}/B_\nu$). By Lemma 4.4 there is a stationary subject E' contained in E such that for all $\nu, \eta \in E'$ $A_{\nu+1}/A_\nu \nsim B_{\eta+1}/B_\eta$. Let $\mathscr{K}$ be the closure of $\{A_{\nu+1}/A_\nu \mid \nu \in E'\}$ under $\cong$ and $\sim$. Let

$$F_0 = \{\nu \in \varkappa \mid A_{\nu+1}/A_\nu \in \mathscr{K}\} \quad \text{and} \quad F_1 = \{\nu \in \eta \mid B_{\nu+1}/B_\nu \in \mathscr{K}\} .$$

Then $\tilde{F}_0 \neq \tilde{F}_1$ since $\tilde{F}_0 \cap \tilde{E}' = \tilde{E}'$, $\tilde{F}_1 \cap \tilde{E}' = 0$. Since Σ separates points there is an $\tilde{E}^* \in \Sigma$ such that either (I) $\tilde{F}_0 \cap \tilde{E}^* = 0 \langle\neq\rangle \tilde{F}_1 \cap \tilde{E}^* = 0$; or (II) $(1-\tilde{F}_0) \cap \cap \tilde{E}^* = 0 \langle\neq\rangle (1-\tilde{F}_1) \cap \tilde{E}^* = 0$. Suppose (I) holds. Since

$$\{\nu \in \tilde{E}^* \mid A_{\nu+1}/A_\nu \in \mathscr{K}\} = \tilde{F}_0 \cap \tilde{E}^* \quad \text{and} \quad \{\nu \in \tilde{E}^* \mid B_{\nu+1}/B_\nu \in \mathscr{K}\} = \tilde{F}_1 \cap \tilde{E}^*,$$

Theorem 3.4 implies that A is not $\tilde{E}^*$-equivalent to B, a contradiction. So (II) must hold. Let $\overline{\mathscr{K}}$ be the complement of $\mathscr{K}$. Thus

$$\{\nu \in \tilde{E}^* \mid A_{\nu+1}/A_\nu \in \overline{\mathscr{K}}\} = (1-\tilde{F}_0) \cap \tilde{E}^*$$

and

$$\{\nu \in \tilde{E}^* \mid B_{\nu+1}/B_\nu \in \overline{\mathscr{K}}\} = (1-\tilde{F}_1) \cap \tilde{E}^*$$

so again by Theorem 3.4 we obtain a contradiction. ∎

4.5. EXAMPLE. Suppose Σ is a subset of $D(\omega_{n+1})$ which does not separate points, i.e., there exist $\tilde{F}_0, \tilde{F}_1 \in D(\omega_{n+1})$ such that $\tilde{F}_0 \neq \tilde{F}_1$ but for every $\tilde{E}^* \in \Sigma$ (I) and (II) do not hold. Let $\varkappa = \omega_{n+1}$. We shall construct strongly $\varkappa$-free groups A and B of cardinality $\varkappa$ which are not quotient-equivalent but which are $\tilde{E}^*$-equivalent for every $\tilde{E}^*$ in Σ. Let H be an $\mathfrak{F}_n$-group which is not free. By Theorem 1.7 there are groups $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ such that $\Gamma(A) = \tilde{F}_0$, $\Gamma(B) = \tilde{F}_1$ and for every $\nu < \varkappa$, $A_{\nu+1}/A_\nu$ (resp. $B_{\nu+1}/B_\nu$) is either isomorphic to H or free-depending on whether $\nu \in F_0$ (resp. $\nu \in F_1$) or not. Obviously A is not quotient-equivalent to B since $\Gamma(A) \neq \Gamma(B)$. We shall use Theorem 3.4 to prove that A is $\tilde{E}^*$-equivalent to B for every $\tilde{E}^*$ in Σ. Fix $\tilde{E}^* \in \Sigma$. Given a class $\mathscr{K}$ of groups closed under $\cong$ and $\sim$, let

$$A[\mathscr{K}] = \{\nu \in E^* \mid A_{\nu+1}/A_\nu \in \mathscr{K}\}^{\sim} \quad \text{and} \quad B[\mathscr{K}] = \{\nu \in E^* \mid B_{\nu+1}/B_\nu \in \mathscr{K}\}^{\sim}.$$

