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Cell-like decompositions arising from mismatched sewings: applications to 4-manifolds

by

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Abstract. The Mismatch Theorem for sewings of crumpled n-cubes as proved by Eaton for $n = 3$ and Daverman for $n > 4$ can be extended to a larger related class of decompositions and can be reformulated so as to depend only upon a certain 1-ULC Taming Hypothesis that is always true for $n \neq 4$ and often (possibly always) true for $n = 4$. Such a reformulation, with applications to sewings of collarable objects in E^n and to cell-like decompositions of 4-manifolds, is the purpose of this paper. Thereby we extend some results of Daverman about sewings of crumpled n-cubes to the case $n = 4$ and answer some questions raised by Cannon about spun decompositions.

0. Prerequisites. The Mismatch Theorem and its applications developed by Eaton [16] and Daverman [13] provide much of the motivation for this paper. Their results set forth a homotopy-theoretic shrinking criterion for decompositions of S^n ($n \neq 4$) having as nondegenerate elements the arc fibers of a topologically embedded copy of $S^{n-1} \times I$. A new level of generality is introduced by the Mismatch Theorem developed here, which applies to decompositions of S^n ($n \neq 4$) having as non-degenerate elements the arc fibers of a topologically embedded compact space $X \times I$, where $X \times E^1$ is an n-manifold (implying that $X \times (0, 1)$ is open in S^n), and which also applies to certain decompositions of S^4 of the same type.

While not hesitating to deal with non-manifold factors X of n-manifolds $X \times (0, 1)$, we do maintain a simplifying hypothesis concerning the objects under consideration, which circumvents a significant question, by considering only compact spaces Z in S^n that admit collared embeddings in n-manifolds — in other words, not only is the frontier F of Z to be an n-manifold factor, but, in addition, adjunction to Z of an open collar on F must produce an n-manifold. In general, even if $F \times E^1$ is a manifold, whether adjunction of a collar along F produces a manifold is not known. For $n \geqslant 5$, the adjunction of such a collar does yield an n-manifold in two instances: (1) in case the frontier F is an $(n-1)$-manifold [14]; (2) in case Z is contained in an n-manifold N (without boundary) such that $N - Z$ is locally simply connected at each point of F [10].

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The proof for our version of the Mismatch Theorem, appearing in Sections 2 and 3, is self-contained except for some basic facts about sets satisfying ULC properties (see [20], [8], [13]) and for Bing's Shrinking Criterion (see [3] and [9, pp. 92 and 108]; the latter of these references contains further references to the literature on this criterion). In Sections 5 though 9, where examples are treated, we tacitly assume quite a bit about cell-like decompositions of 3-manifolds; basic references to the necessary notions and additional sources of information may be found in [16], [13], and [19].

After the results developed in this paper were first discovered, Cannon [11] and Edwards [18] produced powerful homotopy-theoretic methods for determining when a cell-like decomposition of an n-manifold ($n \geqslant 5$) yields the same manifold, which could be used to verify the results contained herein whenever $n \geqslant 5$. The primary advantage of the rather geometric approach which we employ is its applicability to the difficult dimension $n = 4$.

We use S^n, B^n, and E^n to denote the n-sphere, the n-ball, and Euclidean n-space, respectively. We use $\operatorname{Diam} X$ and $\operatorname{Cl} X$ to denote the diameter and closure of a set X. We use $\operatorname{Int} X$ and $\operatorname{Bd} X$ to denote the interior and boundary of a set X; we leave it to context to determine whether these are to denote combinatorial or point set interiors and boundaries, or even, in some cases, the boundary or interior of a crumpled cell or similar object.

Occasionally we use for (our) convenience some of the language of continuous relations [9]. The reader unfamiliar with this language should not have much trouble simply ignoring such material and supplying corresponding (ε, δ) arguments.

We thank Ric Ancel and Dan Everett for reading portions of a preliminary version and suggesting improvements.

1. Setting for the Mismatch Theorem. Throughout the next three sections $X = X \times \{\frac{1}{2}\}$ will denote a compact, connected, bicollared set in S^n with bicollar $X \times I$ ($I = [0, 1]$) such that $S^n - (X \times I)$ has exactly two components, U_0 (bounded

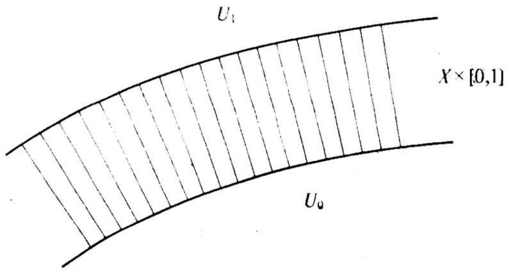

Fig. 1

by $X \times \{0\}$) and U_1 (bounded by $X \times \{1\}$) (see Fig. 1). Associated with $X \times I$ will be the cell-like upper semicontinuous decomposition G of S^n whose nondegenerate elements are the fibers $\{x\} \times I$ of $X \times I$ ($x \in X$), also denoted as $x[1 \leqslant 1]$. As usual, S^n/G will denote the associated decomposition space, and $\pi: S^n \to S^n/G$, the projection map.

2. Mismatch Theorem–simple form. The key decomposition space ideas of the Mismatch Theorem, due essentially to W. T. Eaton [16], are expounded here.

THEOREM 2.1 (Mismatch-simple form). *The spaces S^n and S^n/G are homeomorphic if each neighborhood of the identity* Id: $S^n \to S^n$ *contains a homeomorphism* h_1: $S^n \to S^n$ *such that, for each* $x \in X$, $h_1(\{x\} \times I)$ *misses either* $\operatorname{Cl} U_0$ *or* $\operatorname{Cl} U_1$ (see Fig. 2).

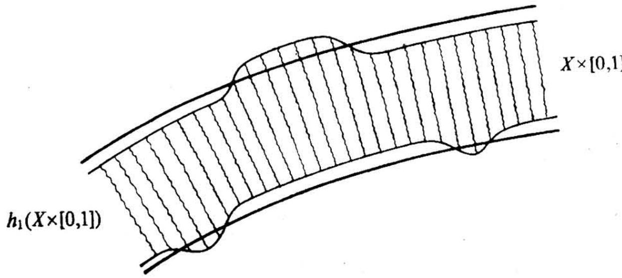

Fig. 2

Proof. The following notions, conventions, and notations will be helpful; let g: $S^n \to S^n$ denote any homeomorphism:

(1) $\pi_g = \pi \circ g^{-1}$: $S^n \to S^n/G$,

(2) $g(X \times I) = g(X) \times I$ (via the identification $g(x, t) \to (g(x), t)$),

(3) $x[j \leqslant k] = \{x\} \times [(j-1)/k, j/k]$ ($x \in g(X)$, $1 \leqslant j \leqslant k$),

(4) $g(X) \times I$ is a $k \times \varepsilon$-product if, for each $x \in g(X)$ and each $j \leqslant k$, $\operatorname{Diam} x[j \leqslant k] < \varepsilon$,

(5) if $\theta = [0 = \theta_0 < \theta_1 < \ldots < \theta_m = 1]$ and $\psi = [0 = \psi_0 < \psi_1 < \ldots < \psi_m = 1]$, then $g_{\theta\psi}$: $S^n \to S^n$ is the homeomorphism fixing $S^n - [g(X) \times I]$, taking (x, θ_i) to (x, ψ_i), and taking $\{x\} \times [\theta_i, \theta_{i+1}]$ "linearly" onto $\{x\} \times [\psi_i, \psi_{i+1}]$.

Suppose now that a positive number ε and a neighborhood U of π: $S^n \to S^n/G$ in $S^n \times (S^n/G)$ are given. By Bing's Shrinking Criterion [9, Theorem A13], it suffices to show the existence of a homeomorphism g: $S^n \to S^n$, with $\pi_g \subset U$, such that, for each $x \in g(X)$, $\operatorname{Diam}\{x\} \times I < \varepsilon$. There is a positive integer k such that $X \times I$ is a $k \times \varepsilon$-product. If k were 1, we could take $g = \mathrm{Id}$. We proceed by induction to reduce k to 1. Assume the following for some $i > 1$:

Inductive hypothesis (i) ($i \leqslant k$). There is a homeomorphism g: $S^n \to S^n$, with $\pi_g \subset U$, such that $g(X) \times I$ is an $(i \times \varepsilon)$-product.

We complete the inductive proof of Theorem 2.1 by defining a homeomorphism h: $S^n \to S^n$, with $\pi_{hg} \subset U$, such that $hg(X) \times I$ is an $(i-1) \times \varepsilon$-product. The homeomorphism h is the composite $h_3 h_2 h_1$ of three homeomorphisms h_1, h_2, and h_3; h_1 is simply a homeomorphism supplied by the hypothesis of Theorem 2.1, very near Id: $S^n \to S^n$, and satisfying, for each $x \in g(X)$, the requirement that $h_1(\{x\} \times I)$ misses either $g(X) \times \{0\}$ or $g(X) \times \{1\}$; i.e., h_1 frees at least one end of each fiber

$h_1(\{x\}\times I)$ $(x\in g(X))$ so that it can be shortened by h_2; h_2 is simply one of the maps $g_{\theta\psi}$,

$$\theta = [0<a<1/i\leq(1-(1/i))<b<1], \qquad \psi = [0<a'<1/i\leq(1-(1/i))<b'<1],$$

$a<a'<b'<b$ (see (5) above) which shoves fibers $h_1 g(\{x\})\times I$ toward $g(X)$ and thereby shortens them; h_3 is fixed outside $h_2h_1g(X)\times I$ and simply reparameterizes this product so that $hg(X)\times I$ is an $(i-1)\times\varepsilon$-product. The necessary additional details are as follows.

First choose $\alpha>\beta>0$, $a'\in(0,1/i)$, and $b'\in(1-(1/i),1)$ in such a manner that the following conditions are satisfied: Suppose $x\in g(X)$, $1\leq j\leq i$, $a\in(0,a')$, and $b\in(b',1)$ are chosen arbitrarily; then

(6) Diam $N(x[j\leq i];\alpha)<\varepsilon$.

