number of critical points of an m-function extending f? It should be possible to obtain similar results as in our case if one consider the chain complex induced by f in place of homology model R^0 in the definition of $\mathfrak{A}(M)$.

For non-simply connected manifolds the minimalization problems are much more difficult and even for Morse functions no satisfactory calculations are known. However, Theorems 2 and 3 are uneffective enough to have straightforward generalizations to that case. Our arguments ought to work if the homology groups of M and ∂M are replaced by the homology groups of universal covering considered as modules over the integer group rings $Z\pi_1 M$ and $Z\pi_1 \partial M$, at least when $i_*: \pi_1 \partial M \to \pi_1 M$ is an isomorphism.

References

[1] D. Braess, *Morse-Theorie für berandete Mannigfaltigkeiten*, Math. Annalen 208 (1974), pp. 133–148.

[2] S. Cairns and M. Morse, *Critical Point Theory in Global Analysis and Differentiable Topology*, New York 1969.

[3] J. Derwent, *Handle decompositions of manifolds*, J. Math. and Mech. 15 (1966), pp. 329–345.

[4] B. Hajduk, *Comparing handle decompositions of homotopy equivalent manifolds*, Fund. Math. 95 (1977), pp. 35–47.

[5] M. Hirsch, *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. 73 (1961), pp. 566–571.

[6] A. Jankowski and R. Rubinsztein, *Functions with non-degenerated critical points on manifolds with boundary*, Comment. Math. 16 (1972), pp. 99–112.

[7] M. Kervaire, *Le theoreme de Barden-Mazur-Stallings*, Comm. Math. Helv. 40 (1965), pp. 31–42.

[8] S. Maumary, *Type simple d'homotopie*, Lecture Notes in Math. 48 (1967), pp. 37–55.

[9] B. Mazur, *Differential topology from the point of view of simple homotopy theory*, Publ. IHES 15 (1963).

[10] J. Milnor, *Lectures on h-cobordism Theory*, Princeton 1965.

[11] S. Smale, *On the structure of manifolds*, Amer. J. Math. 84 (1962), pp. 387–399.

Accepté par la Rédaction le 18. 10. 1978

FUNDAMENTA MATHEMATICAE
CXI (1981)

Special bases for compact metrizable spaces

by

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Abstract. Each compact metrizable space has a base $\mathcal{B}$ such that

(1) for every finite $\mathcal{A} \subseteq \mathcal{B}$, if any two members of $\{\bar{A}: A \in \mathcal{A}\}$ intersect then $\bigcap \mathcal{A} \neq \emptyset$; and

(2) if $\mathcal{R}$ is the ring generated by $\mathcal{B}$, then $\mathcal{R}$ consists of regularly open sets and $(\bigcap \mathcal{F})^- = \bigcap \{\bar{A}: A \in \mathcal{F}\}$ for every finite $\mathcal{F} \subseteq \mathcal{R}$.

This implies that every compact metrizable space is regularly supercompact. The construction of $\mathcal{B}$ is complicated but elementary.

0. Conventions and definitions. As usual, if X is a space, $^-$, ^{0} and ^{c} denote the closure operator, the interior operator and the complementation operator in X; if $\mathcal{F}$ is a family of subsets of X we write e.g. $\bar{\mathcal{F}}$ for $\{\bar{F}: F \in \mathcal{F}\}$.

If X is a space and $\mathcal{F}$ is a family of subsets, then $\mathcal{F}$ is called a *closed subbase* if it is a subbase for the closed sets, i.e. $\mathcal{F}^c$ is a subbase for the open sets, a *ring* if $F \cap G \in \mathcal{F}$ and $F \cup G \in \mathcal{F}$ for all $F, G \in \mathcal{F}$, *linked* if $F \cap G \neq \emptyset$ for any $F, G \in \mathcal{F}$ (not necessarily distinct), *binary* if every linked subfamily has nonempty intersection.

A space is called *supercompact* if it has a binary closed subbase, *regularly supercompact* if it has a binary closed subbase $\mathcal{S}$ such that the ring generated by $\mathcal{S}$ consists of regularly closed sets, *regularly Wallman* if it has a closed subbase which is a ring and which consists of regularly closed sets.

