Characterizing Hilbert space topology

by

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Abstract. It is shown that a complete separable $X \in$ ANR is homeomorphic to an open subset of a Hilbert space iff any map $\otimes \mathbb{P} \to X$ is strongly approximable by maps sending $\{\mathbb{P}\}_{n \in \mathbb{N}}$ to discrete families. Corresponding characterizations of non-separable Hilbert manifolds are also included which imply, in particular, that any Fréchet space is homeomorphic to a Hilbert space.

This paper establishes characterizations of manifolds modelled on infinite-dimensional Hilbert spaces, analogous to the following for $Q$-manifolds, proved by the author in [37]:

(0) A locally compact ANR, $X$, is a manifold modelled on the Hilbert cube $Q$ iff any two maps $f, g : Q \to X$ can be arbitrarily closely approximated by maps with disjoint images.

That result was obtained by using Edward’s theorem stating that $X \times Q$ is a $Q$-manifold, for any locally compact $X \in$ ANR, and by using Bing’s shrinking criterion to show that, under the assumption of (0), $X \cong X \times Q$.(1) The approach of [37] was adopted by Mogilski, who proved in his Ph. D. thesis that if $X \times Q$ is locally homeomorphic to the Hilbert space $l_2$, then so is $X$ and, more generally, that among ANR’s the CE-images of $l_2$-manifolds are $l_2$-manifolds. (See [28].) However, the compactness of the $Q$-axis was essential for the examination of the projection $X \times Q \to X$ in both [37] and [28] and, even though an analogue of Edward’s theorem is known for complete ANR’s (with $Q$ replaced by a Hilbert space), the lack of local compactness of the Hilbert spaces makes the characterizations of Hilbert manifolds more complicated and less accessible than that of $Q$-manifolds.

Here we apply a simple variation of Bing’s shrinking criterion, stated in §1, which is valid for non-proper maps, to examine the projection $X \times H \to H$ where $X \times H$ is a manifold modelled on a Hilbert space $H$ of infinite dimension. In this way we show in §2 that $X \cong H$ iff any map $H \to X$ can strongly be approximated by embeddings sending $H$ to a $Z$-set in $X$. This preliminary characterization is improved

(1) $\cong$ means homeomorphism.
Maps in $B(\mathcal{U}, \mathcal{V})$ are said to be $\mathcal{V}$-close to $f$. We topologize $C(Y, Z)$ by the limitation topology $\tau$ in which each $f \in C(Y, Z)$ has $B(\mathcal{U}, \mathcal{V}) = \{B(f, \mathcal{V}) : \mathcal{V} \in \text{cov}(Z)\}$ as basis of neighbourhoods. If $Z$ is metrizable then $\tau$ coincides with the topology of uniform convergence with respect to all metrics for $Z$; if $\mathcal{G}$ is a fixed metric for $Z$ then the sets $B_\mathcal{G}(f, a) = \{v \in C(Z, 0, \infty)\}$, for a basis of $(\mathcal{G})$-neighbourhoods of $f$.

If not stated otherwise, spaces $X, Y, Z$ are assumed to be metrizable and all function spaces are considered in the limitation topology. $C(Y, Z)$ is in general not metrizable. We shall use the following properties of $C(Y, Z)$:

1.1. **Lemma** (see [37], [33]). Let $Z$ be complete-metrizable, $F$ a subspace of $C(Y, Z)$ and $U_n, n \in \mathbb{N}$, open subspaces of $C(Y, Z)$. If $U_n \cap F$ is dense in $F$ for each $n$ then maps $F$ are approximable by elements (i.e. are in the closure) $\cap U_n \cap F_\mathcal{G}$, where $F_\mathcal{G}$ denotes the $\mathcal{G}$-closure of $F$ and $\mathcal{G}$ is any metric for $Z$. In particular, countable intersections of dense $G_\delta$-sets in $C(Y, Z)$ are dense in $C(Y, Z)$.

1.2. **Lemma**. If $\mathcal{A}$ is a family of subsets of $Y$ then $\{f \in C(Y, Z) : f(\mathcal{A})$ is locally finite in $Z\}$ and $\{f \in C(Y, Z) : f(\mathcal{A})$ is discrete in $Z\}$ are open subspaces of $C(Y, Z)$.

1.3. **Lemma**. If $Z$ is an ANR and $A$ is a closed subset of $Y$ then the restriction $f \mapsto f\mid A$ is an open map from $C(Y, Z)$ to $C(A, Z)$.

The proofs of 1.2 and 1.3 are omitted ([37], §1, contains a proof of 1.3 for the case where $Y$ is compact).

Given $\mathcal{U} \in \text{cov}(Y)$, a map $f : Y \rightarrow Z$ is said to be a $\mathcal{U}$-map if there exist $\mathcal{V} \in \text{cov}(Z)$ with $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.

1.4. **Lemma**. (a) For each $\mathcal{U} \in \text{cov}(Y)$ the set of all $\mathcal{U}$-maps $f : Y \rightarrow Z$ is open in $C(Y, Z)$.

(b) Let $\mathcal{G}$ be a complete metric for $Y$ and $\mathcal{U}_n \in \text{cov}(Y)$ be such that $\text{diam} U_n < 1/n$ for $U \subseteq \mathcal{U}_n$ and $n \in \mathbb{N}$. Then $f : Y \rightarrow Z$ is an embedding if it is a $\mathcal{U}_n$-map for all $n \in \mathbb{N}$. In particular, the set of embeddings $Y \rightarrow Z$ is a $G_\delta$-set in $C(Y, Z)$.

(c) If $f : A \rightarrow Z$ is a $\mathcal{U}$-map, where $f : C(Y, Z)$ and $A$ is closed in $Y$, then there is a neighbourhood $P$ of $A$ in $Y$ such that $f \mid P$ is a $\mathcal{U}$-map.

**Proof of (c).** Choose a locally finite $\mathcal{V}_0 \in \text{cov}(Z)$ with $f^{-1}(\mathcal{V}_0) \cap A \subseteq \mathcal{U}$ for all $V \in \mathcal{V}_0$ and let $\mathcal{V} \in \text{cov}(Z)$ be a star-refinement of $\mathcal{V}_0$, i.e. $\mathcal{V} \cap Z$ is the same realization equipped with the metric topology induced by the Hilbert space in which $[K]$ is naturally embedded, see [10], [22]. Given families $\mathcal{A}, \mathcal{B}$ of sets in a space $X$ we write $\mathcal{A} \subseteq \mathcal{B}$ if $\mathcal{A}$ refines $\mathcal{B}$ and, if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are some maps, we let $f^{-1}(\mathcal{A}) = \{f^{-1}(A) : A \in \mathcal{A}\}$ and $g(\mathcal{A}) = \{g(A) : A \in \mathcal{A}\}$. All the undefined notions have the meaning of [10] or [18].
homeomorphism if, given \( \forall \in \text{cov}(Z) \), there is a homeomorphism \( f: Y \to Z \) \( \forall \)-close to \( \pi \). The set of all near-homeomorphisms \( Y \to Y \) will be denoted by NH(\( Y \)).