(7) $g_{\theta\psi}N(x[j\leq i];\beta)\subset N(x[j\leq i];\alpha)$.

(8) $\{g(X)\times[0,b']\}\cap g_{\theta\psi}N(x[i\leq i];\beta)\subset N(x[i-1\leq i];\alpha)$.

(9) $\{g(X)\times[a',1]\}\cap g_{\theta\psi}N(x[1\leq i];\beta)\subset N(x[2\leq i];\alpha)$.

Next choose a neighborhood V of Id: $S^n\to S^n$ as follows. Since

$$\pi_g\circ \mathrm{Id}\circ(\pi_g^{-1}\pi_g)=\pi_g\subset U,$$

there is by [9, Theorem A 12] a neighborhood V of Id such that $\pi_g\circ V^{-1}\circ(\pi_g^{-1}\pi_g)\subset U$.

Choose h_1 in V so that h_1 moves no point as far as β, $a\in(0,a')$ and $b\in(b',1)$ so that, for each $x\in g(X)$, $h_1(\{x\})\times I$ misses either $g(X)\times\{a\}$ or $g(X)\times\{b\}$. Let $h_2=g_{\theta\psi}$. It follows from (6) and (7) that $h_2h_1g(X)\times I$ is an $i\times\varepsilon$-set. It follows from (8) and (9) that, for each $x\in h_2h_1g(X)$, either $\{x\}\times[0,2/i]$ or $\{x\}\times[1-(2/i),1]$ has diameter less than ε. By the lemma below, it follows that there is a homeomorphism h_3: $S^n\to S^n$, fixed outside $h_2h_1g(X)\times I$ and preserving the fibers of $h_2h_1g(X)\times I$, such that $hg(X)\times I$ is an $(i-1)\times\varepsilon$-product ($h=h_3h_2h_1$). It remains only to check that $\pi_{hg}\subset U$:

$$\pi_{hg}=\pi_{h_2h_1g}\circ h_3^{-1}\subset\pi_{h_2h_1g}\circ(\pi_{h_2h_1g}^{-1}\pi_{h_2h_1g})=\pi_{h_2h_1g}$$
$$=\pi_g\circ h_1^{-1}\circ h_2^{-1}\subset\pi_g\circ V^{-1}\circ(\pi_g^{-1}\pi_g)\subset U.$$

LEMMA 2.2. *Suppose that $X\times I$ is an $i\times\varepsilon$-product $(i>1)$, and suppose that, for each $x\in X$, either $\{x\}\times[0,2/i]$ or $\{x\}\times[1-(2/i),1]$ has diameter less than ε. Then there is a homeomorphism h_3: $S^n\to S^n$ in $\pi^{-1}\circ\pi$ such that $h_3(X)\times I$ is an $(i-1)\times\varepsilon$-product* (see Fig. 3).

Proof. Let

$$U_0=\{x\in X|\ \mathrm{Diam}\{x\}\times[0-(2/i),1]<\varepsilon\}$$

and

$$U_1=\{x\in X|\ \mathrm{Diam}\{x\}\times[1-(2/i),1]<\varepsilon\}.$$

Then U_0 and U_1 are open sets whose union is X. The proof is now an exercise in the use of Urysohn's Lemma. Choose $\delta>0$ so small that if $[a,b]\subset I$, $b-a<\delta$,

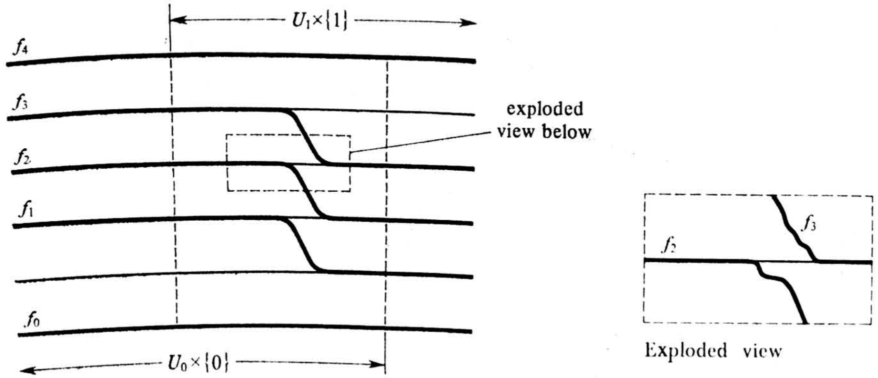

Fig. 3

and $x\in X$, then Diam$\{x\}\times[a,b]<\varepsilon$. There exist continuous functions $(0\equiv f_0)<f_1<\ldots<(f_{i-1}\equiv 1)$: $X\to[0,1]$ (see Fig. 3) such that

(1) If $j\neq 0$, $i-1$, Im$f_j\subset[j/i,(j+1)/i]$, $f_j(X-U_0)=j/i$, and $f_j(X-U_1)=(j+1)/i$; and

(2) If $0<j<j+1<i-1$ and $f_j(x)<(j+1)/i<f_{j+1}(x)$, then $f_{j+1}(x)-f_j(x)<\delta$ (see exploded view in Figure 3).

Define $h_3|X\times I$: $X\times I\to X\times I$ on each fiber $\{x\}\times I$ by defining $h_3(x,j/(i-1))=(x,f_j(x))$ and by extending "linearly". Then $h_3(X)\times I$ is an $(i-1)\times\varepsilon$-product.

3. Mismatch Theorem–general form.

THEOREM 3.1 (Mismatch Theorem). *Assume the* 1-ULC *Taming Hypothesis below. Suppose that X contains disjoint sets F_0 and F_1 such that $U_0\cup(F_0\times\{0\})$ and $U_1\cup(F_1\times\{1\})$ are* 1-ULC. *Then S^n and S^n/G are homeomorphic.*

(The condition concerning F_0 and F_1 is called a mismatch condition; we say that $X\times I$ satisfies a mismatch condition; see Fig. 4.)

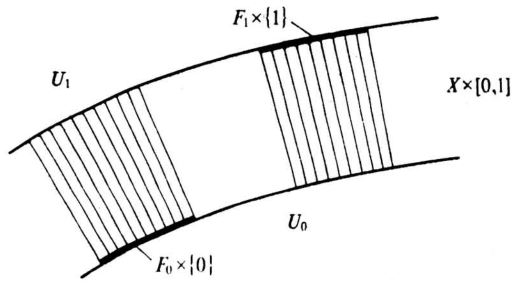

Fig. 4

1-ULC TAMING HYPOTHESIS. Suppose (f: $X\to[0,1])\subset(X\times[0,1])$ is a continuous function such that S^n-f is 1-ULC. Then each neighborhood of Id: $S^n\to S^n$

contains a homeomorphism $h: S^n \to S^n$, fixed outside an arbitrarily small neighborhood of f such that $(f \cap h(f^{-1}(\{0, 1\})) \times I) = \emptyset$ (see Fig. 5).

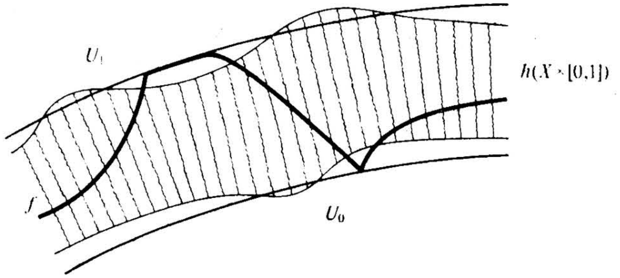

Fig. 5

Remark. The Taming Hypothesis is true for $n \neq 4$ (see [10]). Thus we recover the Eaton and Daverman Mismatch Theorems [16] [13]. If f were bicollared in S^n, the Taming Hypothesis would be obvious. If $n \neq 4$, then f is in fact bicollared in S^n [10]; however, this is more difficult to prove than merely the Taming Hypothesis itself. In Section 7 we shall establish the Taming Hypothesis for certain 4-dimensional examples. For the remainder of Section 3 we assume the hypothesis of Theorem 3.1.

The connection between the Taming Hypothesis and the Mismatch Theorem is forged by the following easy lemma, whose proof we omit. (See [20] and [8, Section 2. for the arguments of the type needed for the proof.)

Lemma 3.2. *Suppose* $(f: X \to [0, 1]) \subset (X \times [0, 1])$ *is a continuous function*] *If* $f^{-1}(0) \cap F_0 = \emptyset = f^{-1}(1) \cap F_1$, *then* $S^n - f$ *is* 1-ULC. *In fact,* $S^n - f$ *is* 1-ULC *if and only if* $\mathrm{Cl}(U_0) - f$ *and* $\mathrm{Cl}(U_1) - f$ *are* 1-ULC.

As a consequence of Lemma 3.2 and the Taming Hypothesis, we deduce the following key technical lemma, the statement of which is essentially that of [13, Approximation Theorem 3.1]. It is in our proof of the lemma that our work most markedly differs from [13].

Lemma 3.3. *Given* $\varepsilon > 0$, *there is a closed subset* F *of* X *satisfying the following conditions*:

(1) $\mathrm{Cl}(U_0) - (F \times \{0\})$ *is* 1-ULC.

(2) *If* V_0 *is a neighborhood of* F *in* X, *then there is an* $\frac{1}{2}\varepsilon$*-homeomorphism* h *of* S^n *fixed outside an* $\frac{1}{2}\varepsilon$*-nbd of* $X \times \{1\}$ *such that* $h((X - V_0) \times I) \cap \mathrm{Cl}\, U_1 = \emptyset$ (see Fig. 6).

Proof. Ric Ancel suggested substantial improvements for this proof, and many of the details of the present proof are due to him.

