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Minimal m-functions

by

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Abstract. Given a compact manifold M, let $\nu M = \min_F\{\nu F$: F is an m-function, νF is the number of critical points of F and $F|\partial M\}$. We prove, by using an appropriate form of the handle theory, that if M and ∂M are simply connected and $\dim M \geqslant 6$, then νM depends only on the map i_*: $H_*(\partial M; Z) \to H_*(M; Z)$ induced by the inclusion $\partial M \subset M$ and on a family of homology operations. In particular, it is an invariant of the homotopy type of the pair $(M, \partial M)$. In some cases νM is explicitly determined.

1. Introduction. A real-valued smooth function F on a compact smooth manifold M is called an *m-function* if both F and $F|\partial M$ have only non-degenerate critical points and F has no critical points in a neighbourhood of the boundary.

It follows from the Morse inequalities that for a Morse function on a closed manifold M the number of critical points of index q is not less than $b_q M + \tau_q M + \tau_{q-1} M$, where $b_q(X, A)$ denotes the rank of $H_q(X, A)$ and $\tau_q(X, A)$ is the number of torsions in $H_q(X, A)$. By the celebrated theorem of Smale [11], there exists a Morse function on M which realizes the lower bounds for all q provided M is simply connected and $\dim M \geqslant 6$. The same is true if we consider a compact, simply connected manifold with boundary, $\dim M \geqslant 6$, and define Morse functions to be constant and maximal on the boundary.

The case of m-functions is more delicate, although the Morse inequalities are still available [2], [6]. Denote by $c_q F$ the number of critical points of F of index q, by $d_q^+ F$ the number of critical points of $F|\partial M$ of index q with the gradient of F pointing outwards, and by $d_q^- F$ the number of critical points of $F|\partial M$ of index q with the gradient of F pointing inwards. The Morse–Cairns proof can easily be improved (cf. Proposition 1 below) to give the following Morse inequalities for m-functions:

$$b_q(M, \partial M) + \tau_q(M, \partial M) + \tau_{q-1}(M, \partial M) \leqslant c_q F + d_{q-1}^+ F,$$
$$b_q M + \tau_q M + \tau_{q-1} M \leqslant c_q F + d_q^- F.$$

For some manifolds, however, these are strict for any m-function F. Let $\nu M = \min_F \sum_q (c_q F + d_q^- F + d_q^+ F)$, and let μM be the Morse number of M, i.e., the minimal number of critical points of Morse functions on M (constant on ∂M). Our main aim is to find the number

$$\varkappa M = \mu M + \mu(\partial M) - \nu M .$$

An easy argument shows that $\varkappa M \geqslant 0$, but it may be non-zero (e.g. $\varkappa D^n \neq 0$).

The standard technique of handle theory yields a reduction of our problem to algebra provided M and ∂M are simply connected and $\dim M > 6$. Sections 2 and 3 contain preparatory material, the most important item there being Theorem 1 extending Smale's Cancellation Theorem. In Section 4 we characterize $\varkappa M$ on the chain level (Theorem 2) and in Section 5 we determine it, using homology groups of M and ∂M, the homomorphism induced by the embedding $i\colon \partial M \to M$, and some homology operations. If the (integral) homology of ∂M or cohomology of M has no torsion, then $\varkappa M$ can be explicitly shown to be

$$\max_{\beta}\{\operatorname{card}(\beta \cap \operatorname{Im} i_*)\colon \beta \text{ is a minimal set of generators of } H_* M\}$$

in the first case and

$$\max_{\gamma}\{\operatorname{card}(\gamma \cap \operatorname{Im} i^*)\colon \gamma \text{ is a minimal set of generators of } H^*(\partial M)\}$$

in the second case. Finally, in Section 6, we compute the minimal number of critical points of a Morse function on ∂M which extends to an m-function with no interior critical points.

2. Some results concerning m-functions. We shall work in the category of smooth compact manifolds. Let F be an m-function on M. By definition, critical points of F lie in the interior of M and we shall call them *interior critical points*. The index of each such point is defined as usual [10]. A point x is called a *boundary critical point* if it is a critical point of $F|\partial M$. Its index is a pair (q, ε), where q is the usual index of x and $\varepsilon = 1$ if the gradient of F at x points outwards or $\varepsilon = -1$ if the gradient points inwards.