1.5. Theorem (Bing's shrinking criterion). Let \( Y \) and \( Z \) be complete-metrizable spaces. A map \( \pi: Y \to Z \) is a near-homeomorphism iff \( \pi(Y) \) is dense in \( Z \) and the following condition is satisfied:

\[
(\text{bi}) \quad \text{given } \forall \in \text{cov}(Y) \text{ and } \forall \in \text{cov}(Z) \text{ there are } \forall \in \text{cov}(Z) \text{ and } f \in \text{NH}(Y) \text{ with } \forall \in B(\pi, \forall) \text{ and } f^{-1}(\forall) \prec \forall. 
\]

Proof. Let \( \forall_n \in \text{cov}(Y), n \in N \) be such that mesh(\( \forall_n \)) \( \to 0 \) with respect to some metric on \( Y \), and let \( U_n \) be the set of all \( \forall_n \)-maps \( Y \to Z \). We fix a metric \( g \) for \( Z \) and write \( F = \{ z: h \text{ is a homeomorphism of } Y \} \). Given \( h \in F \) and \( \forall \in \text{cov}(Y) \) there are, by (bi), \( \forall \in \text{cov}(Z) \) and a homeomorphism \( f \) of \( Y \) with \( f^{-1}(\forall) \prec h(\forall) \) and \( f^{-1} \) as close to \( \pi \) as we wish. Then \( f^{-1} \) is a \( \forall \)-map in \( F \) which closely approximates \( \pi h \). By 1.4 and 1.1 it thus follows that the set of embeddings \( Y \to Z \) belonging to \( F_n \) contains \( F \), hence \( \forall \), in its closure. As any map in \( F_n \) is a uniform limit of maps whose images are dense in \( Y \), the maps in \( F_n \) have dense images and embeddings in \( F_n \) are homeomorphisms. This completes the proof.

1.6. Remark. If \( \pi \) is a closed map then condition (bi) is equivalent to the following more familiar

\[
(\text{bi}^\prime) \quad \text{given } \forall \in \text{cov}(Y) \text{ and } \forall \in \text{cov}(Z) \text{ there is an } f \in \text{NH}(Y) \text{ with } \forall f \in B(\pi, \forall) \text{ and } f^{-1}(\forall) \text{ is } \forall \text{-close.} 
\]

Proof. If \( f^{-1}(\forall) \prec \forall \) then \( f^{-1}(\forall) \prec f^{-1}(\forall) \), whence \( \pi^{-1}(\forall) \prec f^{-1}(\forall) \), for \( \forall = \{ Z \times \pi(Y \setminus U): U \in f^{-1}(\forall) \} \in \text{cov}(Z) \). Thus (bi) \( \Rightarrow \) (bi), provided \( \pi \) is closed.

Condition (bi) is the one which Bing showed in [11] to characterize near-homeomorphisms between compacta. Later, Bing characterization has been extended in [26] and [17], [25] to show that (bi) distinguishes near-homeomorphisms among proper surjections of locally compact metric spaces and of complete metric spaces, respectively. (Actually, in all these papers a parametric version of (bi) was considered to give a necessary and sufficient condition on \( \pi \) to be conjugate to the 1-level of a small pseudo-isotopy of \( X \).)

§ 2. Characterizing Hilbert manifolds. I. We recall that a closed subset \( K \) is said to be a \( Z \)-set in \( X \), written \( K \in \mathcal{Z}(X) \), iff the set \( \{ f \in C(Q, X): f(Q) \cap \mathcal{K} = \emptyset \} \) is dense in \( C(Q, X) \). Embeddings whose images are \( Z \)-sets are called \( Z \)-embeddings. In this section we prove.

2.1. Proposition. A complete connected ANR, \( X \), is a manifold modelled on the infinite-dimensional Hilbert space \( H \) of the same weight as \( X \) iff the following condition is satisfied

\[
(\ast) \quad \text{for any complete-metrizable space } Y \text{ with } wY \subseteq wX \text{ the set of } Z \text{-embeddings } Y \to X \text{ is dense in } C(Y, X).
\]

Proof. Assume that \( X \) is a manifold modelled on \( H \) and fix \( Y \) and \( u \in C(Y, X) \). If \( w: X \times H \times H \to X \) is any homeomorphism and \( v: Y \to H \) is any embedding then \( w(u \times v \times 0) \) is a \( Z \)-embedding of \( Y \) into \( X \); therefore the necessity of (\( \ast \)) follows from the fact that \( p_X: X \times H \times H \to X \) is a near-homeomorphism ([5], [29]) and \( Y \) embeds into \( H \) ([10], p. 193).

The proof of the sufficiency part involves the following ideas: as \( H \) and \( Y \) are ANR's and Hilbert spaces, any given closed subset \( K \) of \( X \) can be obtained from \( X \times H \times H \) by the machinery of ANR; and the embedding \( \rho_K: X \times H \times H \to X \times H \) is a near-homeomorphism ([5], [29]) and \( Y \) embeds into \( H \) ([10], p. 193).

2.2. Lemma. If \( K \in \mathcal{Z}(X) \) then \( p_X: X \times H \to (X \times H)_K \) is a near-homeomorphism; moreover \( K \in \mathcal{Z}(\mathcal{Z}(X \times H)_K) \).

Proof. A homeomorphism \( g: X \times H \to (X \times H)_K \) with \( g(x, 0) = x \) for \( x \in K \) is defined in the proof of Proposition 5.1 of [36]; using the formulas there it is easy to see that \( g \) may be taken to approximate \( p_X \) as close as we wish. Since \( K \times \emptyset \in \mathcal{Z}(X \times H) \) we have \( K = g(K \times \{0\}) \in \mathcal{Z}(\mathcal{Z}(X \times H)_K) \).

Let \( J \) denote \([0, \infty)\) and \( C = (H \times J(0)) \), the metric cone over \( H \).