By standard arguments concerning ULC properties [20] [8, Section 2], [13, Section 2], we may assume that F_0 is the union of countably many closed sets $X_0 = \emptyset, X_1, X_2, \ldots$ We note that, since $U_1 \cup (F_1 \times \{1\})$ is 1-ULC and since $X_i \cap F_1 = \emptyset$ for each i, it is also true that $\mathrm{Cl}(U_1) - (X_i \times \{1\})$ is 1-ULC for each i.

We may therefore iteratively apply Lemma 3.2 and the Taming Hypothesis with respect to functions $(f_1, f_2, \ldots: X \to I) \subset (X \times I)$ such that $f_i^{-1}(\{0, 1\}) = f_i^{-1}(1) = X_i$ for each i. Such an iteration will be used to prove the following claim from which Lemma 3.3 follows easily.

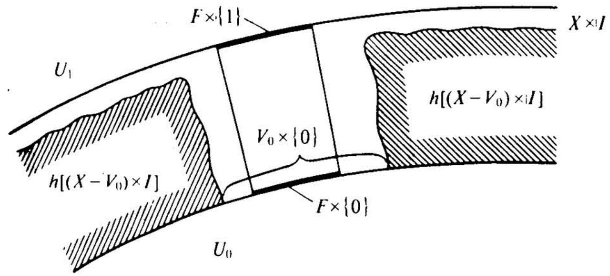

Fig. 6

Notation for the claim. If $A \subset S^n$, then PA is the image of $A \cap (X \times I)$ under the natural projection $X \times I \to X$.

Claim. *Given* $\varepsilon > 0$, *there exist* $\frac{1}{2}\varepsilon$*-homeomorphisms* $g_0, g_1, \ldots$ *of* S^n *and open subsets* $N_0, N_1, \ldots$ *of* X *such that*

(i) g_i *is fixed outside an* $\frac{1}{2}\varepsilon$*-neighborhood of* $X \times \{1\}$,

(ii) $Pg_i(\mathrm{Cl}\, U_1) \subset N_i$;

(iii) $(\mathrm{Cl}\, N_i) \cap (X_0 \cup X_1 \cup \ldots \cup X_i) = \emptyset$; *and*

(iv) $(\mathrm{Cl}\, N_{i+1}) \subset N_i$.

To prove the lemma from the claim, let

$$F = (\bigcap_{i=0}^{\infty} N_i) = (\bigcap_{i=0}^{\infty} \mathrm{Cl}\, N_i) \subset (X - \bigcup_{i=0}^{\infty} X_i).$$

Since $U_0 \cup (\bigcup_{i=0}^{\infty} X_i \times \{0\})$ is 1-ULC the set $\mathrm{Cl}\, U_0 - (F \times \{0\})$ is also 1-ULC. Thus requirement (1) of Lemma 3.3 is satisfied. If V_0 is a neighborhood of F in X and if $N_i \subset V_0$ then $h = g_i^{-1}$ satisfies requirement (2) of Lemma 3.3.

Proof of the claim. Choose numbers $\delta_0 < \delta_1 < \delta_2 < \ldots$ in the open interval (0, 1) such that, for each $x \in X$,

$$\mathrm{Diam}(\{x\} \times [\delta_i, 1]) < \tfrac{1}{2}[(\tfrac{1}{2})^{i+1} \cdot \tfrac{1}{2}\varepsilon].$$

Let g_0 = identity: $S^n \to S^n$ and let $N_0 = X$. Assume inductively that $g_0, g_1, g_2, \ldots, g_i$ and $N_0, N_1, \ldots, N_i$ have been chosen so that, in addition to (i), (ii), (iii), and (iv) of the claim, the following additional requirements are satisfied:

(v) if $i > 0$, then g_i moves no point as far as $[1 - (\frac{1}{2})^i] \cdot \frac{1}{2}\varepsilon$, and

(vi) $g_i(\mathrm{Cl}\, U_1) \subset U_1 \cup (X \times (\delta_i, 1])$.

The homeomorphism $g_{i+1}: S^n \to S^n$ may now easily be chosen as a composite $h_2 \circ h_1 \circ g_i$ of homeomorphisms described as follows.

Choose $f_{i+1}\colon X \to (\delta_{i+1}, 1]$ continuous with $f_{i+1}^{-1}(1) = X_{i+1}$. As noted above, $S^n - f_{i+1}$ is 1-ULC. By the Taming Hypothesis, there is an arbitrarily small homeomorphism h_2 of S^n, fixed outside an $\frac{1}{2}\varepsilon$-neighborhood of $X \times \{1\}$, such that $h_2(\mathrm{Cl}\, U_1) \cap f_{i+1} = \emptyset$. We choose h_2 so near the identity that (1) no point of S^n is moved as far as $\frac{1}{2}[(\frac{1}{2})^{i+1} \cdot \frac{1}{2}\varepsilon]$ and (2) no point of the compact set $[Pg_i(\mathrm{Cl}\, U_1) \times I] \cup \cup g_i(\mathrm{Cl}\, U_i)$ is moved into the disjoint compact set $(X - N_i) \times I$.

There is a homeomorphism h_1 of S^n, fixed outside of $X \times [\delta_i, 1]$ and fiber preserving on $X \times [\delta_i, 1]$, which pushes $g_i(\mathrm{Cl}\, U_1)$ so near to $\mathrm{Cl}\, U_i$, that

$$h_2 h_1 g_i(\mathrm{Cl}\, U_1) \cap f_{i+1} = \emptyset .$$

We leave it to the reader to check that an open set N_{i+1} may be chosen so that $g_{i+1} = h_2 h_1 g_i$ and N_{i+1} satisfy conditions (i)–(vi) ($i+1$ replacing i). This completes the proof of the claim and of Lemma 3.3.

Proof of Theorem 3.1. Suppose $\varepsilon > 0$ given, ε small compared to the distance between $X \times \{0\}$ and $X \times \{1\}$. Let F be a subset of X satisfying conditions (1) and (2) of Lemma 3.3. Let $(f\colon X \to [0, 1]) \subset X \times I$ be a continuous function very close to the identically zero function such that $f^{-1}(0) = F$. By (1), Lemma 3.2 and the Taming Hypothesis, there is an $\frac{1}{2}\varepsilon$-homeomorphism $h_1\colon S^n \to S^n$, fixed outside an $\frac{1}{2}\varepsilon$-neighborhood of f, such that $f \cap h_1(f^{-1}(\{0, 1\}) \times I) = \emptyset$. Hence, for some neighborhood V_0 of F in X, $f \cap h_1(V_0 \times I) = \emptyset$. By condition (2) there is an $\frac{1}{2}\varepsilon$-homeomorphism $h_2\colon S^n \to S^n$ of S^n, fixed outside an $\frac{1}{2}\varepsilon$-neighborhood of $X \times \{1\}$, such that $(X \times \{1\}) \cap h_2((X - V_0) \times I) = \emptyset$. Then $h = h_2 \circ h_1\colon S^n \to S^n$ is an ε-homeomorphism of S^n such that, for each $x \in X$, $h(\{x\} \times I)$ misses either $X \times \{0\}$ or $X \times \{1\}$. We conclude from Theorem 2.1 that S^n and S^n/G are homeomorphic.

4. Mismatch Theorem–extended forms. We catalogue some of the conditions which may be weakened in Theorems 2.1 and 3.1:

(i) S^n may be replaced by any manifold S without boundary.

(ii) $X \times I$ may be replaced by any closed subset X (twist-product) I of S which has the structure of a locally trivial arc-bundle over X and is a neighborhood in S of its $\frac{1}{2}$-section.

(iii) The global conditions may be replaced by local conditions defined on small subsets of X (twist-product) I on which the twist-product is a standard product. Similarly, the global 1-ULC Taming Hypothesis may be replaced by a local version.

(iv) The conclusion of the 1-ULC Taming Hypothesis need only be checked for functions $(f\colon X \to I)$ arising in the proof and application of Lemma 3.3.

An extraordinarily significant extension, to be used repeatedly in later sections, it warrants an explicit statement.

THEOREM 4.1. *Suppose for each $\varepsilon > 0$ there exists a closed set F in X such that*

(1) *for each continuous function $(f\colon X \to [0, 1))$ in $X \times [0, 1)$ with $F = f^{-1}(0)$, every neighborhood of the identity $\mathrm{Id}\colon S^n \to S^n$ contains a homeomorphism $h\colon S^n \to S^n$ fixed outside an arbitrarily small neighborhood of f such that $f \cap h(F \times I) = \emptyset$,*

(2) *for each neighborhood V_0 of F in X, there exists an $\frac{1}{2}\varepsilon$-homeomorphism $g\colon S^n \to S^n$ fixed outside the $\frac{1}{2}\varepsilon$-neighborhood of $X \times 1$ such that*

$$g((X - V_0) \times I) \cap \mathrm{Cl}\, U_1 = \emptyset .$$

Then S^n and S^n/G are homeomorphic.

Remark. Condition (1) holds if there exists some continuous $f_0\colon X \to [0, 1)$ with $F = f_0^{-1}(0)$ such that f_0 is bicollared.

5. Collarable objects and sewings. A *collared set* C in an n-manifold C is a closed proper subset of $\mathrm{Int}\, M$ such that the frontier $\mathrm{Fr}\, C$ of C is collared from $M - C$; in particular, such a collar is required to contain a neighborhood of $\mathrm{Fr}\, C$ relative to $\mathrm{Cl}(M - C)$. Consequently, $\mathrm{Fr}\, C \times E^1$ is an n-manifold.

An *n-dimensional collarable object* is a space C that can be embedded in some n-manifold as a collared set. Alternatively, for a space C and positive integer n, let F denote the subset of C consisting of all points having no neighborhood homeomorphic to E^n, and abstractly attach an open collar $F \times [0, 1)$ to C along F in the obvious manner to form a space M; then C is an n-dimensional collarable object iff M is an n-manifold.

Accordingly, an n-dimensional collarable object C has an intrinsically defined interior ($\mathrm{Int}\, C$) consisting of those points having neighborhoods homeomorphic to E^n and an intrinsically defined *boundary* ($\mathrm{Bd}\, C$) defined by $\mathrm{Bd}\, C = C - \mathrm{Int}\, C$.