1. Introduction. The notion of supercompactness was introduced by de Groot in [dG]. It is a trivial consequence of Alexander's Subbase Lemma, [A], that every supercompact space is compact. An easy example of a compact T_1-space that is not supercompact was given by Verbeek, [V, II. 2.2(8)]. The question of whether all compact Hausdorff spaces are supercompact was settled in the negative by Bell, [B], this is a nontrivial result in spite of the fact that the answer was to be expected, [dG]. Subsequently van Douwen and van Mill showed that every infinite supercompact Hausdorff space has many nontrivial convergent sequences, [vDvM]; this gives a rich supply of compact Hausdorff spaces that are not supercompact.

This paper deals with de Groot's conjecture that all compact metrizable spaces are supercompact, [dG]. The first result is due to de Groot who proved that compact polyhedra are supercompact. An erroneous proof of de Groot's conjecture was published in [O'C]; several people have noticed that this proof does not work for spaces which have an isolated point, Jeroen Bruyning has discovered that the proof is entirely incorrect, even for spaces without isolated points.

The first correct proof of de Groot's conjecture was published by Strok and Szymański, [SSz]. Their very complicated proof is inspired by de Groot's proof that compact polyhedra are supercompact, and follows his suggestion to consider a compact metrizable space as an inverse limit of a sequence of polyhedra, [dG].

The purpose of this paper is to give an elementary proof of de Groot's conjecture which is based on a totally different idea. In fact we prove something more.

THEOREM. *Every compact metrizable space X has a base $\mathscr{B}$ such that*

(1) *for every finite $\mathscr{F}\subseteq\mathscr{B}$, if $\bar{\mathscr{F}}$ is linked then $\bigcap\mathscr{F}\neq\emptyset$;*

(2) *if $\mathscr{R}$ is the ring generated by $\mathscr{B}$, then $\mathscr{R}$ consists of regularly open sets and has the property that*

$$(*)\qquad (\bigcap\mathscr{F})^- = \bigcap\bar{\mathscr{F}}\quad \textit{for every finite } \mathscr{F}\subseteq\mathscr{R}.$$

COROLLARY 1. *Each compact metrizable space X is regularly supercompact.*

Proof. Let $\mathscr{B}$ and $\mathscr{R}$ be as in the theorem. $\bar{\mathscr{B}}$ is a closed subbase since X is compact and $\mathscr{B}$ is a base. $\bar{\mathscr{B}}$ is binary by (1) since X is compact. Clearly $\bar{\mathscr{R}}$ consists of regularly closed sets. $\bar{\mathscr{R}}$ is closed under finite union since $\mathscr{R}$ is, and $\mathscr{R}$ is closed under finite intersections because of $(*)$ in (2). ■

Corollary 1 is best possible in the sense that the implicit condition that X have a countable base cannot be weakened. Indeed, there is a compact Hausdorff space with weight ω_1 (in fact even with cardinality ω_1) which is not supercompact, [vDvM].

Regularly supercompact spaces were introduced by van Mill, [vM, 1.4], in analogy with regular Wallman spaces, introduced by Steiner, [St]. A consequence of Corollary 1 is that every compact metrizable space is a superextension (this we will not define) of every dense subspace, [vM, 1.4.2]. This is analoguous to the fact a regular Wallman space is a Wallman compactification (this we will not define) of every dense subspace, [St, Thm. 4]. The proof of Corollary 1 also yields the following.

COROLLARY 2. *Each compact metrizable space is regular Wallman.*

This was first proved by Steiner and Steiner by a totally different method, [SS]. The consequence that every metrizable compactification is a Wallman compactification was also obtained by Aarts, [Aa]. It is a recent result of Ul'janov that not every compactification is a Wallman compactification, [U].

We prove the theorem from the following lemma, which in fact is a corollary to the theorem.

LEMMA. *Every compact metrizable space has a base $\mathscr{U}$ which is a ring consisting of regularly open sets, and which has the property that $(\bigcap\mathscr{F})^- = \bigcap\bar{\mathscr{F}}$ for every finite $\mathscr{F}\subseteq\mathscr{U}$.*

We prove the theorem from the lemma by constructing a suitable subfamily $\mathscr{B}$ of $\mathscr{U}$.