2.3. Lemma. Given \( \forall_0 \in \text{cov}(X) \) there is \( K = \bar{K} \subset X \) and \( g \in \text{NH}(X \times C) \) such that

(iii) \( p_X g \in B(p_X, \forall_0), g(K \times \{0\}) = X \times H \times \{ \infty \} \) and

(iv) the sets \( g(P \times C), \text{ where } P \text{ open in } X, \text{ form a basis of neighbourhoods of points of } X \times H \times \{ \infty \} \) in \( X \times C \).

Proof. Let \( \forall \in \text{cov}(X) \) satisfy \( s^2(\forall) \prec \forall_0 \); since \( X \in \text{ANR} \) there is by (\( \ast \)) a \( Z \)-embedding \( u: X \times H \times \{0\} \to X \) \( \forall \)-homotopic to \( p_X \). With \( K = \text{im}(u) \) there is by 2.2 a homeomorphism \( f: X \times C \to (X \times C)_K \) such that \( p_X f \) \( \forall \)-homotopic to \( p_X \); then \( f^{-1}(K) \in \text{NH}(X \times C) \) and \( f^{-1}(K) \in \mathcal{Z}(X \times C) \). Since \( X \times C \) is a Hilbert manifold [34], by the unknotting theorem for \( Z \)-sets there is a homeomorphism \( h \) of \( X \times C \) with \( p_X h \) \( s^2(\forall) \)-close to \( p_X \) and

\[
h[X \times H \times \{ \infty \}] = f^{-1} u: X \times H \times \{ \infty \} \to f^{-1}(K)
\]

(see [4], [12], [33]). We let \( g = h^{-1} f^{-1} p_X \).

Proof of 2.1. We shall apply 1.5 to show that \( X \times C \to X \) is a near-homeomorphism. This will conclude the proof since \( X \times C \cong X \times H \times J \) by 2.2 and \( X \times H \times J \) is an \( H \)-manifold by [34].

Given \( \forall \in \text{cov}(X) \) and \( \forall \in \text{cov}(X \times C) \). Take \( z: X \to (0, \infty) \) and \( \forall \in \text{cov}(X) \) such that \( s(\forall) \prec \forall \) and

(i) \( \{ (x, h, t) \in X \times (H \times J)_0: t \leq z(x), \forall x \in V \} \) refines \( \forall \), for each \( V \in \forall_0 \).
Then, take $\Psi_0 \in \text{cov}(X \times H)$ and $\gamma: H \to (0, \infty)$ such that

$$(ii) \quad \{(x, h, t) \in X \times H \times (0, \infty): (x, h) \in U \text{ and } t-\alpha(x, h) \in [\gamma(x, h), (i+3)\gamma(x, h)]\} \quad \text{refines } \Psi_0 \quad \text{for each } U \in \Psi_0 \text{ and } i \in \mathbb{N}.$$ (Construction. By compactness of $[0, \infty)$ there is a locally finite $\Psi_0 \in \text{cov}(X \times H)$ and positive reals $\delta_0, U \in \Psi_0$, such that $\text{inf}_{x} |U| > 0$ and $U \times [t, t+\delta_0] \not\subset \Psi$ for each $U \in \Psi_0$ and $t \geq \text{inf}_{x} |U|$. By the well known theorem on separation of semicontinuous functions by continuous ones ([18], p. 236) there is a $\gamma: X \times H \to (0, \infty)$ whose graph misses the closed subset $\bigcup \{U \times [\delta_0, \infty): U \in \Psi_0\}$ of $X \times H \times \mathbb{R}$.) Let $g \in \text{NH}(X \times C)$ satisfy (iii) and (iv). Then, there are an open neighbourhood $P_0$ of $K$ in $X$ and $\theta \in \text{cov}(P_0)$ such that

$$(v) \quad P_0 \times H \times (0, \infty) \subset \Psi_0 \quad \text{for } P \in \mathcal{P}.$$ Let $\beta_1: X \times H \to (0, \infty)$ be a map with $S_1 = \{(x, h, t): t \geq \beta_1(x, h)\}$ isomorphic to $(\mathbb{R}, H \times X)$ and the separation theorem again we get an open $\beta_2: X \times H \to (0, \infty)$, $\beta_2 \geq \beta_1$, with $c(\beta_2) = P_0$, $g(P_1 \times H \times X) = S_1$, and $S_2 = \{(x, h, t): t \geq \beta_2(x, h)\} \subset g(P_1 \times H \times X)$. Inductively, we construct maps $\beta_i: X \times H \to (0, \infty)$ and open neighbourhoods $P_i$ of $K$ in $X$ such that $c(\beta_i) \subset P_{i-1}$, $\beta_i \geq \beta_{i-1}$ and

$$(vi) \quad f(x, h, t) = \beta_{i+1}(x, h) \subset g(P_i \times H \times X) \subset (x, h, t): t \geq \beta_i(x, h) \quad \text{for} \quad i = 1, 2, \ldots.$$ Let $g$ be a homeomorphism of $X \times (H \times J)_{0,1}$ preserving the sets $\{x\} \times \{h\} \times J$ and carrying the graph of $\beta_1$ onto that of $\alpha_{i+1} + \gamma_{i+1}$, for each $i \in \mathbb{N}$. We put $f = \beta_2$; then $p x \in B(p_x, s(t'/s))$ and it remains to show that, given $x \in X$, there is a neighbourhood $G$ of $x$ such that $f(G \times C) \not\subset \Psi$. We consider 3 cases

$1^0$ $x \not\in P_0$. Take $V \in \Psi_0$ containing $x$ and put $G = \mathcal{V} \cap P_0$. Then $p x \in B(p_x, s(t'/s))$ and $f(G) = \{(x, h, t) \in X \times H \times (0, \infty): t \leq \alpha(x, h)\}$. Hence $f(G \times C) \not\subset \Psi$, by (i).

$2^0$ For some $i, x \in P_i \times P_{i+1}$. Take $P \in \mathcal{P}$ containing $x$ and put $G = P \cap P_{i+1} \cap P_{i+2}$. Then $p x \in B(p_x, s(t'/s))$ and $f(G \times C) \not\subset \Psi_0$ and

$$(x, h, t): t - \alpha(x, h) \in [\gamma(x, h), (i+3)\gamma(x, h)]$$

by (v) and (vi). Hence $f(G \times C) \not\subset \Psi$, by (i).

$3^0$ $x \in K$. The existence of the required neighbourhood $G$ of $x$ follows from (iii).