A *sewing* of two n-dimensional collarable objects C_0 and C_1 is a homeomorphism h between their boundaries. Associated with any sewing h is the *sewing space*, denoted as $C_0 \cup_h C_1$, namely, the identification space obtained from the disjoint union of C_0 and C_1 under identification of each point c_0 from $\mathrm{Bd}\, C_0$ with the corresponding point $h(c_0)$ from $\mathrm{Bd}\, C_1$.

THEOREM 5.1. *Suppose h is a sewing of n-dimensional collarable objects C_0 and C_1. Then there exist an n-manifold M (without boundary), a bicollar $X \times I$ in M for which X is homeomorphic to $\mathrm{Bd}\, C_0$, and a decomposition G of M having for its nondegenerate elements the fibers $\{x\} \times I$ of $X \times I$, as in the setting of Section 1 for the Mismatch Theorem, such that $C_0 \cup_h C_1$ is naturally homeomorphic with M/G.*

To establish this we set $X = \mathrm{Bd}\, C_0$ and obtain M from the disjoint union of C_0, C_1 and $X \times I$ by identifying each point c of $\mathrm{Bd}\, C_0$ with $\langle c, 0\rangle$ in $X \times I$ and each point $h(c)$ in $\mathrm{Bd}\, C_1$ with $\langle c, 1\rangle$ in $X \times I$.

The following corollary involves a simple application of the Generalized Schoenflies Theorem.

COROLLARY 5.2. *Suppose that the n-dimensional collarable objects C_0 and C_1 can be embedded in S^n as collared sets and that $\mathrm{Bd}\, C_i \times E^1$ is homeomorphic to $S^{n-1} \times E^1$ ($i = 0, 1$). Then for each sewing h of C_0 and C_1 there is a bicollar $X \times I$ in S^n and there is a decomposition G of S^n having as its nondegenerate elements the fibers $\{x\} \times I$ of $X \times I$ such that $C_0 \cup_h C_1$ is homeomorphic to S^n/G.*

A sewing h on n-dimensional collarable objects C_0 and C_1 is said to satisfy the Homotopical Mismatch Property if there exist subsets F_0 and F_1 of $\mathrm{Bd}\, C_0$ and

$\mathrm{Bd}\, C_1$, respectively, such that $F_i \cup \mathrm{Int}\, C_i$ is 1-LC at each point of C_i $(i = 0, 1)$ and $h(F_0) \cap F_1 = \emptyset$.

THEOREM 5.3 (Homotopical Mismatch Theorem). *Let C_0 and C_1 denote n-dimensional collarable objects ($n \neq 4$) and let h denote a sewing of C_0 and C_1 that satisfies the Homotopical Mismatch Property. Then $C_0 \cup_h C_1$ is homeomorphic to the n-manifold $M = C_0 \cup (X \times I) \cup C_1$ of Theorem* 5.1.

Remarks. This is nothing more than a reinterpretation of Theorem 3.1. For $n = 3$ it is a consequence of Eaton's work [16] because all 3-dimensional collarable objects are bounded by 2-manifolds. For $n \geq 5$ it extends Daverman's work [13] about sewings of crumpled n-cubes.

An additional strong and significant feature of the Homotopical Mismatch Property in dimension 3 is that it is both necessary and sufficient for a sewing to yield a 3-manifold [16], while in higher dimensions, even in the simple crumpled n-cube case, a sewing can yield S^n without satisfying the Homotopical Mismatch Property [13, Example 13.2]. Nevertheless, in some important situations, such as that set forth in the following result, the Homotopical Mismatch Property is both necessary and sufficient for the sewing to yield an n-manifold. (Necessity follows, even for $n = 4$, just as in [13, Theorem 10.1].)

COROLLARY 5.4. *Let C an n-dimensional collarable object, $n \geq 5$, and* $\mathrm{Id}\colon \mathrm{Bd}\, C \to \mathrm{Bd}\, C$ *the identity map. Then $C \cup_{\mathrm{Id}} C$ is an n-manifold if and only if* $\mathrm{Bd}\, C$ *contains disjoint sets F_0 and F_1 such that $F_i \cup \mathrm{Int}\, C$ is* 1-LC *at each point of C* $(i = 0, 1)$.

Elementary transformation group theory finds some use for the $C \cup_{\mathrm{Id}} C$ construction, for there exists an obvious involution of $C \cup_{\mathrm{Id}} C$ having the seam corresponding to $\mathrm{Bd}\, C$ as its fixed point set and having C as its orbit space.

COROLLARY 5.5. *Let h denote a sewing of n-dimensional collarable objects C_0 and C_1. Then $(C_0 \cup_h C_1) \times E^1$ is an $(n+1)$-manifold.*

Proof. For $n = 3$ this is announced in [14], so we assume $n \geq 4$. Clearly $(C_0 \cup_h C_1) \times E^1$ is equivalent to $(C_0 \times E^1) \cup_{h \times \mathrm{Id}} (C_1 \times E^1)$. Choose disjoint dense subsets D_0 and D_1 of E^1. Then $F_0 = \mathrm{Bd}\, C_0 \times D_0$ and $F_1 = \mathrm{Bd}\, C_1 \times D_1$ show that $h \times \mathrm{Id}$ satisfies the Homotopical Mismatch Property.

A result more general than Corollary 5.5 is established in Section 9.

An n-dimensional collarable object C is said to be *universal* if, for any n-dimensional collarable object C^* and any sewing h of C and C^*, $C \cup_h C^*$ is an n-manifold.

COROLLARY 5.6. *If C is an $(n-1)$-dimensional collarable object ($n \geq 5$), then $C \times E^1$ is universal.*

Note that, because C is collarable, $\mathrm{Bd}(C \times E^1)$ is an n-manifold. Thus, the proof of Theorem 8.6 of [13] shows that any sewing of $C \times E^1$ to another n-dimensional collarable object satisfies the Homotopical Mismatch Property.

6. Inflations and inflated decompositions. The results of this section improve upon those of [13, Section 11] and serve as an auxiliary aid for the spinning construction described in Section 8.

Let C be a space, B a closed subset of C, and $f\colon C \to [0, 1)$ a map such that $B = f^{-1}(0)$. By the *inflation of C relative to B* we mean the space

$$\mathrm{Infl}(C, B) = \{\langle c, t\rangle \in C \times E^1 \mid |t| \leq f(c)\}.$$

The topological type of $\mathrm{Infl}(C, B)$ does not depend on the particular map f employed.

For simplicity we confine our attention to a relatively standard setting. Consider a cell-like upper semicontinuous decomposition G of B^n, every nondegenerate element of which intersects S^{n-1} in a cell-like subcontinuum. There is an *inflated decomposition* $I(G)$ of B^{n+1} and there is a doubled decomposition $2G$ of $S^n = \mathrm{Bd}\, B^{n+1}$ whose nondegenerate elements are, respectively, the sets $\{(g \times E^1) \cap B^{n+1}\}$ and $\{(g \times E^1) \cap S^n\}$, where g is a nondegenerate element of G and $E^{n+1} = E^n \times E^1$ (see Fig. 7).

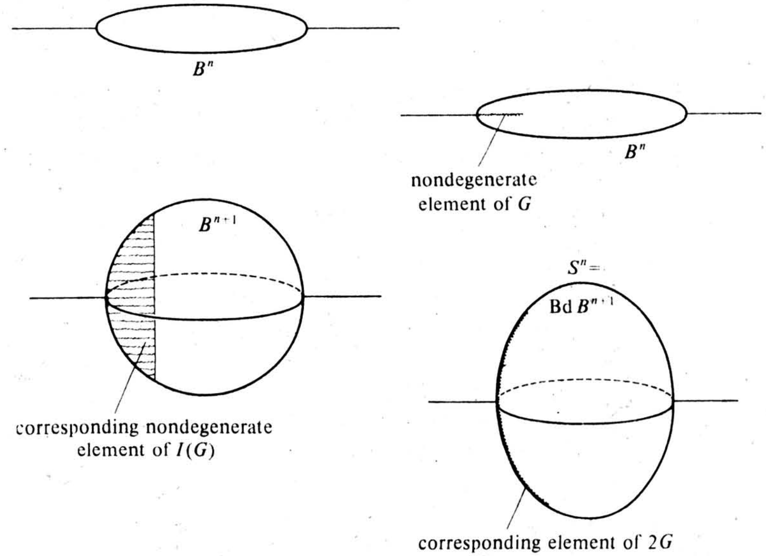

Fig. 7

Our next result relates inflations and inflated decompositions. We leave the proof tracing through the definitions to the reader.

THEOREM 6.1. *Suppose G is a cell-like upper semicontinuous decomposition of B^n, each nondegenerate element of which intersects S^{n-1} in a cell-like subcontinuum; let $C = B^n/G$ and let B denote the image of S^{n-1} in C. Then $B^{n+1}/I(G)$ is homeomorphic to* $\mathrm{Infl}(C, B)$, *and $S^n/2G$ is homeomorphic to $C \cup_{\mathrm{Id}(B)} C$, where* $\mathrm{Id}(B)$ *denotes the identity map of B to itself.*

Remark. Even when C is not an n-dimensional collarable object, it makes sense to consider $C \cup_{\mathrm{Id}(B)} C$ as a sewing space.

The following is like Theorem 11.1 of [13].

THEOREM 6.2. *There is a collared embedding $h\colon B^{n+1}/I(G) \to E^{n+1}$ if the following two conditions are satisfied:*

(1) *The decomposition* $I(G)$, *extended trivially to all of* E^{n+1}, *is a shrinkable decomposition of* E^{n+1} (see definition below).

(2) *The decomposition* $(2G)\times E^1$ *of* $S^n\times E^1$ (*with elements of the form* $g\times\{t\}$, $g\in 2G$, $t\in E^1$) *is a shrinkable decomposition of* $S^n\times E^1$.