We start with a simple lemma which shows how to replace a critical point of index (q, ε) by a point of index $(q, -\varepsilon)$ at the price of adding one critical point in the interior of M. The lemma is proved in [2] and [1], but we present a proof which we shall exploit in the sequel.

LEMMA 1. *Let* $V = \{(x_1, \ldots, x_n) \in R^n\colon x_n \leqslant -x_1^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_{n-1}^2\}$ *and let U be a neighbourhood of the point* $(0, \ldots, 0)$. *Then there exists an m-function F on V such that:*

(i) $F(x_1, \ldots, x_n) = x_n$ *for* $(x_1, \ldots, x_n) \in (V-U) \cup \partial V$,

(ii) *F has one interior critical point of index* $q+1$,

(iii) *the gradient of F at* $(0, \ldots, 0)$ *points inwards.*

Proof. Let $U_\varepsilon = \{x_1, \ldots, x_n) \in V\colon x_1^2 + \ldots + x_{n-1}^2 < \varepsilon, -\varepsilon < x_n\}$. Choose ε such that $U_{2\varepsilon} \subset U$ and a smooth non-increasing function $\eta\colon R \to R$ equal to 1 for $x \leqslant 0$ and to 0 for $x \geqslant \varepsilon$. It is not difficult to find another smooth function $\xi\colon R \to R$ such that:

$$\xi(x) = 0 \quad \text{for} \quad x = 0 \text{ and } x \geqslant \varepsilon,$$
$$0 < \xi(x) < (\max|\eta'| + 1)^{-1} \quad \text{for} \quad 0 < x < \varepsilon$$

and also

$$\text{(A)} \quad \begin{cases} \xi'(0) > 1, \\ \text{the equation } \xi'(t) = 1 \text{ has only one solution } x_0 \text{ in the interval } (0, \varepsilon), \\ \xi''(x) < 0 \text{ for } 0 < x \leqslant x_0 . \end{cases}$$

One can check that the function

$$F(x_1, \ldots, x_n) = x_n + \eta(x_1^2 + \ldots + x_{n-1}^2)\xi(-x_1^2 - \ldots - x_q^2 + x_{q+1}^2 + \ldots + x_{n-1}^2 - x_n)$$

has the required properties.

Condition (i) is easy.

Straightforward computation shows that an interior critical point of F must have coordinates $(0, \ldots, 0, t)$ where t satisfies the equation $\xi'(-t) - 1 = 0$, but this has exactly one solution by (A). The hessian at the interior critical point is a diagonal matrix with q elements on the diagonal equal to -2, $n-q-1$ elements equal to 2 and one equal to $\xi''(x_0) < 0$. The gradient at $(0, \ldots, 0)$ has coordinates $(0, \ldots, 0, 1-\xi'(0))$, and hence points inwards. The lemma follows.

Lemma 1 and the results of [6] give the following result, equivalent to Theorem 10.1 in [2].

PROPOSITION 1. *Let* $F\colon M \to R$ *be an m-function and let* $p \in \partial M$ *be a boundary critical point of index* $(q-1, 1)$. *If p is the unique critical point in an open set W, then there exists an m-function G such that:*

(i) $G(x) = F(x)$ *for* $x \in (M-W) \cup \partial M$,

(ii) *G has one interior critical point in W and it is of index q,*

(iii) *p is a critical point of index* $(q-1, -1)$.