§ 3. Characterizing Hilbert manifolds II. Throughout this section we assume that $X$ is a connected complete metrizable ANR and $A$ is a discrete space of cardinality $\kappa X$. Our purpose is to establish the following characterization:

3.1. Theorem. $X$ is a Hilbert manifold iff the following two conditions are satisfied:

$$(1) \quad \text{for each } n \in \mathbb{N} \text{ the set of maps } A \times X \to X \text{ sending } \{(a) \times X\}_{a \in A} \text{ to a discrete family in } X \text{ is dense in } C(A \times X, X);$$

$$(2) \quad \text{for any sequence } (K_n) \text{ of finite-dimensional simplicial complexes having not more than } n \text{ vertices the set }$$

$$\{f \in C(\bigoplus_{n \in \mathbb{N}} |K_n|, X): \{f(K_n)\}_{n \in \mathbb{N}} \text{ is locally finite in } X\}$$

is dense in $C(\bigoplus_{n \in \mathbb{N}} |K_n|, X)$. 3.2. Corollary. $X$ is an $l_2$-manifold iff it is separable and

$$\{f \in C(\bigoplus_{n \in \mathbb{N}} I^n, X): \{f(I^n)\}_{n \in \mathbb{N}} \text{ is discrete in } X\}$$

is dense in $C(\bigoplus_{n \in \mathbb{N}} I^n, X).$ 3.3. Corollary. $X$ is a Hilbert manifold iff $X \times l_2 \cong X$ and there are metrics $g_n$ of $X$ such that, for all $n \in \mathbb{N}$, the set

$$\{f \in C(A \times X^n, X): \{f(a) \times X^n\}_{a \in A} \text{ is } \sigma\text{-discrete in } X\}$$

is $g_n$-dense in $C(A \times X^n, X).$

The necessity of all of the conditions mentioned follows from 2.1. The proofs of the sufficiency parts involve the following properties of metric simplicial complexes (they are presumably known but the author could not find them in the literature):

3.4. Lemma. Let $Y \in \text{ANR}$ and $\Psi \in \text{cov}(Y)$. Then, there is a locally finite-dimensional simplicial complex $K$ with not more than $Y / W$ vertices and maps $v: Y \to |K|, w: |K| \to Y$ such that $w \in B(\mathbb{R}, \Psi)$ and, writing $\Psi_0$ for the cover of $|K|$ by open stars of the vertices of $K$, we have $v^{-1}(|\Psi_0|) \not\subset \Psi$. Moreover, $K$ may be taken to be finite-dimensional if $Y$ is, and locally finite if $Y$ is separable.

Proof. Assume first that $Y$ is an open subset of a normed space $E(= \mathbb{R})$. Passing to a refinement we may assume that $\Psi$ consists of convex subsets of $E$. As $\Psi$ is a star-refinement of $\Psi$ such that the nerve $K$ of $\Psi$ is locally finite-dimensional; the existence of $\Psi$ follows from a result of Dowker [16], p. 209. If $Y$ is separable (finite-dimensional) then $\Psi$ may be taken star-finite (finite order). Define $v: Y \to |K|$ and $w: \mathbb{R}, \Psi_0$ to $E$ by

$$w(x) = (\lambda_\Psi(x))_{x \in \Psi} \quad \text{and} \quad (\lambda_\Psi(x))_{x \in \Psi} = \sum_{W \in \Psi} x_{\Psi}$$

$$W \in \Psi$$

where $(\lambda_{\Psi})$ is a locally finite partition of unity on $Y$ with $\lambda_\Psi(y, W) = \{0\}$ for all $W \in \Psi$, and $(x_\Psi: W \in \Psi)$ is a system of points of $Y$ with $x_\Psi \in W$, for all $W \in \Psi$. It is easy to see that $w(|K|) \subset X$ and $v$ and $w$ satisfy the required conditions.

The general case now follows by considering $Y$ as a retract of an open subset of a normed linear space (see [10], p. 68).

3.5. Lemma. Let $K$ be an $n$-dimensional simplicial complex, $n < \infty$, and let $P$ be an open neighbourhood of $|K_{n-1}|$ in $|K|$. Then, the convex hulls $C_\sigma$ of $|\sigma \setminus P, \sigma \in K_{n}$, form a discrete family in $|K|$. 253
Proof. Otherwise, there is an \( x \in C, C \subseteq \mathbb{C} \), where \( C = \bigcup \{ C_\sigma : \sigma \in K^{(n)} \} \). Then, \( x \in \tau \) for some \( \tau \in K^{(n-1)} \). By simple Hilbert-space geometry, \( \text{dist}(\theta \cap P, |\tau|) = \text{dist}(C_\tau, |\tau|) \) for all \( \sigma \in \text{st}(\tau, K) \). Since \( |\tau| \subseteq P \), we get \( \text{dist}(C \cap |\text{st}(\tau, K)|, |\tau|) > 0 \), contrary to the fact that \( x \in C \cap |\tau| \).

The reduction of 3.1 to 2.1 is divided into three lemmas:

3.6. LEMMA. Assume that \( X \) satisfies *(+1)* and, in addition, \( X \times X \approx X \). If \( K \) is a finite-dimensional simplicial complex with \( \leq_w X \), \( \theta \) the cover of \( K \) by open stars of the simplices of \( K \), then the set of \( \theta \)-maps \( [K] \to X \) is dense in \( C([K], X) \).

Proof. If \( \dim K = 0 \) then the assertion follows from *(+1)*. Suppose that \( \dim K = n \) and the lemma is proved for complexes of dimension \( n-1 \); we shall show that, given \( f : [K] \to X \) and a neighbourhood \( G \) of \( f \) in \( C([K], X) \), there is a \( \theta \)-map \( g \) in \( G \).

By 1.3 and the inductive assumption we may assume that \( f \) is a \( \theta \)-map on \( K^{(n-1)} \). Let \( P \) be a neighbourhood of \( K^{(n-1)} \) in \( [K] \) with \( f \mid P \) \( \theta \)-map, and let \( \theta \subseteq \text{cov}(X) \) be such that

(2) \( g : [K] \to X \) is \( \theta \)-close to \( f \) then \( g \in G \) and \( g \mid P \) is a \( \theta \)-map.