Note that Corollary 2 follows from the lemma in the same way Corollary 1 follows from the theorem.

I am grateful to Jan van Mill for supplying me with some information, given above, that does not occur in print.

2. Proof of the lemma. We first show that if suffices to prove the following

CLAIM. *X has a base $\mathscr{V}$ consisting of regularly open sets, satisfying*

$(*)$ *for any two disjoint finite $\mathscr{F}$, $\mathscr{G}\subseteq\mathscr{V}$, if $\bigcap(\mathscr{F}\cup\mathscr{G}^{-c})=\emptyset$ then $\bigcap((\mathscr{F}\cup\mathscr{G}^{-c})^-)=\emptyset$.*

Given $\mathscr{V}$ as in the claim, we will show that

$$\mathscr{W} = \{\bigcup\mathscr{G}:\ \mathscr{G}\subseteq\mathscr{V} \text{ is finite}\}$$

consists of regularly open sets and satisfies

$$(1)\qquad \bigcap\bar{\mathscr{F}} = (\bigcap\mathscr{F})^-\quad\text{for every finite } \mathscr{F}\subseteq\mathscr{W}.$$

Since the intersection of finitely many regularly open sets again is regularly open, we then can define our base $\mathscr{U}$ by

$$\mathscr{U} = \{\bigcap\mathscr{F}:\ \mathscr{F}\subseteq\mathscr{W} \text{ is finite}\}.$$

In order to show that $\mathscr{W}$ is as stated, we need the following strengthening of $(*)$:

$$(**)\quad \bigcap((\mathscr{F}\cup\mathscr{G}^{-c})^-) = (\bigcap(\mathscr{F}\cup\mathscr{G}^{-c}))^-\ \text{for any two disjoint finite } \mathscr{F},\mathscr{G}\subseteq\mathscr{V}.$$

To see that $(**)$ holds, let $\mathscr{F},\mathscr{G}\subseteq\mathscr{V}$ be finite and disjoint. Clearly $(\bigcap(\mathscr{F}\cup\mathscr{G}^{-c}))^-\subseteq\bigcap((\mathscr{F}\cup\mathscr{G}^{-c})^-)$. Let $x\notin(\bigcap(\mathscr{F}\cup\mathscr{G}^{-c}))^-$ be arbitrary. It is easy to see that $x\notin\bigcap((\mathscr{F}\cup\mathscr{G}^{c})^-)$ if x is isolated, so we assume that x is not isolated. Then there is a $V\in\mathscr{V}-\mathscr{G}$ with $x\in V$ and $V\cap\bigcap(\mathscr{F}\cup\mathscr{G}^{-c})=\emptyset$. It now follows from $(*)$ with $\mathscr{F}\cup\{V\}$ instead of $\mathscr{F}$ that $x\notin\bigcap((\mathscr{F}\cup\mathscr{G}^{-c})^-)$.

Evidently (1) is nothing but the special case $\mathscr{G}=\emptyset$ of $(**)$.

Since the members of $\mathscr{V}$ are regularly open, we see from the special case $\mathscr{F}=\emptyset$ of (1) that for all finite $\mathscr{G}\subseteq\mathscr{V}$

$$\begin{aligned}(\bigcup\mathscr{G})^{-0} &= (\bigcup\mathscr{G})^{-c-c} = (\bigcup\mathscr{G}^-)^{c-c} = (\bigcap\mathscr{G}^{-c})^{-c} = (\bigcap\mathscr{G}^{-c-})^c\\ &= \bigcup\mathscr{G}^{-c-c} = \bigcup\mathscr{G}^{-0} = \bigcup\mathscr{G}.\end{aligned}$$

Hence the members of $\mathscr{W}$ are regularly open.

It remains to prove the claim.