LEMMA 2. *Let F be an m-function on M, and U an open subset of ∂M. If the indices of all boundary critical points in the closure of U have signs $+$, then there is an m-function G with the same set of critical points, such that the gradient of G points outwards in U.*

Proof. Let ξ be a gradient-like vector field for F (see [10]). Cover $\bar{U}$ with a finite number of closed cells such that their interiors cover U. Choose a Riemannian metric such that $\xi(x)$ is orthogonal to the boundary if and only if x is a boundary critical point. We shall proceed by induction. Suppose that $\xi(x)$ is an exterior vector for $x \in \bigcup_{j \leqslant i-1} B_j$. Using the trivialization given by the Riemannian metric, we may consider $\xi|B_i$ as the map $\xi_i\colon B_i \to R^n$. Assume that the lower half-space R^n_+ of R^n corresponds to the exterior tangent vectors and the vectors orthogonal to ∂M correspond to the x_n-axis. By assumption, ξ_i maps B_i to R^n minus the non-positive part of the x_n-axis. A deformation of R^n to R^n_+ fixed on a neighbourhood of the positive part of the x_n-axis induces a deformation of ξ_i to a map sending B_i to R^n_+ and such that:

(1) there is no change in a neighbourhood of critical points of F,

(2) x is sent to a vector orthogonal to ∂M if and only if it is a critical point of F,

(3) the property "x is mapped to an exterior tangent vector" is preserved during the deformation.

It extends to a deformation of ξ also satisfying the three conditions and we may assume that the deformation is supported in a neighbourhood of the boundary without interior critical points. Finally we get a deformation of ξ to a vector field with no interior vectors in U. The deformation induces a deformation of the m-function as described in [10]. In a neighbourhood of the critical points we have avoided changes and by (2), new critical points can not appear, whence we get an m-function with the required properties.

It follows from [6] that an m-function yields a decomposition analogous to a handle decomposition. We shall now describe such decompositions in some detail (omitting the discussion of smoothing corners, however). If X is a smooth manifold with a submanifold V, $X = V \cup D^q \times D^{n-q}$ and $V \cap D^q \times D^{n-q} = S^{q-1} \times D^{n-q} \subset \partial V$, then $D^q \times D^{n-q}$ is called a *handle of index q on V*. The embedding $S^{q-1} \times D^{n-q} \to \partial V$ is called the *attaching map*. A handle of index $(q, -1)$ on the pair (V, W), where $W \subset \partial V$, is a disc $D^q \times D^{n-q-1} \times D^1$ such that

$$D^q \times D^{n-q-1} \times D^1 \cap V = S^{q-1} \times D^{n-q-1} \times D^1 \subset \partial V$$

and

$$S^{q-1} \times D^{n-q-1} \times D^1 \cap W = S^{q-1} \times D^{n-q-1} \times \{-1\} \subset \partial W.$$

A handle of index $(q, 1)$ on the pair (V, W) is a disc $D^q \times D^{n-q-1} \times D^1$ such that

$$D^q \times D^{n-q-1} \times D^1 \cap V = S^{q-1} \times D^{n-q-1} \times D^1 \cup D^q \times D^{n-q-1} \times \{1\} \subset \partial V$$

and

$$D^q \times D^{n-q-1} \times D^1 \cap W = S^{q-1} \times D^{n-q-1} \times \{-1\} \subset \partial W.$$

In both relative cases the disc $D^q \times D^{n-q-1}$ is a handle on W which we call the *restricted handle*

If M is a smooth compact manifold with boundary, then an m-decomposition is a sequence of submanifolds

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \ldots \subset M_k = M$$

such that

(i) $M_i = M_{i-1} \cup h$, where h is a handle of index q on $M_{i-1} - \partial M$ or a handle of index $(q, \pm 1)$ on $(M_{i-1}, M_{i-1} \cap \partial M)$,

(ii) the sequence $M_{-1} \cap \partial M \subset M_0 \cap \partial M \subset \ldots \subset M_k \cap \partial M = \partial M$ is a handle decomposition of ∂M.

The handles of indices q, $0 \leqslant q \leqslant \dim M$, are called *interior handles*, the others are *boundary handles*. The left (right) sphere of an interior handle is $S^{q-1} \times \{0\}$ ($\{0\} \times S^{n-q-1}$), its left disc is $D^q \times \{0\}$ and the a-disc of a boundary handle of index $(q, 1)$ is $D^q \times \{0\} \times D^1$.