By 3.5 there is a cover \( C_\sigma : \sigma \in K^n \) of \( K \setminus \{ \sigma \} \), discrete in \( K \), such that \( C_\sigma \approx I^n \) and \( C_\sigma \subseteq \text{int}(\sigma) \) for each \( \sigma \in K^{(n)} \). By *(+1)* and 1.2 there is a map \( g_\sigma : [K] \to X \) sending \( C_\sigma \to \theta \)-discrete family and \( \theta \)-close to \( f \). Let \( D_\sigma : \sigma \in K^{(n)} \) be a discrete family of open subsets of \( [K] \) such that \( C_\sigma \subseteq \text{int}(\sigma) \) for each \( \sigma \in K^{(n)} \). Take \( \lambda : [K] \to [0,1] \) with \( \lambda(y) = 1 \) if \( y \in \text{int}(\sigma) \) and \( \lambda(y) = 0 \) if \( y \in \partial \sigma \). Define \( g : [K] \to X \times [0,1] \) by \( g(y) = (g_\sigma(y), \lambda(y)) \) for \( y \in [K] \). Then, \( g \) is a \( \theta \)-map: if \( \theta \subseteq \text{cov}(X) \) is such that, for each \( W \) \( \in \theta \), \( g^{-1}(W) \cap P \in \theta \) and \( W \) intersects at most one member of \( \{ g_\sigma(D_\sigma) : \sigma \in K^{(n)} \} \), then \( g^{-1}(W \times [s, t]) \subseteq \theta \) for all \( W \in \theta \) and \( s, t \in [0,1] \) with \( |s-t| < 1 \).

Since \( X \times [0,1] \) is homeomorphic to \( X \) \( \times [0,1] \), there is a homeomorphism \( h : X \times [0,1] \to X \) \( \theta \)-close to \( p_x \) (see [49], [29]). We let \( g = h \circ g \).

3.7. LEMMA. Let \( K \) be a simplicial complex with \( \leq_w X \), \( \theta \) the cover of \( |\tau| \) by the stars of the vertices of \( K \). If \( X \) satisfies *(+1)* then there is a dense \( \theta \)-set \( F \) in \( C([K], X \times X) \) such that \( (\theta \cap X) \) is dense in \( X \). \( \theta \)-close to \( f \) for all \( g \in \{ p_X f : f \in F \} \).

Proof. Given integers \( m, n \), consider for each \( \sigma \in K^{(m)} \setminus K^{(n-1)} \) the image of \( \sigma \) under the \( (1-n) \)-homotheory with respect to the barycenter of \( \sigma \), and denote by \( \sigma_{m,n} \) the discrete family of so obtained subsets of \( K \). By *(+1)* and the compactness of \( X \), the sets \( F_{m,n} = \{ f \in C([K], X \times X) : p_X f(\sigma_{m,n}) \text{ is \( \theta \)-discrete in } X \} \) are open and dense in \( C([K], X \times X) \). We let \( F = \bigcap F_{m,n} \).

3.8. LEMMA. Assume \( X \) satisfies *(+1)* and *(+2)* and, in addition, \( X \times X \approx X \). Then, for any complete metrizable space \( Y \) with \( w X \subseteq w X \), the set of all embeddings \( Y \to X \) is a dense \( G_\delta \) in \( C(Y, X) \).

Proof. By 1.1 and 1.4 it suffices to show that

(3) \( \forall \theta \in \text{cov}(X), \theta \in \text{cov}(X) \) and \( f : Y \to X \) there is a \( \theta \)-map \( g : Y \to X \) \( \theta \)-close to \( f \).

Since \( X \in \mathcal{N} \) there is an extension \( \bar{f} : Y \to f \) such that \( \bar{f} \in \mathcal{R} \). By 3.4 there are a locally finite-dimensional simplicial complex \( K \) and maps \( v : \bar{f} : [K] \to \bar{f} \) with \( [K] \to \bar{f} \) with \( f \) for close as to \( f \) as we wish and \( v^{-1}(\mathcal{R}) \), where \( \mathcal{R} \) is the cover of \( K \) by open stars of the vertices of \( K \). Then, \( g \) is a \( \theta \)-map for any \( \theta \)-map \( g_0 : [K] \to X \); this observation reduces the proof of (3) to the case where \( Y = [K] \), for some l.f.d. simplicial complex \( K \), and \( \theta \) is the cover of \( K \) by open stars of its vertices.

Let \( \theta = \theta_0 \subseteq \theta_1 \subseteq \ldots \) be open finite-dimensional subsets of \( [K] \) which cover \( [K] \), without loss of generality we assume that \( [K] \subseteq \theta_1 \). For all \( i \), replace \( \theta_i \) by \( \{ \theta_i \} \in \mathcal{R} \) where \( \theta_i \) is a discrete family \( \{ \theta_i \} \) of open finite-dimensional subsets of \( [K] \) such that \( \theta_i \subseteq \theta_{i+1} \subseteq \theta_i \). Then \( \theta_i \in \mathcal{R} \) for all \( i \) (see [22]), whence, by 3.4, (2) and 1.3, there is a map \( h : [K] \to X \) with \( h \in \mathcal{R} \) and \( \{ h^{-1}(\theta_i) \} \). By 3.7 and 1.3.2 we require in addition that the sets \( \{ h^{-1}(\theta_i) \} \) are pair-wise disjoint and hence form a discrete collection. There are \( \theta \)-maps \( h_\sigma : \theta \subseteq \text{cov}(X) \to X \) for all \( \theta \) \( \theta \)-close to \( f \) (see 3.6, 1.2 and 1.3). With \( n(i) = \dim \theta_i+2 \) for \( i \geq 0 \), we extend the map \( \theta_i \to X \) \( \theta \)-close to \( f \) (see 3.6, 1.2 and 1.3). With \( n(i) = \dim \theta_i+2 \) for \( i \geq 0 \), we extend the map \( \theta_i \to X \) \( \theta \)-close to \( f \) (see 3.6, 1.2 and 1.3). With \( n(i) = \dim \theta_i+2 \) for \( i \geq 0 \), we extend the map \( \theta_i \to X \) \( \theta \)-close to \( f \) (see 3.6, 1.2 and 1.3). With \( n(i) = \dim \theta_i+2 \) for \( i \geq 0 \), we extend the map \( \theta_i \to X \) \( \theta \)-close to \( f \) (see 3.6, 1.2 and 1.3).
Applying this with \( S = Y \oplus A \times Q \) we infer that, for any complete metric space \( Y \) with \( wY \leq wX \), the sets

\[
F_a = \{ f \in C( Y, X ) : f \text{ is an embedding and there is an } 1/n\text{-net } \{ g_\phi \}_{\phi \in \mathcal{A}} \text{ in } C( Q, X ) \text{ with } \{ f( Y ) \} \cup \{ g_\phi( Q ) \}_{\phi \in \mathcal{A}} \text{ discrete in } X \}
\]

are dense in \( C( Y, X ) \). Since the \( F_a \)'s are of type \( G_\delta \) it follows that \( \bigcap F_a \) is a dense subset of \( C( Y, X ) \) consisting of \( Z \)-embeddings. By 2.1, \( X \) is a Hilbert manifold.