If $H, Z \in \mathscr{K}$ then $A[\mathscr{K}] = E^* = B[\mathscr{K}]$ so 3.4(2) holds. If $H \in \mathscr{K}$, but $Z \notin \mathscr{K}$ then

$A[\mathscr{K}] = \tilde{F}_0 \cap \tilde{E}^*$, $B[\mathscr{K}] = \tilde{F}_1 \cap \tilde{E}^*$ so 3.4(2) holds because (I) fails. If $H \notin \mathscr{K}$, but $Z \in \mathscr{K}$, then $A[\mathscr{K}] = (1-\tilde{F}_0) \cap E^*$ and $B[\mathscr{K}] = (1-\tilde{F}_1) \cap E^*$ so 3.4(2) holds because (II) fails. Finally, if $H \notin \mathscr{K}$ and $Z \notin \mathscr{K}$ then $A[\mathscr{K}] = 0 = B[\mathscr{K}]$. ■

4.6. COROLLARY. $\Sigma \subseteq D(\omega_{n+1})$ *separates points if and only if whenever A and B are strongly ω_{n+1}-free groups of cardinality ω_{n+1} which are $\tilde{E}$-equivalent for all $\tilde{E} \in \Sigma$, then $\Gamma(A) = \Gamma(B)$.* ■

We could have defined a language $L_{\infty\varkappa}(D(\varkappa))$ in which we introduce a new quantifier $\mathrm{aa}_{\tilde{E}}$ for each $\tilde{E}$ in $D(\varkappa)$; thus a formula of $L_{\infty\varkappa}(D(\varkappa))$ may involve infinitely many different quantifiers of the form $\mathrm{aa}_{\tilde{E}}$. The semantics of $\mathrm{aa}_{\tilde{E}}$ is defined just as before. The following result shows that we do not obtain, at least in our setting, any additional strength by doing so.

4.7. THEOREM. *Let A and B be strongly $\varkappa$-free groups of cardinality $\varkappa$. The following are equivalent.*

(1) *A is $\tilde{E}$-equivalent to B for all $\tilde{E} \in D(\varkappa)$;*

(2) *A and B satisfy the same sentences of $L_{\infty\varkappa}(D(\varkappa))$.*

Proof. (2) $\Rightarrow$ (1) is trivial. Now suppose (1) holds; so by 4.3 A and B have $\varkappa$-filtrations $A = \bigcup_{\nu<\varkappa} A_\nu$ and $B = \bigcup_{\nu<\varkappa} B_\nu$ such that for all $\nu \in \varkappa$, $A_{\nu+1}/A_\nu \cong B_{\nu+1}/B_\nu$. There is an obvious back-and-forth criterion for $L_{\infty\varkappa}(D(\varkappa))$; namely, we require a set I of partial isomorphisms (f, σ) — as in 3.1 — which satisfies 3.2(1) and for every $\tilde{E} \in D(\varkappa)$ satisfies 3.2(2) and (2′). But inspection of the proof of Theorem 3.4 shows that we have such an I, namely the set of all partial isomorphisms (f, σ) such that σ is the identity on its domain. ■

5. The construction of quotient-equivalent groups. We shall show, under the assumption of the axiom of constructibility, the existence of many quotient-equivalent non-isomorphic groups. Recently Shelah has shown how to construct such groups in ZFC. However we do not know if the theorem from which we derive our result as a corollary is provable in ZFC.

In our proof we shall make use of the terminology and methods of the solution of the Whitehead Problem in L (see [21], or [4]).

5.1. THEOREM. (V = L) *Let A be a $\varkappa$-free group of cardinality $\varkappa$ and G a group of cardinality $<\varkappa$ such that* $\mathrm{Ext}(A, G) \neq 0$. *There are $2^\varkappa$ different groups B_i $(i<2^\varkappa)$ such that there is a short exact sequence*

$$(*) \qquad 0 \to G \xrightarrow{e_i} B_i \xrightarrow{\pi_i} A \to 0 .$$

Before proving the theorem let us derive two corollaries.