Addendum. If F is a closed set in S^n which misses the nondegenerate elements of G, then the embedding h: $B^{n+1}/I(G)\to E^{n+1}$ may be chosen in such a manner that

$$F\subset B^{n+1}\to B^{n+1}/I(G)\xrightarrow{h} E^{n+1}$$

is the identity and such that the fibers $[1,2]\cdot\{x\}$, for $x\in F$, are fibers of the collar on $h(B^{n+1}/I(G))$ ($\cdot$ denotes scalar product in the vector space E^n).

DEFINITION. A decomposition G of a space X is *shrinkable* if the following condition is satisfied: let U be any G-saturated open subset of X, let V be any neighborhood of Id: $U\to U$, and let W be any neighborhood of π: $U\to U/(G|U)$; then there is a homeomorphism f: $U\to U$ such that $(\pi_f^{-1}\cdot\pi_f)\subset V$ and $\pi_f\subset W$ ($\pi_f=\pi\cdot f^{-1}$: $U\to U/(G|U)$). Under rather general conditions [9, Theorem A 13], the shrinkability of G is sufficient to prove the existence of a closed map p: $X\to X$, the identity on any given closed subset of X missing the nondegenerate elements, such that the point preimages of p are precisely the elements of G.

Remark. In general it is easier to prove a slightly weaker shrinkability condition, and this course is followed by almost all authors. However, a careful examination of most proofs shows that an adaptation of the argument proves shrinkability in the sense defined above.

Proof of Theorem 6.2. Consider the decomposition G' of E^{n+1} whose nondegenerate elements are either of the form $g\subset B^{n+1}$ ($g\in I(G)$, g nondegenerate) or of the form $t\cdot g\subset t\cdot S^n$ ($t\in(1,\infty)$, $g\in 2G$, g nondegenerate). The set $U=E^{n+1}-[1,\infty)\cdot F$ is a G'-saturated open subset of E^{n+1} containing the nondegenerate elements of G'. Using (1) and (2) and the definition of shrinkability, one can show the existence of a closed surjective map p: $E^{n+1}\to E^{n+1}$, fixed on $[1,\infty)\cdot F$, such that the point preimages of p are precisely the elements of G' (see, for example, [9, Theorem A13]). This shows that E^{n+1} is homeomorphic with E^{n+1}/G', and the desired result follows.

COROLLARY 6.3. *Let* G *denote a cell-like upper semicontinuous decomposition of* B^n ($n\geq 4$), *each nondegenerate element of which intersects* $S^{n-1}=\mathrm{Bd}\,B^n$ *in a cell-like subcontinuum. Suppose that* $(B^n/G)\times E^1$ *is a* $(n+1)$-*dimensional collarable object. Then* $B^{n+1}/I(G)$ *has a collared embedding in* E^{n+1}.

Proof. The hypothesis implies that the trivial extension $G'\times E^1$ of the product decomposition $G\times E^1$ yields E^{n+1}, and, therefore, by the Approximation Theorems of Siebenmann [29] or Cannon [9, Theorem 56], $G'\times E^1$ is shrinkable. From this it follows that the decomposition of E^{n+1} whose nondegenerate elements are those of the form $g\times\{0\}$, $g\in G$, is shrinkable, and then $I(G)$, trivially extended, is also shrinkable.

To verify Condition (2), we set $C=B^n/G$ and B equal to the image of S^{n-1} in C. By Theorem 6.1, $S^n/2G=C\cup_{\mathrm{Id}(B)}C$. By the proof of Corollary 5.6, $(C\cup_{\mathrm{Id}(B)}C)\times E^1$ is an $(n+1)$-manifold. Once again the approximation theorems imply that $2G\times E^1$ is shrinkable.

COROLLARY 6.4. *Let* G *denote a cell-like upper semicontinuous decomposition of* B^n ($n\geq 4$), *each nondegenerate element of which intersects* $S^{n-1}=\mathrm{Bd}\,B^n$ *in a cell-like subcontinuum, and such that* B^n/G *is an* n-*dimensional collarable object. Then* $B^{n+1}/I(G)$ *has a collared embedding in* E^{n+1}.

COROLLARY 6.5. *If* C *is a crumpled* 3-*cube, then* $\mathrm{Infl}(C,\mathrm{Bd}\,C)$ *is a* 4-*dimensional collarable object.*

Proof. We obtain a decomposition G of B^3 into points and arcs, where the set of arcs consists of the fibers from some collar $c(S^2\times I)$ on $\mathrm{Bd}\,B^3$ in B^3, such that $C=B^3/G$ [22] [24]. Again it is easy to see that Condition (1) of Theorem 6.2 holds, and it follows from the proof of Theorem 9.1 or, as hinted in [15], from techniques of Bryant [7], that $2G\times E^1$ is a shrinkable decomposition of $S^3\times E^1$. Thus, there exists a collared embedding of $B^4/I(G)=\mathrm{Infl}(C,\mathrm{Bd}\,C)$ in E^4.

COROLLARY 6.6. *If* G *is a cell-like, closed*-0-*dimensional decomposition of* B^3, *each nondegenerate element of which intersects* $S^2=\mathrm{Bd}\,B^3$ *in a singleton, then* $B^4/I(G)$ *admits a collared embedding in* E^4.

The main result of [17] or [19] implies that both $G'\times E^1$ and $2G$ are shrinkable decompositions of $E^3\times E^1$ and $S^3\times E^1$, and, as in Corollary 6.3, the shrinkability of $G'\times E^1$ implies the shrinkability of the trivially extended $I(G)$.

7. Some crumpled 4-cubes, 4-dimensional collarable objects and sewings. Fix a space C for which there is a cell-like upper semicontinuous decomposition G of B^n, every nondegenerate element of which intersects S^{n-1} in a cell-like subcontinuum, and there is a closed surjective map π_G: $B^n\to C$ whose point preimages are precisely the elements of G. We say that C has *simple* 1-ULC *sets* if, whenever F is a closed subset of $\mathrm{Bd}\,C$ such that $C-F$ is 1-ULC, it is possible to find a decomposition G and projection π_G: $B^n\to C$ as above such that $\pi_G|\pi_G^{-1}F$: $\pi_G^{-1}F\to F$ is a homeomorphism (i.e., no nondegenerate element of G hits $\pi_G^{-1}(F)$).

LEMMA 7.1. *If* C *is a crumpled cube in* S^3, *then* C *has simple* 1-ULC *sets.*

Proof. This is well-known. See [16].

LEMMA 7.2. *If* C *arises from a cell-like, closed*-0-*dimensional decomposition of* B^3 *such that each nondegenerate element intersects* S^2 *in a cell-like continuum, then* C *has simple* 1-ULC *sets. Moreover, the decompositions* G *which show that* C *has simple* 1-ULC *sets may all be chosen to be cell-like, closed*-0-*dimensional decompositions of* B^3.

Proof. Let G be a closed-0-dimensional decomposition of B^3 and π_G: $B^3\to C$ a closed map as indicated by the hypothesis. By R. L. Moore's Theorem and its refinements ([27], [9]), we may assume, after a pseudoisotopy of B^3, that each nondegenerate element of G hits $S^2 = \mathrm{Bd}\,B^3$ in a single point. Let $F_0 = \pi_G^{-1}(F)$ and let $\varepsilon>0$. Since G is a closed-0-dimensional, it follows from the proofs in [26] that there exist finitely many disjoint cells-with-handles $H_1, \ldots, H_k$ in B^3, intersecting S^2 is single boundary disks $D_1, \ldots, D_k$ of diameter less than ε, such that

(1) $F_0 \subset \bigcup \mathrm{Int}\,H_i \subset N(F_0, \varepsilon)$ and

(2) No element of G in the closure of the union of the nondegenerate elements of G hits $(\mathrm{Bd}\,H_i) - \mathrm{Int}\,D_i$.

We fix our attention on one of the cells with handles $H = H_i$. Let $E_1, \ldots, E_j$ be a complete set of meridional disks for H, each missing $D = D_i$. Since $C-F$ is 1-ULC, it is easy to find singular disks $E_1', \ldots, E_j'$ in $H-D$, having the same boundaries as $E_1, \ldots, E_j$, but missing F_0. After applying Dehn's Lemma [28], and cut-and-paste arguments, we find that we may assume that $E_1, \ldots, E_j$ originally missed F_0. Thus, $H-(E_1 \cup \ldots \cup E_j)$ contains a cell H' intersecting S^2 precisely in the set $D \subset \mathrm{Bd}\,H'$ such that $H \cap F_0 \subset H'$. With the cells $H_1', \ldots, H_k'$ obtained by the argument above, it is easy to pull each element of G in F_0 to a set of size less than ε. Standard decomposition space techniques now complete the proof [9, Theorem A13].