Proof of 3.2. If \( X \) is separable then the complex occurring in the proof of 3.8 may be taken to be star-finite and the sets \( W_i \)'s in that proof to be compact polyhedra. Therefore in order that \( X \) be an \( I_2 \)-manifold it suffices that the set (1) be dense in \( C( \bigoplus K_n, X ) \), for any sequence \( ( K_n ) \) of finite complexes. Considering \( \bigoplus K_n \) as a subset of \( \bigoplus I^a \), a map \( \bigoplus K_n \to X \) extends to a map \( \bigoplus I^a \to X \) provided each \( g( K_n ) \) is contractible in \( X \). Therefore, under condition of 3.2, it follows from the local contractibility of \( X \) that each \( x \in X \) has an \( I_2 \)-manifold neighbourhood.

The proof of 3.3 involves the following

**Lemma.** Let \( Y_a, a \in A \), be metrizable spaces and let \( f : \bigoplus Y_a \to X \) be a map such that \( \{ f( Y_a ) \}_{a \in A} \) is \( \sigma \)-discrete in \( X \). If \( X \times I_2 \cong X \) then \( f \) is approximable by maps \( g : \bigoplus Y_a \to X \) with \( \{ g( Y_a ) \}_{a \in A} \) \( \sigma \)-discrete in \( X \).

**Proof.** Let \( ( A_a )_{a \in A} \) be a decomposition of \( A \) such that \( \{ f( Y_a ) \}_{a \in A} \) is discrete in \( X \), for all \( n \). We let \( g = \bigcup \{ v \colon X \times (0, \infty) \to X \) is a homeomorphism close to \( p_X \) (see [48], [29]) and

\[
\gamma( y ) = ( f( y ), y ) \in X \times (0, \infty) \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad y \in \bigcup \{ Y_a : a \in A \}.
\]

Proof of 3.3. It follows from the lemma that \( X \) satisfies \((\ast)\) and that to verify \((\ast)\) it suffices to check if, given \( f : A \times I^a \to X \) and \( a : X \to (0, \infty) \), there is a \( g \in B_\gamma( f, 2a ) \) with \( \{ g( a \times I^a ) \}_{a \in A} \) \( \sigma \)-discrete in \( X \).

To this end let

\[
A_i = \{ a \in A : \inf_{x \in X} a( x ) < 1/i, 1/i < 1 \} \quad \text{in} \quad \mathbb{N}.
\]

By assumption, there are \( g_i : A_i \times I^a \to X \), \( i \in \mathbb{N} \), with \( g_i( a, f( A_i \times I^a ) ) < 1/i \) and \( \{ g_i( a \times I^a ) \}_{a \in A} \) \( \sigma \)-discrete in \( X \). We let \( g( a, q ) = g_i( a, q ) \) for \( i \in \mathbb{N} \) and \( ( a, q ) \in A_i \times I^a \).

§ 4. Spaces finely dominated by Hilbert manifolds. We say that \( p : M \to X \) is a \( \mathcal{W} \)-domination, where \( \mathcal{W} \subset \text{cov}(X) \), if \( p \) is proper and there is a map \( q : X \to M \) with \( pq \text{-homotopic to } id_X \).

**Theorem.** Let \( X \) be a complete ANR. If, for every \( \mathcal{W} \subset \text{cov}(X) \), \( X \) is \( \mathcal{W} \)-dominated by a Hilbert manifold then \( X \) is a Hilbert manifold itself.

4.2. Remark. Let \( p : M \to X \) be a proper map of ANR's. If \( p \) is either a retraction or a CE-map then it is a \( \mathcal{W} \)-domination for each \( \mathcal{W} \subset \text{cov}(X) \). (The latter case follows from infinite-dimensional versions of Lacher's theorem, see [23] and [33].)

**Proof of 4.1.** We check that \( X \) satisfies \((\ast)\). Given \( f : A \times I^a \to X \) and \( \mathcal{W} \subset \text{cov}(X) \).

Let \( p : M \to X \) and \( g : X \to M \), where \( M \) is a Hilbert manifold, be as in the definition of a \( \mathcal{W} \)-domination. If \( f : A \times I^a \to M \) is a sufficiently close approximation to \( qf \) with \( \{ f_0( a \times I^a ) \}_{a \in A} \) \( \sigma \)-discrete in \( M \), then \( g = p_0 \mathcal{W} \)-approximates \( f \); moreover \( \{ g( a \times I^a ) \}_{a \in A} \) consists of compacta and is locally finite (we use the properness of \( p \)). Therefore the relation

\[
a \sim b \quad \text{iff there are } a_1, \ldots, a_n \in A \text{ with } a = a_1, a_n = b \text{ and } g( a_i \times I^a ) \cap g( a_{i+1} \times I^a ) \neq \emptyset \quad \text{for } i < n
\]

decomposes \( A \) into countable sets \( A_\lambda, \lambda \in A \), such that \( \{ g( A_\lambda \times I^a ) \}_{\lambda, a} \) is discrete in \( X \). Moreover, it follows from assumptions that

\[
\{ h \in C( I^a \times \{ 1, 2 \}, X ) : h( I^a ) \cap h( I^a ) = \emptyset \}
\]

dense in \( C( I^a \times \{ 1, 2 \}, X ) \); therefore one can apply Baire property of \( C( A_\lambda \times I^a, X ) \) to get \( \mathcal{W} \)-approximations \( \{ h_\lambda \} : A_\lambda \times I^a \to g( A_\lambda \times I^a ) \) such that still \( h = \bigcup \{ h_\lambda \} \) has \( \{ g( A_\lambda \times I^a ) \}_{\lambda, a} \) \( \sigma \)-discrete and \( \{ h( a \times I^a ) \}_{a, \lambda} \) locally finite for each \( \lambda \), but in addition each \( \{ h( a \times I^a ) \}_{a, \lambda} \) consists of disjoint sets. Then, \( h \) \( \mathcal{W} \)-approximates \( f \) and sends \( \{ a \times I^a \}_{a, \lambda} \) to a discrete family.