5.2. COROLLARY. (V = L) *Let A be a strongly $\varkappa$-free group of cardinality $\varkappa$ which is not free. Then there are $2^\varkappa$ different strongly $\varkappa$-free groups B_i $(i<2^\varkappa)$ of cardinality $\varkappa$ such that each B_i is $\tilde{E}$-equivalent to A for all $\tilde{E} \in D(\varkappa)$.*

Proof. Let $A = \bigcup_{\nu<\varkappa} A_\nu$ be a $\varkappa$-filtration of A. We apply 5.1 with $G = Z$. Assuming V = L, we have that A not free implies $\mathrm{Ext}(A, Z) \neq 0$ [22]. For any short exact sequence (*) define a $\varkappa$-filtration of B_i by $B_{i,\nu} = \pi_i^{-1}(A_\nu)$. Then it is clear that $B_{i,\nu+1}/B_{i,\nu} \cong A_{\nu+1}/A_\nu$, so B_i is quotient-equivalent to A. ■

We also have the following immediate corollary (see [13], [14], [15], [19] for various versions) which is not a theorem of ZFC. (The method of proof of 5.1 is a generalization of the method used to prove 5.3.)

5.3. COROLLARY. (V = L) *If A is a $\varkappa$-free group of cardinality $\varkappa$ and G a group of cardinality $<\varkappa$, then either* $\mathrm{Ext}(A, G) = 0$ *or* $|\mathrm{Ext}(A, G)| = 2^\varkappa$. ■

Proof of 5.1. Given A and G as in the hypotheses, fix a $\varkappa$-filtration $A = \bigcup_{\nu<\varkappa} A_\nu$ of A such that either $\mathrm{Ext}(A_{\nu+1}/A_\nu, G) \neq 0$ or $\mathrm{Ext}(A_\mu/A_\nu, G) = 0$ for all $\mu \geqslant \nu+1$. If we let

$$E = \{\nu \mid \mathrm{Ext}(A_{\nu+1}/A_\nu, G) \neq 0\}$$

then E is stationary in $\varkappa$ (Theorem 2.1 of [5]). We can write E as a disjoint union, $E = \bigcup_{i<\varkappa} E_i$, of stationary subsets of $\varkappa$ ([24]).

We claim that there are $2^\varkappa$ pairwise non-isomorphic pairs (B_i, e_i) such that

(5.1.1) There is a short exact sequence (*) and, moreover, such that; $B_i = A \times G$ as sets; π_i: $B \to A$ is projection on A; and e_i is defined by $e_i(x) = (0, x)$.

(where by an isomorphism of pairs (B_i, e_i) and (B_j, e_j) we mean a group isomorphism φ: $B_i \to B_j$ such that $\varphi \circ e_i = e_j$). If the claim is true, then the theorem follows since each B_i can appear only $\varkappa$ times as the first coordinate of a pair.

Suppose to the contrary that there are up to isomorphism, at most $\varkappa$ pairs (B_i, e_i), $i<\varkappa$. We shall obtain a contradiction by constructing a pair (C, e) satisfying the conditions (5.1.1) which is not isomorphic to any (B_i, e_i). As a set C will be $A \times G$. Let $A = \bigcup_{\nu<\varkappa} A_\nu$ be a $\varkappa$-filtration of A. We shall define by induction on ν a group structure C_ν on $A_\nu \times G$ such that

$$0 \to G \xrightarrow{e} C_\nu \xrightarrow{\pi} A_\nu \to 0$$

is a short exact sequence (where e: $G \to C$: $\nu \mapsto (0, x)$ and π: $C \to A$: $(a, x) \mapsto a$). To insure that (C, e) is not isomorphic to any (B_i, e_i) we make use of $\diamondsuit_\varkappa(E_i)$ for each $i<\varkappa$. For pair (B_i, e_i) fix a short exact sequence (*) satisfying the conditions (5.1.1). Let $B_{i,\nu} = \pi_i^{-1}(A_\nu)$. Let $\{f_{i,\nu}$: $B_{i,\nu} \to A_\nu \times G\}$ be a $\diamondsuit_\varkappa(E_i)$-sequence (see, for example, Theorem 0.2 of [4]).