THEOREM 7.3. *Suppose that C_i $(i = 0, 1)$ is either a crumpled 3-cube or a space arising from a closed-0-dimensional decomposition of B^3 as in Lemma 7.2 and suppose h is a sewing of* $\mathrm{Infl}(C_0, \mathrm{Bd}\,C_0)$ *and* $\mathrm{Infl}(C_1, \mathrm{Bd}\,C_1)$. *Then there exists a bicollared compact set X in S^4 with bicollar $X \times I$ satisfying the* 1-ULC *Taming Hypothesis such that, for the associated decomposition G of S^4, S^4/G is naturally homeomorphic to*

$$\mathrm{Infl}(C_0, \mathrm{Bd}\,C_0) \cup_h \mathrm{Infl}(C_1, \mathrm{Bd}\,C_1).$$

Proof. In light of Theorem 5.1 and Corollaries 6.5 and 6.6, we need only check the Taming Hypothesis. Let $f\colon X \to I$ be a continuous function with 1-ULC complement in S^n. We set $C_i^* = \mathrm{Infl}(C_i, \mathrm{Bd}\,C_i)$ and identify C_i^* with $\mathrm{Cl}\,U_i$ $(i = 0, 1)$. By Lemma 3.2, C_0^*-f and C_1^*-f are 1-ULC. In order to satisfy the Taming Hypothesis, we must show how to use a small homeomorphism of S^4 to pull those fibers of $X \times I$ which hit f at $X \times \{0\}$ or at $X \times \{1\}$ away from f. Let us concentrate on the fibers hitting f at $X \times \{0\}$ for the moment. Let $F = f \cap (X \times \{0\})$. By Theorems 6.1 and 6.2 we may think of $C_0^* \cup (X \times [0, \frac{1}{2}])$ as arising from maps

$$B^4 \cup ([1, 2]\cdot S^3) \xrightarrow{\pi} (B^4/I(G_0)) \cup (([1, 2]\cdot S^3)/([1, 2]\cdot 2G_0)) \xrightarrow{h} C_0^* \cup (X \times [0, \tfrac{1}{2}]).$$

Thus $\pi^{-1}h^{-1}(F)$ is a closed, $2G_0$-saturated subset of S^3. Let $F_0 = [\pi^{-1}h^{-1}(F)] \cap S^2$. It is easy to check that (B^3/G_0)-image (F_0) is 1-ULC. By Lemmas 7.1 or 7.2, B^3/G_0 has simple 1-ULC sets. Hence, we may assume that no nondegenerate element of G_0 hits F_0. Thus by the Addendum to Theorem 6.2 we may assume that $F_0 \subset F$ and that the fiber $\{x\} \times [0, \frac{1}{2}]$ through a point $y \in F$ is of the form $[1, 2]\cdot\{y\}$. It is thus an easy matter to pull these fibers toward $X \times \{\frac{1}{4}\}$. This shortening pulls all of the fibers through F away from f and proves one half of the Taming Hypothesis. A similar argument near $X \times \{1\}$ completes the proof of the Taming Hypothesis.

COROLLARY 7.4. *Suppose that, for $i = 0, 1$, C_i is a crumpled 3-cube such that $C_i^* = \mathrm{Infl}(C_i, \mathrm{Bd}\,C_i)$ is a crumpled 4-cube (i.e., C_i^* is bounded by a 3-sphere) and h is a sewing of C_0^* and C_1^*. Then $C_0^* \cup_h C_1^*$ is homeomorphic to S^4.*

Proof. The homotopy theoretic property equivalent to $C_0^* = \mathrm{Infl}(C_0, \mathrm{Bd}\,C_0)$ being a crumpled 4-cube is that $\mathrm{Bd}\,C_0$ contains disjoint sets F_0 and F_1 such that $F_e \cup \mathrm{Int}\,C_0$ is 1-ULC $(e = 0, 1)$ [16]. It follows that in $\mathrm{Bd}\,C_0^*$, which is equivalent to $C_0 \cup_{\mathrm{Id}} C_0$, the set Σ corresponding to the seam of $C_0 \cup_{\mathrm{Id}} C_0$ contains disjoint sets F_0^* and F_1^* such that $F_e^* \cup \mathrm{Int}\,C_0^*$ is 1-ULC $(e = 0, 1)$. Without loss of generality, F_0^* is a countable union of compact 0-dimensional sets $\{Z_i\}$. But since each Z_i is contained in Σ and $\mathrm{Bd}\,C_0^* - \Sigma$ is 1-ULC in $(\mathrm{Bd}\,C_0^* - \Sigma) \cup F_1^*$, each Z_i is tame relative to $\mathrm{Bd}\,C_0^*$. Moreover, under the natural embedding of C_1^* in S^4, each set $h(Z_i)$ is locally flat modulo its intersection with the flat 2-sphere corresponding to the seam of $C_1 \cup_{\mathrm{Id}} C_1$ in $\mathrm{Bd}\,C_1^*$. Hence, $h(Z_i)$ is flat (it is sufficient simply to see that $S^4 - h(Z_i)$ is 1-ULC). As in [13, Proposition 12.2], h satisfies the Mismatch Property because $C_1^* - h(F_0^*)$ is 1-ULC.

A crumpled n-cube C is *self-universal* if, for each sewing h of C to itself, $C \cup_h C$ is homeomorphic to S^n.

COROLLARY 7.5. *Any crumpled n-cube obtained by inflating a crumpled $(n-1)$-cube is self-universal.*

For $n \geqslant 5$ this is Corollary 12.5 of [13].

We shall derive a few more (admittedly weak) 4-dimensional analogues to sewing results from [3]. Repeating a technical definition given there, we say that a crumpled n-cube C is a *closed n-cell-complement* if C can be embedded in S^n so that $\mathrm{Cl}(S^n - C)$ is an n-cell.

THEOREM 7.6. *Let C_0 be a closed 4-cell-complement in S^4 such that each tame Cantor set in $\mathrm{Bd}\,C_0$ is tame in S^4, and let C_1 be a closed 4-cell-complement in S^4 such that $\mathrm{Bd}\,C_1$ is locally flat modulo a 1-dimensional polyhedron K that is tamely embedded in S^4. Then, for each sewing h of C_0 and C_1, $C_0 \cup_h C_1$ is homeomorphic to S^4.*

Proof. We think of $C_0 \cup_h C_1$ as arising from the decomposition G of S^4 with fibers in $X \times I$ $(= S^3 \times I)$, where S^4 is represented as the union of C_0, C_1 and $X \times I$, as in Theorem 5.1. We shall prove that the Conditions of Mismatch Theorem 4.1 are satisfied.

Fix $\varepsilon>0$. Choose t_0 in $(0, 1)$ such that the diameter of $\{x\} \times [t_0, 1]$ is less than $\frac{1}{12}\varepsilon$ for each $x \in X$. Determine a triangulation T of S^4 of very small mesh so that some subcomplex N of T is a compact 4-manifold containing $\mathrm{Bd}\,C_1$ in its interior and $N \cap (C_0 \cup X \times [0, t_0]) = \emptyset$. Let P denote the 2-skeleton of N.

For any (curvilinear) triangulation S of X, we can adjust S slightly so that the 2-skeleton $S^{(2)} \times \{1\}$ intersects K in a 0-dimensional set contained in $S^{(2)} - S^{(1)}$. It follows from [23] that $S^{(2)} \times [t_0, 1]$ is locally tame in S^4. Hence, there exists an $\frac{1}{12}\varepsilon$-homeomorphism q of S^4 fixed outside the union of $X \times [t_0, 1]$ and a small neighborhood of $\mathrm{Bd}\,C_1$ such that $q(P) \cap X \times [t_0, 1]$ does not intersect $S^{(2)} \times I$. We choose a sequence of triangulations S_i of X, with S_{i+1} subdividing S_i, $S_i^{(1)}$ disjoint from K, and $S_i^{(2)} \cap K$ 0-dimensional. Then, as in the proof of Lemma 3.3, we obtain a compact set F in $X - \bigcup S_i^{(2)}$ such that, for any neighborhood V of F in X, there exists an $\frac{1}{6}\varepsilon$-homeomorphism h^* of S^4 to itself such that

$$P \subset h^*((V \times I) \cup \operatorname{Int} C_1)$$

and h^* moves no point of $C_0 \cup X \times [0, t_0]$. Fix a neighborhood V_0 of F in X and name such an h^*. By standard engulfing techniques, there exists an $\frac{1}{6}\varepsilon$-homeomorphism h of S^4 to itself such that, for the 1-skeleton Q of N (subdivided) dual to P,

$$Q \subset h(C_0 \cup X \times [0, 1))$$

and h moves no point of C_0. There exists an $\frac{1}{6}\varepsilon$-homeomorphism θ of S^4, fixed outside N, stretching across the join structure of the subdivided triangulation of N, so that

$$\theta h(C_0 \cup X \times [0, 1)) \cup h^*((V_0 \times I) \cup \operatorname{Int} C_1) \supset N.$$

Observe that the two open sets above cover S^4. We define g as $h^{-1}\theta^{-1}h^*$. Then

$$(C_0 \cup X \times [0, 1)) \cup g((V_0 \times I) \cup \operatorname{Int} C_1) = \theta^{-1}h^{-1}(S^4) = S^4,$$

implying that g is an $\frac{1}{2}\varepsilon$-homeomorphism of S^4 such that $g((X - V_0) \times I) \cap C_1 = \emptyset$. Hence, F satisfies Condition (2) of Theorem 4.1.

To see that it satisfies Condition (1), we note that F is contained in a tame Cantor set in X because it is contained in $X - S_i^{(2)}$ for all i. Thus, if f is a map of X into $[0, 1)$ with $F = f^{-1}(0)$, then f is locally flat modulo $F \times 0$, which is tame both in X and in S^4 (by hypothesis), implying that f is bicollared [23] and that Condition (1) holds. Hence, $C_0 \cup_h C_1$ is topologically S^4.

COROLLARY 7.7. *Suppose C is a closed 4-cell-complement in S^4 such that* Bd C *is locally flat modulo a 1-dimensional polyhedron in* Bd C *that is tamely embedded in S^4. Then C is self-universal.*

THEOREM 7.8. *Suppose C is a closed 4-cell-complement in S^4 such that* Bd C *is locally flat modulo a 1-dimensional polyhedron K tamely embedded as a subset of* Bd C. *Then $C \cup_{\mathrm{Id}} C$ is homeomorphic to S^4.*

Because the argument is quite similar to that of Theorem 7.6, we focus on the dissimilarities. We think of $C_0 = C = C_1$, $X = \operatorname{Bd} C$ and S^4 as the union of C_0, C_1 and $X \times I$ reflecting the "identity" sewing of C_0 to C_1. We fix $\varepsilon > 0$ and choose t_0, T, N, P, and Q as before, excepting that we adjust P so that $P \cap (K \times 1)$ is a compact 0-dimensional set and $P \cap (K \times [0, 1))$ is locally finite (read countable). Hence, K contains a compact 0-dimensional set F such that

$$F \times I \supset P \cap (K \times I)$$

and each fiber $\{f\} \times I$, $f \in F$, meets P. Then F satisfies Condition (2) of Theorem 4.1 as before. To see that it satisfies Condition (1), note that $F \times 1$, with the exception of countably many points, lies in P and therefore is tame as a subset of S^4, implying

that the symmetric set $F \times 0$ is tame as well; since $F \times 0 \subset K \times 0 \subset X \times 0$, $F \times 0$ is also a tame subset of $X \times 0$, and Condition (1) follows as well.