Proof of the claim. Let $\mathscr{A}$ be any countable base for X. Enumerate

$$\{\langle A,B\rangle\in\mathscr{A}\times\mathscr{A}:\ \bar{A}\subseteq B\}\quad\text{as}\quad\langle\langle A(n,0),B(n,0)\rangle:\ n\in\omega\rangle,$$

and

$$\{\langle F,G\rangle:\ F,G\subseteq\omega;\ F\cap G=\emptyset,\ |F\cup G|<\omega\}\quad\text{as}\quad\langle\langle F_k,G_k\rangle:\ k\in\omega\rangle.$$

It is a straightforward exercise in normality to construct with recursion on $1 \leq k < \omega$ open set $A(n,k)$ and $B(n,k)$ for all $n \in \omega$ in such a way that for all $n \in \omega$

(a) $A(n,k)^- \subseteq A(n,k+1)$, $A(n,k+1)^- \subseteq B(n,k+1)$, and $B(n,k+1)^- \subseteq B(n,k)$, for $k \in \omega$; and

(b) if $\bigcap \{A(i,k)^-: i \in F_k\} \cap \bigcap \{B(j,k)^c: j \in G_k\} = \emptyset$, then
$$\bigcap \{B(i,k+1): i \in F_k\} \cap \bigcap \{A(j,k+1)^c: j \in G_k\} = \emptyset .$$

For each $n \in \omega$ define
$$V_n = \Big(\bigcup_{k \in \omega} A(n,k)\Big)^{-0},$$
and let $\mathscr{V} = \{V_n: n \in \omega\}$.

It follows from (a) that

(c) $A(n,k)^- \subseteq V_n$ and $\overline{V}_n \subseteq B(n,k)$ for $k \in \omega$, $n \in \omega$.

Hence $\mathscr{V}$ is a base for X. Obviously $\mathscr{V}$ consists of regularly open sets. We check (*). Let $k \in \omega$ be arbitrary. Write $\bigcap_i$ and $\bigcap_j$ for the intersections with $i \in F_k$ and $j \in G_k$, respectively. If
$$\bigcap_i V_i \cap \bigcap_j V_j^{-c} = \emptyset ,$$
then using (c), (b) and (c) in turn, we see that
$$\bigcap_i A(i,k)^- \cap \bigcap_j B(j,k)^c = \emptyset ,$$
hence
$$\bigcap_i B(i,k+1)^- \cap \bigcap_j A(j,k+1)^c = \emptyset .$$
hence
$$\bigcap_i V_i^- \cap \bigcap_j V_j^{-c-} = \emptyset .$$
(In the last step $V_j^{-c-} \subseteq A(j,k+1)^c$ since $A(j,k+1) \subseteq V_j^-$ and $A(j,k+1)$ is open.)

Remark 1. Note that (*) implies that

(↘) for all $U, V \in \mathscr{V}$, if U is a proper subset of V, then $\overline{U} \subseteq V$.

(For $U \subseteq \overline{V}$, hence $U \cap \overline{V}^c = \emptyset$, hence $\overline{U} \cap \overline{V}^{c-} = \emptyset$, hence $\overline{U} \subseteq \overline{V}^{c-c} = \overline{V}^0 = V$.) It was shown in [G] that every metacompact Moore space has a basis satisfying (↘).

Remark 2. Since every metrizable space is perfectly normal and has a σ-discrete base, for every metrizable space X one can find for each $n \in \omega$ a collection $\langle\langle A_\gamma, B_\gamma\rangle: \gamma \in \Gamma_n\rangle$, where $\langle \Gamma_n: n \in \omega\rangle$ is a pairwise disjoint collection of index sets, such that

A_γ is closed, B_γ is open and $A_\gamma \subseteq B_\gamma$ for $\gamma \in \Gamma_n$;
$\langle B_\gamma: \gamma \in \Gamma_n\rangle$ is a discrete family; and
if $\mathscr{U} = \langle U_\gamma: \gamma \in \bigcup_{n\in\omega} \Gamma_n$ is any open family such that $A_\gamma \subseteq U_\gamma \subseteq B_\gamma$ for all $\gamma \in \bigcup_{n\in\omega} \Gamma_n\rangle$, then $\mathscr{U}$ is a base for X.

One can probably use this to show that the lemma holds for all metrizable spaces. Since I do not know an application I did not bother to check this. (I recall that I once verified the special case that every metrizable space has a base $\mathscr{V}$ satisfying both (↘) and

(↔) for all $U, V \in \mathscr{V}$, if $U \cap V = \emptyset$ then $\overline{U} \cap \overline{V} = \emptyset$.)