The crucial case in the construction is when C_ν has been constructed, $\nu \in E_i$, and the function $f_{i,\nu}$ is an isomorphism of the pairs $(B_{i,\nu}, e_i)$ and (C_ν, e). We consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & Z & \xrightarrow{e_i} & B_{i,\nu} & \xrightarrow{\pi_i|B_{i,\nu}} & A_\nu & \to & 0 \\ & & \| & & \downarrow{\scriptstyle f_{i,\nu}} & & \downarrow{\scriptstyle \bar{f}} & & \\ 0 & \to & Z & \to & C_\nu & \xrightarrow{\pi|C_\nu} & A_\nu & \to & 0 \end{array}$$

where $\bar{f}$ is induced by $f_{i,\nu}$ i.e., $\bar{f}(\pi_i(b)) = \pi(f_{i,\nu}(b))$. Choose a splitting $\varrho\colon A_{\nu+1} \to B_{i,\nu+1}$ of $\pi_i|B_{i,\nu+1}$. Let $p = f_{i,\nu} \circ (\varrho|A_\nu) \circ f_{i,\nu}^{-1}\colon A_\nu \to C_\nu$. Then p is a splitting of $\pi|C_\nu$, so since $\mathrm{Ext}(A_{\nu+1}/A_\nu, G) \neq 0$ the usual argument shows that there is an extension

$$0 \to G \to C_{\nu+1} \xrightarrow{\pi|C_{\nu+1}} A_{\nu+1} \to 0$$

of $0 \to G \to C_\nu \to A_\nu \to 0$ such that p does not extend to a splitting of $\pi|C_{\nu+1}$ (see Lemma 1.3 of [4]).

The other cases of the construction are routine (cf. proof of Lemma 1.4 of [4]). Now having constructed C we must check that (C, e) is not isomorphic to any (B_i, e_i). Suppose to the contrary that for some i there is an isomorphism $\varphi\colon B_i \to C$ such that $\varphi \circ e_i = e$. Let $\bar{\varphi}\colon A \to A$ be the automorphism induced on A. There is a cub S such that for $\nu \in S$, $\varphi(B_{i,\nu}) = C_\nu$. By $\diamondsuit_\varkappa(E_i)$ there is a $\nu \in E_i \cap S$ such that $\varphi|B_{i,\nu} = f_{i,\nu}$. Thus we are in the crucial case of the construction. Choose $\tau > \nu$ such that $\tau \in S$. Since $A_\tau/A_{\nu+1}$ is free, ϱ extends to a splitting $\varrho'\colon A_\tau \to B_{i,\tau}$ for $\pi_i|B_{i,\tau}$. Let $p' = (\varphi|B_{i,\tau}) \circ \varrho' \circ (\bar{\varphi}|B_{i,\tau})^{-1}\colon A_\tau \to C_\tau$; then p' extends p since $\varphi|B_{i,\nu} = f_{i,\nu}$ and is a splitting for $\pi|A_\tau$. But this contradicts the construction of $C_{\nu+1}$. ■

By combining the methods of the above theorem and the methods of [7], we obtain the following results which are *not* theorems of ZFC. (For the second part we use [11].)

5.4. THEOREM. (i) (V = L) *For every $\varkappa$ which is regular and not weakly compact, there are $2^\varkappa$ strongly $\varkappa$-free indecomposable groups B_i $(i < 2^\varkappa)$ of cardinality $\varkappa$ which are pairwise quotient-equivalent and pairwise non-isomorphic.*

(ii) (GCH) *For $\varkappa = \omega_{n+2}$ $(n \in \omega)$ we obtain the same result as in part* (i). ■

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