COROLLARY 7.9. *Suppose C is a closed 4-cell-complement in S^4 such that* Bd C *is locally flat modulo a Cantor set tamely embedded in* Bd C. *Then $C \cup_{\mathrm{Id}} C$ is homeomorphic to S^4.*

In particular, this encompasses a result of Broussard [5].

THEOREM 7.10. *Suppose C is a closed 4-cell-complement in S^4 such that to each $\varepsilon > 0$ there corresponds $\delta > 0$ such that for any polyhedral 2-complex P in $N_\delta(C)$ there exists a pseudo-isotopy ψ_t of S^4 fixed off $N_\varepsilon(\operatorname{Bd} C)$ and moving points less than ε such that $\psi_1(P) \subset C$ and $\psi_1(P) \cap \operatorname{Bd} C$ is countable. Then C is universal.*

Proof. Let C_0 be another closed 4-cell-complement and h be a sewing of C_0 and $C = C_1$. As usual, arrange S^4 as the union of C_0, C_1 and $X \times I$ so that the result of the associated decomposition is $C_0 \cup_h C_1$. Again we shall show that the conditions of Theorem 4.1 are satisfied.

Fix $\varepsilon > 0$. As in the proof of Theorem 7.6, choose $t_0 \in (0, 1)$ such that $\operatorname{Diam}(\{x\} \times [t_0, 1]) < \frac{1}{12}\varepsilon$ for each $x \in X$. Let ε' denote the distance between C_1 and $X \times [0, t_0]$. Triangulate S^4 with very small simplices so that some subcomplex N is a compact 4-manifold containing $\operatorname{Bd} C_1$ in its interior, $N \subset N_\delta(\operatorname{Bd} C)$, and $N \cap (C_0 \cup X \times [0, t_0]) = \emptyset$. Let P denote the 2-skeleton of N. By hypothesis there exists an $\frac{1}{6}\varepsilon$-pseudoisotopy ψ_t of P such that $\psi_1(P) \subset C_1$ and $\psi_1(P) \cap \operatorname{Bd} C_1$ is a countable set F, and ψ_t is fixed on $C_1 - N$. Then, for any neighborhood V_0 of F in X, there exists $s \in [0, 1)$ such that $P \subset h^*((V \times I) \cup \operatorname{Int} C_1)$, where $h^* = \psi_s^{-1}$. The rest of the argument verifying Condition (2) proceeds exactly as in Theorem 7.6.

Kirby's result [23] shows that Condition (1) also holds for this countable set F. Thus, by Theorem 4.1, $C_0 \cup_h C_1$ is topologically S^4.

COROLLARY 7.11. *If C is a closed 4-cell-complement in S^4 such that* Bd C *is locally flat modulo a Cantor set tamely embedded in* Bd C *and embedded in S^4 as a Blankinship example* [4], *then C is universal.*

We leave to the geometrically inclined reader the proof that 2-complexes near C can be pseudo-isotopically deformed until, in the limit, their intersection with Bd C is a finite set. (The argument is just a pseudoisotopically obtained verification of a remark in [5, p. 284].)

8. Spins of collarable objects. Cannon defined spun decompositions of E^n_+ in [9, Appendix III], and Daverman introduced an alternative definition for spins of crumpled cubes, which we extend to the broader setting considered here: let $C = B^n/G$, where G is a decomposition of B^n as in paragraph 3 of Section 6; then the *k-spin* $\operatorname{Sp}^k(C)$ $(k \geqslant 0)$ of C is the quotient space of $C \times S^k$ resulting when each set of the form $\{x\} \times S^k$, $x \in \operatorname{Bd} C$, is identified to a point.

The results of this section are low dimensional versions of theorems conjectured in [9] and established (for higher dimensions) in [13]. They portray almost completely the interconnections among spins, inflations, and identity sewings, with the single unresolved exception for 0-spins of crumpled 4-cubes now revealed in Theorem 8.2 below.

Let C denote B^n/G, as above, and suppose C is an n-dimensional collarable object. The basic idea of [13, Section 11] is the observation that $\mathrm{Sp}^0(C) \approx C \cup_{\mathrm{Id}} C$; $\mathrm{Sp}^1(C) \approx C^* \cup_{\mathrm{Id}} C^*$, where $C^* = \mathrm{Infl}(C, \mathrm{Bd}\, C)$; $\mathrm{Sp}^2(C) \approx C^{**} \cup_{\mathrm{Id}} C^{**}$, where $C^{**} = \mathrm{Infl}(C^*, \mathrm{Bd}\, C^*)$; etc. Moreover if any one of these identity sewings satisfies the Homotopical Mismatch Property, they all do.

As a result, our earlier work throughout provides the underpinnings allowing mismatch arguments of [13, Section 11] to be applied for the following theorems.

THEOREM 8.1. *Let C be a crumpled 3-cube. Then $\mathrm{Sp}^0(C) \approx S^3$ if and only if $\mathrm{Sp}^1(C) \approx S^4$. Thus, for all nonnegative integers k and m, $\mathrm{Sp}^k(C) \approx S^{k+3}$ if and only if $\mathrm{Sp}^m(C) \approx S^{m+3}$.*

THEOREM 8.2. *Let C denote a closed 4-cell-complement in S^4. Then the identity sewing of C to itself satisfies the Homotopical Mismatch Property if and only if $\mathrm{Sp}^k(C) \approx S^{k+4}$ for all $k \geq 1$.*

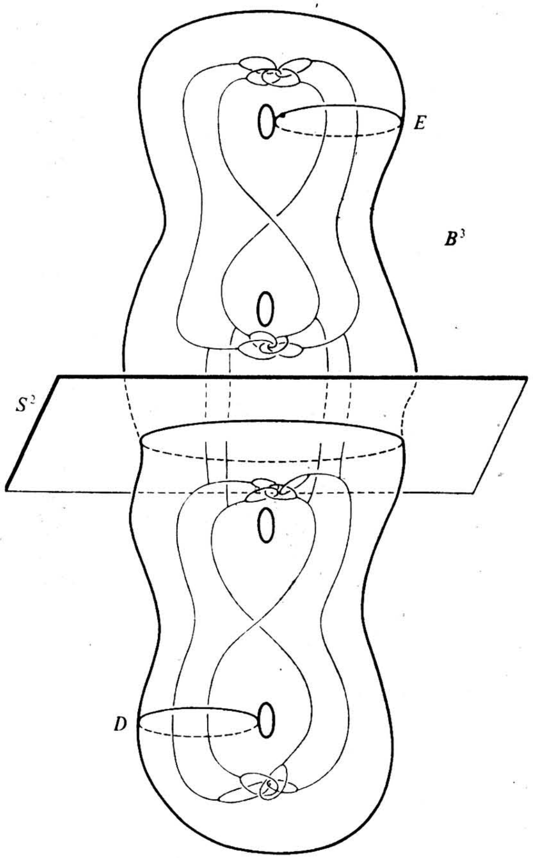

Fig. 8

THEOREM 8.3. *Let G_0 denote Bing's dogbone decomposition [2], so described that each nondegenerate element is a polygonal arc with two bends spanning the annulus between S^2 and $\frac{1}{2} \cdot S^2$, let G be the decomposition of B^3 with the same nondegenerate elements as G_0, and let $C = B^3/G$. Then $\mathrm{Sp}^0(C)$ is not a 3-manifold but $\mathrm{Sp}^k(C) \approx S^{k+3}$ for all $k \geq 1$.*

This is proved exactly like the other theorems of this section, except for the fact that C does not embed in a 3-manifold. Nonetheless, $\mathrm{Sp}^1(C)$ corresponds to $C^* \cup_{\mathrm{Id}} C^*$, where $C^* = \mathrm{Infl}(C, \mathrm{Bd}\, C)$. We leave it to the reader to verify that the necessary mismatch conditions are satisfied for $\mathrm{Sp}^1(C)$. The key to the proof that $\mathrm{Id}(\mathrm{Bd}\, C)$ and $\mathrm{Id}(\mathrm{Bd}\, C^*)$ satisfy the Homotopical Mismatch Property is that no element of the decomposition in Figure 8 yielding $\mathrm{Sp}^0(C)$ intersects both of the disks D and E. Thus $\mathrm{Sp}^1(C) \approx C^* \cup_{\mathrm{Id}} C^* \approx S^4$. Furthermore, $C^{**} = \mathrm{Infl}(C^*, \mathrm{Bd}\, C^*)$ and $\mathrm{Sp}^k(C) \approx S^{3+k}$ for $k \geq 2$, as before.

9. Near-product decompositions. The results of this section generalize R. H. Bing's result that, if D is the dogbone space, then $D \times E^1 \approx E^4$ [3] and complement the Edwards–Miller [19] Eaton–Pixley [17] result that closed-0-dimensional cell-like decompositions of E^3 are E^4 factors. The results were used to prove Corollary 5.5. They illustrate the remarks of Section 4, particularly remark (iv).

Setting. Let $X_0 = S^2 \times E^1$, and let $g\colon X_0 \times I \to S^3 \times E^1$ be an embedding such that, for each $t \in E^1$, $g(S^2 \times \{t\} \times I) \subset S^3 \times \{t\}$. Define $X = g(X_0 \times \{\frac{1}{2}\})$ and identify $g(X_0 \times I)$ with $X \times I$ in the natural way. Then X is a bicollared set with bicollar $X \times I$ in $S^3 \times E^1$ which separates $S^3 \times E^1$ into two open complementary domains U_0 (bounded by $X \times \{0\}$) and U_1 (bounded by $X \times \{1\}$). Let G be the associated decomposition of $S^3 \times E^1$ as in Section 1. Note that $(S^3 \times E^1)/G$ may be thought of as resulting from one parameter family of sewings of crumpled cubes.