Remark 3. The technique of constructing a base by successive approximation is not new, the earliest reference I am aware of is [Os, Lemma 4]. Also, Aarts proves that every metrizable compactification is a Wallman compactification using a one-step approximation, in [Aa].

Remark 4. There is an easy example of a compact metrizable space which has a base $\mathscr{U}$ which is a ring consisting of regularly open sets, yet $(\bigcap \mathscr{F})^- \neq \bigcap \overline{\mathscr{F}}$ for some finite $\mathscr{F} \subseteq \mathscr{U}$. Indeed, let $\omega+1$, the ordinals $\leq \omega$ have the order topology. Define
$$U = \{3n: n \in \omega\}, \qquad V = \{3n+1: n \in \omega\};$$
and
$$\mathscr{B} = \{\{n\};\ n \in \omega\} \cup \{[n,\omega]: n \in \omega\} \cup \{U, V\} .$$
Then $\mathscr{B}$ is a base for $\omega+1$. One easily verifies that the ring generated by $\mathscr{B}$ consists of regularly open sets. However, $(U \cap V)^- \neq \overline{U} \cap \overline{V}$.

3. Proof of the theorem. Let X be a (nonempty) compact metrizable space, let d be a compatible metric for X, and for $x \in X$ let $S(x,\varepsilon)$ be the ε-sphere about x. By the lemma there is a base $\mathscr{U}$ for X such that

(A) $\mathscr{U}$ is a ring consisting of regularly open sests; and

(B) $(\bigcap \mathscr{F})^- = \bigcap \overline{\mathscr{F}}$ for all finite $\mathscr{F} \subseteq \mathscr{U}$.

We construct our base $\mathscr{B}$ by finding for each $n \in \omega$ a finite $\mathscr{B}_n \subseteq \mathscr{U}$ such that

(C) $\bigcup \mathscr{B}_n = X$;

(D) $\operatorname{diam}(B) \leq 2^{-n}$ for $B \in \mathscr{B}_n$; and

(E) $\bigcup \{\mathscr{B}_k: k \leq n\}$ is binary;

for all $n \in \omega$. Then $\mathscr{B} = \bigcup \{\mathscr{B}_n: n \in \omega\}$ is a base for X by (C) and (D). Clearly (1) follows from (E), and (2) follows from (A) and (B) since $\mathscr{B} \subseteq \mathscr{U}$.

We construct the $\mathscr{B}_n$'s with recursion on $n \in \omega$, using the following claim which we prove later.

Claim. *Let $n \in \omega$. If $\mathscr{A}$ is a finite binary subfamily of $\mathscr{U}$, then there is a finite closed cover $\mathscr{E}$ of X such that*

$\mathscr{A} \cup \mathscr{E}$ is binary;
for all $E \in \mathscr{E}$ and $A \in \mathscr{A}$ if $E \cap A = \emptyset$ then $E \cap \overline{A} = \emptyset$; and
$\operatorname{diam}(E) < 2^{-n}$ for $E \in \mathscr{E}$.

Let $n \in \omega$, and suppose $\mathscr{B}_k$ to be constructed for $k < n$. Then $\mathscr{A} = \bigcup\{\mathscr{B}_k: 0 \leq k < n\}$ is a (possibly empty) finite subcollection of $\mathscr{U}$. Let $\mathscr{E}$ be as in the claim. Since $\mathscr{U}$ is closed under finite unions, $\{U\in\mathscr{U}\colon U\supseteq E\}$ is a neighborhood base for E, for $E\in\mathscr{E}$. Therefore we can find for each $E\in\mathscr{E}$ a $U(E)\in\mathscr{U}$ such that

$E\subseteq U(E)$ for $E\in\mathscr{E}$;
for all $E\in\mathscr{E}$ and $A\in\mathscr{A}$, if $E\cap A=\emptyset$ then $U(E)\cap A=\emptyset$;
for any two $E, E'\in\mathscr{E}$, if $E\cap E'=\emptyset$ then $U(E)\cap U(E')=\emptyset$; and
$\operatorname{diam}(U(E))<2^{-n}$ for $E\in\mathscr{E}$.