The verification of \((\ast)\) is trivial.

§ 5. Infinite products which are Hilbert spaces.

5.1. **Theorem.** Let \( X_1, X_2, \ldots \) be complete AR's. In any of the following cases \( X = \prod X_i \) is homeomorphic to a Hilbert space:

(a) \( wX = k_0 \) and infinitely many of the \( X_i \)'s are non-compact,
(b) \( wX > k_0 \) and sup \( wX_i = wX \), for each \( n \in \mathbb{N} \).

**Remark.** Let \( Y \) be a complete non-compact AR. By 5.2, the product \( Y \) is homeomorphic to a Hilbert space; in particular this is true for \( Y = J(\infty) \), the \( \mathcal{W} \)-hedgehog (see [18], pp. 172, 197), as was conjectured by de Groot. That \( J(\infty) \) is already been shown by Curtis and Vo-Thanh-Liem in a recent paper [14] which covers also some other special cases of 5.2.

In notation of 5.1 equip \( X \) with the metric \( q((x_i), (y_i)) = \max_{i \leq n} q_i( x_i, y_i ) \), where \( q_i \)

is a metric for \( X_i \) with \( q_i \leq 2^{-i} \), for each \( i \in \mathbb{N} \).

**Lemma.** Let \( H \) be the Hilbert space of weight \( wX \). If all the \( X_i \)'s contain closed subsets homeomorphic to \( H \), then \( X \cong H \).

**Proof.** By 3.1 and [20] it suffices to show that, given a complete metric space \( Y \) with \( wY \leq wX \) and maps \( f : Y \to X, a : X \to (0, 1) \), there is an embedding \( g : Y \to X \) with \( g \in B_\gamma(f, a) \).
By assumption, for each \( i \in \mathbb{N} \) there is an embedding \( \varphi_i : Y \to X_i \). Since \( X_i \in \mathbb{R} \) there are maps \( c_i : X_i \times [0, \infty) \to X_i, \ i \in \mathbb{N} \), such that \( c_i(x_1, x_2, t) = x_1 \) for \( t < 1 \) and \( c_i(x_1, x_2, t) = x_2 \) for \( t \geq 2 \). We define \( g \) by the formulas

\[
p_g(y) = c_i(p_f(y), \varphi_i(y), 2^n g(y)) \quad \text{for } y \in Y,
\]

where \( p_f : X \to X_i \) denotes the natural projection. Given \( y \in Y \), if \( g(y) \in \{2^{-n-1}, 2^{-n}\} \) then \( 2^n g(y) \leq 1 \) for \( i \leq n \), whence \( p_f(y) = p_g(y) \) for \( i \leq n \) and

\[
g(f(y), g(y)) \leq 2^{-n-1} \leq 2^n g(y).
\]

Moreover, if \( g(y_n) \) converges for a given sequence \( (y_n) \) then \( x = \lim_{n \to \infty} g(y_n) \) and consider \( \varepsilon = \inf g(y_n) \). If \( \varepsilon = 0 \) then \( (f(y_{i+n})) \) converges to \( x \) for any subsequence \( (y_{i+n}) \) of \( (y_n) \) with \( \lim_{n \to \infty} (y_{i+n}) = 0 \); this however yields \( \varepsilon(x) = 0 \) which is impossible. Thus \( \varepsilon > 0 \) and for \( n \) so large that \( 2^n \varepsilon \geq 2 \) we get \( p_f(y_n) = \varphi_i(y_n) \) for all \( i \). Since \( \varphi_i(y_n) \) is an embedding \( (y_n) \) converges and \( g \) is an embedding.

Proof of 5.1. Let \( A \) be a discrete space of cardinality \( \kappa \). Using the lemma and considering products of infinitely many of the \( X_i \)'s instead of the \( X_i \)'s themselves, we reduce the problem to demonstrating that \( X \) contains closed subsets homeomorphic to \( I_\varepsilon(A) \). In case (a) simply observe that each \( X_i \times X_{i+1} \) contains a closed copy of \([0, \infty) \) (cf. [14]) and separable metric spaces embed into \([0, \infty) \) by means of a sequence of partitions of unity. In case (b) assume without loss of generality \( wX = \lim_{i \to \infty} X_i \) and write \( X = \bigsqcup_{i \in \mathbb{N}} X_i \). By the preceding argument \( X_i \times I_\varepsilon \) is homeomorphic to a closed subset of \( X \) and by 3.3 it remains to show that, given \( n \in \mathbb{N} \) and \( f: A \times Q \to X_i \), there is a \( g: A \times Q \to X_i \) with \( \{g([a] \times Q): a \in A\} \) \( \sigma \)-discrete in \( X \) and \( g(f, .) \leq 2^n \). Put \( Y = \prod_{i \in \mathbb{N}} X_i \) and \( Z = \prod_{i \in \mathbb{N}} X_i \); then \( wZ = wX \). Pick a point from each member of a \( \sigma \)-discrete basis in \( Z \) to get a set \( \{z(a): a \in A\} \) \( \sigma \)-discrete in \( Z \) and define \( g: A \times Q \to X_i \) by \( p_fg = p_f \varphi f \) and \( p_g([a] \times Q) = [z(a)] \) for \( a \in A \) to complete the proof.

§ 6. The topological classification of Fréchet spaces. By a Fréchet space we mean any locally convex complete-metrizable topological vector space. The purpose of this section is to prove the following

6.1. Theorem. Any Fréchet space, \( X \), is homeomorphic to a Hilbert space.

The separable version of 6.1, to which we shall refer as to the Kadeč–Anderson theorem, was obtained by combined efforts of Kadeč [24], Anderson [2] and Blessaga and Pelczyński [8]. Shorter proofs of the results of Kadeč and Anderson were given in [3], [9] and [10].

For non-separable spaces many special cases of 6.1 have been established, including results of Blessaga [6], Trojanski [32], Gutman [19], Terry–Toruńczyk ([31] and [35]) stating, respectively, that Banach spaces which are either reflexive or are of the form \( c_0(A) \) or are weakly compactly generated are homeomorphic to a Hilbert space, as is any Fréchet space homeomorphic to its own countable product.

Other results on this subject proved till 1975 were contained in [7] and in [10], Chapter VII.