THEOREM 9.1. *The spaces $S^3 \times E^1$ and $(S^3 \times E^1)/G$ are homeomorphic.*

The analogous result for $S^n \times E^1$, $n > 3$, is an immediate consequence of the Mismatch Theorem (Theorem 3.2). For $S^3 \times E^1$, the proof relies on a pair of lemmas, the first establishing a weak version of the Taming Hypothesis, the second supplying a version of Lemma 3.3.

LEMMA 9.2. *Let $(f\colon X \to [0, 1]) \subset (X \times [0, 1])$ be a continuous function such that, for some closed-0-dimensional subset C of E^1, $f^{-1}(\{0, 1\}) = f^{-1}(\{1\}) \subset (S^3 \times C)$. Then there is a homeomorphism $h\colon S^3 \times E^1 \to S^3 \times E^1$, arbitrarily near the identity and fixed outside an arbitrarily small neighborhood of $f \cap (X \times \{1\})$ such that $f \cap h(f^{-1}(\{1\}) \times I) = \emptyset$.*

Proof. To avoid some of the notational and expositional difficulties of the general case, we assume for simplicity that the closed set C is compact and that $f \cap (X \times \{1\})$ is all, and not just a part, of $(S^3 \times C) \cap (X \times \{1\})$; that is, f intersects $X \times \{1\}$ precisely in the "Cantor set" C of horizontal levels in $S^3 \times E^1$. The lemma requires that we show how to pull $(S^3 \times C) \cap (X \times I)$ into $(S^3 \times E^1) \cap (X \times [0, 1))$ by a small homeomorphism of $S^3 \times E^1$.

Fix a single level $t \in C$ for consideration. Then set $D = (S^3 \times \{t\}) - U_1$ is a 3-cell. We use the Canonical Neighborhood Theorem of D. R. Mc Millan, Jr. [25] to find in $S^3 \times \{t\}$ a 3-dimensional annulus $S \times I$, S a 2-sphere, and finitely many disjoint cubes-with-handles $H_1, \ldots, H_k$ such that

(1) $H_i \cap (S \times I) = (\mathrm{Bd}\, H_i) \cap (S \times \{1\})$ is a single disk for each i;

(2) $\mathrm{Bd}\, D \subset \mathrm{Int}((S \times I) \cup H_1 \cup \ldots \cup H_k)$; and

(3) $(S \times \{0\}) \subset \mathrm{Int}\, D$.

In addition, we require that, for some $\delta > 0$ to be chosen below,

(4) each H_i has diameter less than δ;

(5) $S \times \{0\}$ and $S \times \{1\}$ are homeomorphically within δ of $\mathrm{Bd}\, D$; and

(6) each fiber $\{s\} \times I$ of $S \times I$ has diameter less than δ.

Conditions (1)–(5) are parts of McMillan's theorem. Condition (6) was essentially proved by McMillan in the case that concerns us (D a cell). However, for completeness we may quote R. Craggs' difficult theorem [11], which states that, given condition (5) for δ' sufficiently small, condition (6) may be realized for δ simply by changing the fiber structure of $S \times I$.

The idea now is to untangle the cells-with-handles $H_1, \ldots, H_k$ in $S^3 \times$ (Small Interval) so that they can be pulled near $S \times \{1\}$. Then $S \times \{1\}$ can be pulled near to $S \times \{0\}$ along the fibers of $S \times I$. Thereby $(X \times I) \cap (S^3 \times \{t\})$ will be pulled into $X \times [0, 1)$, as desired.

The choice of δ depends on any given $\varepsilon > 0$, on an associated special open cover V_ε of $S^3 \times \{t\}$, and on the 2-sphere $\mathrm{Bd}\, D$ as follows. Let V_ε be a finite open $\frac{1}{4}\varepsilon$-covering of $S^3 \times \{t\}$ that is the union of four discrete collections V_0, V_1, V_2, and V_3. Let $\delta' > 0$ be a Lebesgue number for the covering V_ε. Let $\delta > 0$ be so small that any δ-subset of $\mathrm{Bd}\, D$ lies in a δ'-disk in $\mathrm{Bd}\, D$.

We choose the small vertical interval in which $H_1, \ldots, H_k$ will be untangled as follows. Let $a < a' < t < b' < b$ be real numbers such that

(7) $(b - a) < \varepsilon$,

(8) $[a, a'] \cap C = \emptyset = [b, b'] \cap C$, and

(9) $(X \times \{1\}) \cap (S^3 \times [a, b]) \subset \{\mathrm{Int}((S \times I) \cup H_1 \cup \ldots \cup H_k)\} \times [a, b]$.

The untangling of $H_1, \ldots, H_k$ can be described in terms of 1-dimensional polyhedral spines $Sp_1, \ldots, Sp_k$ of $H_1, \ldots, H_k$, respectively, each Sp_i intersecting $S \times \{1\}$ in a single point x_i in the disk $H_i \cap (S \times \{1\})$. By the choice of δ and since $\mathrm{Diam}\, H_i < \delta$, there is a homotopy of Sp_i in a δ'-subset of $[(S^3 \times \{t\}) - (S \times \{1\})] \cup \{x_i\}$ which fixes $\{x_i\}$ and, at the final stage, takes Sp_i onto a set arbitrarily close to $\{x_i\}$. Note that the image of the homotopy, being a δ'-set, must lie in some element v of V_ε. It is well-known that this homotopy of Sp_i can be approximated arbitrarily closely by an ambient isotopy of Sp_i in $S^3 \times E^1$ which fixes $[(S \times I) \cup \bigcup_{j \neq i} H_j] \times E^1$ and has its support in $v \times [a, b]$.

Finally we are in a position to describe a small homeomorphism h of $S^3 \times E^1$, fixed outside of $S^3 \times [a, b]$, which takes $(X \times I) \cap (S^3 \times [a', b'])$ into $(X \times [0, 1)) \cap (S^3 \times [a, b])$. The homeomorphism h is the composite $h_3 \circ h_2 \circ h_1$ of three homeomorphisms which we describe in reverse order. The homeomorphism $h_3\colon S^3 \times E^1 \to S^3 \times E^1$ pulls $(S \times [0, 1]) \times [a', b']$ near to $(S \times \{0\}) \times [a', b']$ along fibers $\{s\} \times [0, 1] \times \{u\}$ ($s \in S$, $u \in [a', b']$). This pulls a neighborhood A of $S \times [0, 1] \times [a', b']$ into $(X \times [0, 1)) \cap (S^3 \times [a, b])$. The homeomorphism $h_2\colon S^3 \times E^1 \to S^3 \times E^1$ pulls $H_1 \cup \ldots \cup H_k$ into the open set A; h_2 is itself a composition of homeomorphisms which first pull the sets H_i very close to $[H_i \cap (S \times I)] \cup Sp_i$ and then isotope the sets Sp_i, and hence the image of H_i, into A as described in the preceding paragraph; these isotopies must be performed in a very specific order (see below) to ensure that h_2 is a small homeomorphism. Finally, h_1 pulls $(X \times I) \cap (S^3 \times [a', b'])$ into $h_2^{-1}(A)$ by pulling $[(S \times I) \cup \bigcup H_i] \times [a', b']$ close to $[(S \times I) \cup \bigcup Sp_i] \times [a', b']$.

Since h_1 and h_3 can obviously be chosen small, it remains only to note that h_2 will also be small if properly described. The idea is to order $H_1, \ldots, H_k$ so that all H_i's lying in elements of V_0 come first, then those remaining that lie in elements of V_1, then those in elements of V_2, and finally those in elements of V_3. If the isotopies associated with $H_2, \ldots, H_k$ and described two paragraphs previously are then carried out in order, no point will be moved further than $4 \cdot \frac{1}{4}\varepsilon = \varepsilon$ by h_2.

To complete the proof of Lemma 9.2 it remains only to note that C can be covered by finitely many disjoint intervals $[a_1, b_1], \ldots, [a_m, b_m]$ satisfying the conditions satisfied by $[a, b]$ in the above argument.

LEMMA 9.3. *Given $\varepsilon > 0$, there is a closed-0-dimensional subset C of E^1 satisfying the following condition*:

(*) *If V_0 is a neighborhood of C in E^1, then there is an $\frac{1}{2}\varepsilon$-homeomorphism h of $S^3 \times E^1$ fixed outside an arbitrarily small neighborhood of $X \times \{1\}$ such that $h\{(X \times I) \cap [S^3 \times (E^1 - V_0)]\} \cap (X \times \{1\}) = \emptyset$.*

Proof. The proof is exactly like the proof of Lemma 3.3 except that in place of $F_0 = X_1 \cup X_2 \cup \ldots$ we take $F_0 = [X \cap (S^3 \times \{t_1\}] \cup [X \cap (S^3 \times \{t_2\})] \cup \ldots$ where $\{t_1, t_2, \ldots\}$ is a countable dense set in E^1. Lemma 9.2 takes the place of both the Taming Hypothesis and Lemma 3.2 in the proof of Lemma 3.3.

Proof of Theorem 9.1. This follows from Lemmas 9.2 and 9.3 and Theorem 2.1 exactly as Theorem 3.1 follows from the Taming Hypothesis, Lemmas 3.2 and 3.3, and Theorem 2.1. As in the proof of Lemma 9.3, Lemma 9.2 takes the place of both Lemma 3.2 and the Taming Hypothesis.

COROLLARY 9.4 (Daverman–Eaton [15]). *If C_1 and C_2 are crumpled 3-cubes and $h\colon \mathrm{Bd}\, C_1 \to \mathrm{Bd}\, C_2$ is a homeomorphism, then $(C_1 \cup_h C_2) \times E^1 \approx S^3 \times E^1$.*

COROLLARY 9.5 (Bing [3]). *If D is the one point compactification of Bing's dogbone space, then $D \times E^1 \approx S^3 \times E^1$.*

Proof. The space D can be realized as a sewing $C_1 \cup_h C_2$ of crumpled cubes [1] so that Corollary 9.4 applies.

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