Then $\mathscr{B}_n=\{U(E)\colon E\in\mathscr{E}\}$ is as required, as one can easily check.

We assume that $X\in\mathscr{A}$; alternatively accept the convention that $\bigcap\emptyset=X$.

Proof of the claim. The idea is to take a closed cover $\mathscr{F}$, consisting of sufficiently small sets, and then to enlarge each $F\in\mathscr{F}$ to $E(F)$ in such a way that

for all $\mathscr{A}'\subseteq\mathscr{A}$ and $\mathscr{F}'\subseteq\mathscr{F}$, if $\mathscr{A}'\cup\mathscr{F}'$ is linked then $\mathscr{A}'\cup\{E(F)\colon F\in\mathscr{F}'\}$ has nonempty intersection,

making sure that the enlarging does not cause more families to become linked. In other words, we also will require

for all $A\in\mathscr{A}$ and $F, F'\in\mathscr{F}$, if $A\cap F=\emptyset$ then $A\cap E(F)=\emptyset$, and if $F\cap F'=\emptyset$ then $E(F)\cap E(F')=\emptyset$.

For $x\in X$ we define $\mathscr{A}_x\subseteq\mathscr{A}$ and $\delta(x)>0$ by

$$\mathscr{A}_x=\{A\in\mathscr{A}\colon x\notin\bar{A}\}\,;$$

$$S(x)=d(x,\textstyle\bigcup\bar{\mathscr{A}}_x),\quad\text{where}\quad d(x,\emptyset)=1\,.$$

Step 1. We construct a finite closed cover.

For each $x\in X$ define a neighborhood N_x of x by

$$N_x=\begin{cases}\{x\} & \text{if } x \text{ is isolated;}\\ S(x,2^{-n-2})\cap S(x,\tfrac{1}{3}\delta(x)) & \text{if } x \text{ is not isolated.}\end{cases}$$

Since X is compact, there is a finite $Y\subseteq X$ such that $\bigcup\{N_y\colon y\in Y\}=X$. For each $y\in Y$ choose a neighborhood $I_y\subseteq N_y$ such that $I_y\cap I_{y'}=\emptyset$ for distinct $y, y'\in Y$. For $y\in Y$ define

$$F_y=\bar{N}_y-\textstyle\bigcup\{I_{y'}\colon y'\in Y,\ y'\neq y\}\,.$$

One can easily check the following facts:

(a) $\bigcup\{F_y\colon y\in Y\}=X$;
(b) $I_y\subseteq F_y$ for $y\in Y$;
(c) $I_y\cap F_{y'}=\emptyset$ for distinct $y, y'\in Y$;
(d) $F_y\subseteq S(y,2^{-n-2})\cap S(y,\frac{1}{2}\delta(x))$ for $y\in Y$; and
(e) $F_y=\{y\}$ if y is isolated.

Remark. The I_y's will be used when we enlarge the F_y's.

Fact 1. *For all $y, y'\in Y$, if $F_y\cap F_{y'}\neq\emptyset$ then $\mathscr{A}_y\subseteq\mathscr{A}_{y'}$ or $\mathscr{A}_{y'}\subseteq\mathscr{A}_y$.*

Proof of Fact 1. Let $y, y'\in Y$ be arbitrary, and suppose there are $A\in\mathscr{A}_y-\mathscr{A}_{y'}$ and $A'\in\mathscr{A}_{y'}-\mathscr{A}_y$. Then $y\notin A$ but $y'\in A$ hence $d(y,y')\geqslant d(y,A)\geqslant d(y,\bigcup A_y)=\delta(y)$. Similarly, $d(y,y')\geqslant\delta(y')$. Therefore (d) implies that

$$F_y\cap F_{y'}\subseteq S(y,\tfrac{1}{2}d(y,y'))\cap S(y',\tfrac{1}{2}d(y,y'))=\emptyset\,.$$

Step 2. We enlarge the F_y's.