In the proof of 6.1 we use Kadeč–Anderson theorem and results 6.2 and 6.3 below:

6.2. Proposition. Let \( X \) be a complete connected ANR such that \( X \times I_\varepsilon \cong B \).

If \( K \subseteq \mathcal{L}(X) \) for any closed subset \( K \) of \( X \) with \( wK \approx wX \), then \( X \) is a Hilbert manifold.

Proof. Equip \( X \) with a metric \( g \) and let \( A \) be a discrete space of cardinality \( \kappa \). By 3.3 it suffices to show that, given \( \varepsilon > 0 \) and \( f: A \times Q \to X \), there is a \( g: A \times Q \to X \) with \( g(f, .) \leq \varepsilon \) and \( \{g([a] \times Q): a \in A\} \) \( \sigma \)-discrete in \( X \).

Let \( \sigma \) be a well-ordering of \( A \) with card \( [a \in A: a < a'] \approx wX \), for all \( a \in A \). We construct \( g([a] \times Q) \) by induction on \( \sigma \) so that

\[
\delta(a) = \inf \{g([a] \times Q), g([a'] \times Q): a' < a\} > 0,
\]

for all \( a \in A \). Let \[ g([a_0] \times Q) = f([a_0] \times Q) \]
for the minimal element \( a_0 \) of \( [A, <] \) and, if \[ g([a'] \times Q) \] has been defined for a certain \( a' < a_0 \), write \( K \) for the closure of \( g([a'] \times Q) \). Clearly \( wK \approx wX \), whence \( K \subseteq \mathcal{L}(X) \). To complete the construction we set \( g([a] \times Q) = \varepsilon \)-approximation of \( f([a] \times Q) \) whose image misses \( K \).

Write \( A_k = \{a \in A: \delta(a) > 1/k\} \) for \( k \in \mathbb{N} \). Evidently, \( \bigcup_{k \in \mathbb{N}} A_k \) is discrete in \( X \) for all \( k \in \mathbb{N} \); thus \( g \) has the desired properties.

6.3. Lemma. Let \( U \) be a connected open subset of a non-separable metrizable topological group \( X \). Then, \( K \subseteq \mathcal{L}(U) \) whenever \( K \) is a closed set in \( U \) with \( wK \subseteq wU \); in particular, \( X \times I_\varepsilon \) is a Hilbert manifold provided \( X \) is a complete ANR.

Proof. We may assume that \( 1 \times U \). It follows from the homogeneity of \( X \) that \( wK \approx wU \) for any subset \( V \subseteq U \) of \( U \) (see [30], p. 497). Therefore, given \( f: Q \to U \), there is a sequence \( (a_n) \) of points \( a_n = U \times \{x_n\} \in K \times X \) converging to \( 1 \times U \). We write \( g_n = g_n \in Q \) for \( n \in \mathbb{N} \); then \( g_n \circ Q \subseteq U \) for large \( n \)'s and \( g_n(Q) \cap K = \emptyset \) for all \( n \)'s, showing that \( K \subseteq \mathcal{L}(U) \). Hence if \( X \in \mathcal{L}(U) \) then 6.2 applies to any component of \( X \times I_\varepsilon \).

By 6.1. Let \( X \) be an infinite-dimensional Fréchet space. Then \( X \) is a Hilbert space by the Kadeč–Anderson theorem. A theorem of Bartle and Graves hence gives \( X \approx X \times I_\varepsilon \), for some \( X_i \) (see [10], p. 86). Thus \( X \approx X \times I_\varepsilon \times I_\varepsilon \approx X \times I_\varepsilon \), and the result follows from 6.3 and Hennerson's [20] theorem that Hilbert spaces are the only contractible Hilbert manifolds.

In connection with the subject of this section let us ask the following (cf. [10]).

6.1. Question. Let \( X \) be a complete-metrizable topological group. If \( X \) is an ANR, is \( X \) locally homeomorphic to a Hilbert space, of finite or infinite dimension?

Appendix to § 6. A proof of Kadeč–Anderson theorem. For the sake of completeness let us apply 3.2 to give a short argument for the Kadeč–Anderson
theorem which was used in the proof of 6.1 and states that $X \cong l_2$ for any separable infinite-dimensional Fréchet space $X$. We consider 3 cases:

1. $X$ is a Banach space. Let $g$ be the metric induced by the norm $|| \ | |$ of $X$. By 3.2 and Dugundji theorem it suffices to show that, given $\alpha : X \rightarrow (0, \infty)$ and $f : \bigoplus l^n \rightarrow X$, there is a $g \in B(f, \alpha)$ with $(g(l^n))_{n \in \mathbb{N}}$ discrete in $X$.

We construct $g/l^n$ by induction on $n$. Assume that, for a certain $n \geq 1$, $g$ has been defined on $\bigoplus l^k$ so that each $g(l^n)$, $k<n$, is contained in a finite-dimensional linear subspace of $X$ and

$$g(y) - f(y) = \alpha(y)$$

for $y \in l^n$, and let $E$ be the linear span of $g(l^n) \cup \ldots \cup g(l^{n-1}) \cup h(l^n)$; by construction, $\dim(E) \leq n$. Take a vector $x \in X$ of norm $\leq \frac{1}{2}$ which is at distance 1 from $E$ and write $g(y) = h(y) + \alpha(y)x$. Let $y \in l^n$ to complete the inductive step.

We claim that the map $g : \bigoplus l^n \rightarrow X$ satisfies (5), for all $n \in \mathbb{N}$

$$\limsup_{n \to \infty} \sup_{l^k \ni n} \frac{g(l^n) \setminus g(l^k)}{l^k}$$

is a discrete family in $X$. In fact, otherwise there exist integers $k_1 < \ldots < k_m$ and points $x_j \in l^{k_j}$ such that $(g(x_j))$ converges by (5), $\lim_{n \to \infty} g(y) = 0$ and $(f(y))$ converges; hence $\alpha(l(f(y))) = 0$ which is impossible.

2. $X = R^n$. This case is covered by 5.2(1).

3. The general case. We follow an argument of [8]. By a theorem of Eidelheit either $X$ is a Banach space, and then $X \cong l_2$ by 1, or there is a closed linear subspace $X_0$ of $X$ with $X/X_0 \cong R^n$; see [10], p. 184. By 2^0 and the Banach–Graves theorem we then have $X \cong l_2 \times X_0$, whence $X \cong l_2$ by [8] or [34].

References


[3] — and R. H. Bing, A completely elementary proof that the Hilbert space is homeomorphic to the countably infinite product of lines, ibidem, 74 (1968), pp. 771-792.


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