Since any linked family that contains a singleton set has nonempty intersection, we only have to enlarge F_y if y is not isolated, because of (e). Define Z and $\mathscr{L}$ by

$$Z=\{y\in Y\colon y \text{ is not isolated}\};$$
$$\mathscr{L}=\{L\subseteq Z\colon \{F_y\colon y\in L\} \text{ is linked}\}\,.$$

Since each $L\in\mathscr{L}$ is finite, there is by Fact 1 for each $L\in\mathscr{L}$ an

$$\alpha(L)\in L\quad\text{with } \mathscr{A}_z\subseteq\mathscr{A}_{\alpha(L)} \text{ for all } z\in L\,.$$

Since $\mathscr{A}$ is a finite subset of $\mathscr{U}$, it follows from (B) that

$$\alpha(L)\in\textstyle\bigcap\{\bar{A}\colon A\in\mathscr{A}, L\subseteq\bar{A}\}=(\bigcap\{A\in\mathscr{A}\colon L\subseteq\bar{A}\})^{-}$$

for all $L\in\mathscr{L}$. Hence we can pick p_L for $L\in\mathscr{L}$ in such a way that

(f) $p_L\in(I_{\alpha(L)}\cap\bigcap\{A\in\mathscr{A}\colon L\subseteq\bar{A}\})-\bigcup\bar{\mathscr{A}}_{\alpha(L)}$,
(g) $p_L\neq p_{L'}$ for distinct $L, L'\in\mathscr{L}$,

for $\mathscr{L}$ is finite, and for each $L\in\mathscr{L}$ the set $I_{\alpha(L)}$ is a neighborhood of the (non-isolated) point $\alpha(L)$, and $\bar{\mathscr{A}}_{\alpha(L)}$ is a finite collection of closed sets none of which contains $\alpha(L)$.

For each $y\in Y$ define an enlargement E_y of F_y by

$$E_y=\begin{cases}F_y\ (=\{y\}) & \text{if } x \text{ is isolated;}\\ F_y\cup\{p_L\colon x\in L\in\mathscr{L}\} & \text{if } x \text{ is not isolated}\,.\end{cases}$$

Step 3. Proof that this works.

We first note that (b), (d) and the definition of δ imply

(*) $y\in\bar{A}$ iff $F_y\cap A\neq\emptyset$ iff $F_y\cap\bar{A}\neq\emptyset$, for all $y\in Y$ and $A\in\mathscr{A}$.

Fact 2. *For all $\mathscr{A}'\subseteq\mathscr{A}$ and $L\subseteq Y$, if $\mathscr{A}'\cup\{F_y\colon y\in L\}$ is linked, then $\bigcap\mathscr{A}'\cap\bigcap\{E_y\colon y\in L\}\neq\emptyset$.*

We may assume that $\mathscr{A}'\neq\emptyset$ since $X\in\mathscr{A}$, and that $L\neq\emptyset$ since our hypothesis about $\mathscr{A}$ tells that $\bigcap\mathscr{A}'\neq\emptyset$. We also may assume that $L\subseteq Z$, the set of nonisolated points in Y, since $F_y=E_y=\{y\}$ for $y\in Y-Z$.

Hence $L\in\mathscr{L}$, so our construction guarantees that $p_L\in E_y$ for all $y\in L$, and that $p_L\in A$ for all $A\in\mathscr{A}'$ since (*) implies that $L\subseteq\bigcap\bar{\mathscr{A}}'$.

So we did enlarge the F_y's enough. It remains to show that we did not enlarge the F_y's too much.

Fact 3. *For all* $A \in \mathscr{A}$ *and* $y \in Y$, *if* $\bar{A} \cap F_y = \emptyset$ *then* $\bar{A} \cap E_y = \emptyset$.

First observe that $y \notin \bar{A}$ because of (*), hence $A \in \mathscr{A}_y$. So if $L \in \mathscr{L}$ contains y, then $\alpha(L) \notin \bar{A}$ since $\mathscr{A}_y \subseteq \mathscr{A}_{\alpha(L)}$, hence then $p_L \notin \bar{A}$. Since $F_y \cap \bar{A} = \emptyset$ because of (*), it follows that $\bar{A} \cap E_y = \emptyset$, both for isolated y and nonisolated y.

Fact 4. *For all* $y, y' \in Y$, *if* $F_y \cap F_{y'} = \emptyset$ *then* $E_y \cap E_{y'} = \emptyset$.

If $z \in Y$ is isolated then $E_z = I_z = \{z\}$, and if $z \in Y$ is not isolated then $E_z \subseteq F_z \cup \bigcup \{I_{z'}: z' \in Z \text{ not isolated}\}$, hence if one of y and y' is isolated, then $E_y \cap E_{y'} = \emptyset$. Next assume that neither y nor y' is isolated. If $L \in \mathscr{L}$ contains y and $L' \in \mathscr{L}$ contains y', then clearly

$$y \notin L' \quad \text{and} \quad y' \notin L.$$

Hence $p_L \neq p_{L'}$ by (g); and $\alpha(L) \neq y'$ and $\alpha(L') \neq y$, hence $p_L \notin F_{y'}$ and $p_{L'} \notin F_y$ by (c) since $p_L \in I_{\alpha(L)}$ and $p_{L'} \in I_{\alpha(L')}$. Consequently $E_y \cap E_{y'} = \emptyset$.

Fact 5. $\operatorname{diam}(E_y) \leqslant 2^{-n}$ *for* $y \in Y$.

If y is isolated, $\operatorname{diam}(E_y) = \operatorname{diam}(\{y\}) = 0$. If y is not isolated, then for all $L \in \mathscr{L}$, if $y \in L$ then $p_L \in F_{\alpha(L)}$ and $F_{\alpha(L)} \cap F_y \neq \emptyset$, so $d(\alpha(L), y) < 2^{-n-1}$ by (d), hence $E_y \subseteq S(y, 2^{-n-1})$, again by (d). Therefore $\operatorname{diam}(E_y) < 2^{-n}$ since E_y is compact.

It now follows from (a) and Facts 1 through 5 that $\mathscr{E} = \{E_y: y \in Y\}$ has all properties required.

References

[Aa] J. M. Aarts, *Every metric compactification is a Wallman-type compactification*, Proc. Int. Symp. on Top. and its Appl. Herceg-Novi, Yugoslavia, 1968, pp. 29–34, Beograd 1969.

[A] J. W. Alexander, *Ordered sets, complexes and the problem of compactifications*, Proc. Nat. Acad. Sci. USA 25 (1939), pp. 296–298.

[B] M. G. Bell, *Not all compact Hausdorff spaces are supercompact*, Gen. Top. Appl. 8 (1978), pp. 151–155.

[vDvM] E. K. van Douwen and J. van Mill, *Supercompact spaces*, to appear.

[G] J. Wm. Green, *Moore-closed spaces, completeness and centered bases*, Gen. Top. Appl. 4 (1974), pp. 297–313.

[dG] J. de Groot, *Supercompactness and superextensions*, in: J. Flachsmeyer, H. Poppe and F. Terpe, eds., Contributions to extension theory of topological structures, Proc. 1967 Berlin Symp., pp. 89–90, Berlin 1969.

[vM] J. van Mill, *Supercompactness and Wallman spaces*, Ph. D. Thesis Free University of Amsterdam, June 1977 ≡ Mathematical Centre Tract 85, Amsterdam 1977.

[O'C] J. L. O'Connor, *Supercompactness of compact metric spaces*, Indag. Math. 32 (1970), pp. 30–34.

[Os] P. Ostrand, *Covering dimension in general spaces*, Gen. Top. Appl. 1 (1971), pp. 209–221.

[SS] A. K. Steiner and E. F. Steiner, *Products of compact metric spaces are regular Wallman*, Indag. Math. 30 (1968), pp. 428–430.

[St] E. F. Steiner, *Wallman spaces and compactifications*, Fund. Math. 61 (1968), pp. 295–304.

[SSz] M. Strok and A. Szymański, *Compact metric spaces have binary bases*, Fund. Math. 89 (1975), pp. 81–91.

[U] V. M. Ul'janov, *Solution of a basic problem on compactifications of Wallman type*, Dokl. Akad. Nauk SSSR 233 (1977), pp. 1056–1059 ≡ Sov. Math. Dokl. 18 (1977), pp. 567–571.

[V] A. Verbeek, *Superextensions of topological spaces*, Ph. D. Thesis Municipal University of Amsterdam, April 1972 ≡ Mathematical Centre Tract 41, Amsterdam 1972.

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Accepté par la Rédaction le 27. 12. 1978