We have a natural operation on m-decompositions which corresponds to the operation described in Lemma 1. Decompose the handle $D^q \times D^{n-q-1} \times D^1$ of index $(q, 1)$ as $D^q \times D^{n-q-1} \times [0, 1] \cup D^q \times D^{n-q-1} \times [-1, 0]$. The second part is a boundary handle h of index $(q, -1)$ and the other part is an interior handle h' of index $q+1$ whose left sphere intersects the right sphere of h at just one point and transversally. Moreover, the left disc of h' is the part of the a-disc of the given handle contained in h'. An inspection of the proof of Lemma 1 shows that the corresponding operation on m-functions also has these properties, when left spheres, discs, etc. are defined as in the case of Morse functions.

Every m-function determines an m-decomposition. On the other hand, for any m-decomposition of M we can find an m-function F on M such that there is a 1-1 correspondence (preserving indices) between handles of the decomposition and critical points of F. Thus, as long as we deal with quantitative problems, we can replace m-functions by m-decompositions. Also, if we ignore m-functions and consider m-decompositions, the results of the paper are valid in the PL category.

Let h_1, h_2 be a pair of adjacent handles in an m-decomposition, say, $M_{q+1} = M_q \cup h_1$, $M_{q+2} = M_{q+1} \cup h_2$. We call the pair complementary if one of the following conditions is satisfied:

a) h_1 and h_2 are of the same type (both interior or both boundary) and there exist a relative collar (U, U') of $(\partial M_q - \partial M, \partial(M_q \cap \partial M))$ in $(M_q, M_q \cap \partial M)$ and a diffeomorphism $(M_q, M_q \cap \partial M) \to (M_{q+2}, M_{q+2} \cap \partial M)$ which is isotopic to the inclusion $M_q \to M_{q+2}$ through an isotopy supported in $U \cup h_1 \cup h_2$,

b) h_1 has index $(q, -1)$, h_2 is interior of index $q+1$ and $(M_{q+2}, M_{q+2} \cap \partial M)$ is diffeomorphic to $(M_q \cup h', (M_q \cup h') \cap \partial M)$, where h' is a boundary handle of index $(q, +1)$ restricted to the same handle in the boundary as h_1 and the diffeomorphism is supported in a collar as above,

c) h_1 is of index q, h_2 of index $(q, 1)$ and $(M_{q+2}, M_{q+2} \cap \partial M)$ is diffeomorphic to $(M_q \cup h', (M_q \cup h') \cap \partial M)$ for a handle h' of index $(q, -1)$ and the diffeomorphism supported in a collar as in a).

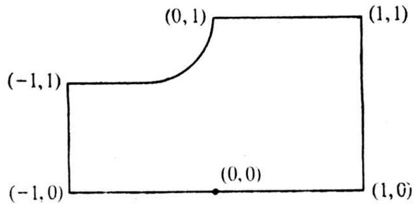
The most important tool for minimalization problems in the case of Morse functions is the Smale–Morse cancellation theorem, which provides a characterization of complementary critical points (cf. [10]). We shall extend this characterization to cover the case of boundary critical points.

THEOREM 1. a) *If h is an interior handle of index q, h' a boundary handle of index (q, 1) the right sphere of h and the a-disc of h' intersect transversally and at a single point, then the pair h, h' is complementary.*

b) *A pair of boundary handles is complementary if and only if the signs of their indices are equal and the restricted handles are complementary.*

Proof. a) Replace the handle h' by the sum of an interior handle k and a boundary handle k' of index $(q, -1)$. We can choose the decomposition of h' into k and k' such that the left sphere of k contains the intersection point of the a-disc of h' with the right sphere of h. By the Smales–Morse theorem ([10], Th. 5.4), the pair h, k is complementary, and this gives the required diffeomorphism.

b) If we add a boundary handle h to a manifold M, then $M \cup h$ has the same homotopy type as M if and only if the index of h has the sign $+$. This implies the necessity of the first condition. The restriction of the diffeomorphism provided by the definition of the complementary handles ensures that the restricted handles are complementary, hence the second condition is also necessary. Let $N = \partial(M_q \cap \partial M)$. There exists a neighbourhood U of N of the form $N \times [-1, 1] \times [0, 1]$, where $U \cap \partial M = N \times \{1\} \times [0, 1]$ and N is identified with $N \times \{1\} \times \{1\}$. It follows from the uniqueness of collars that, given the collar $N \times [0, 1] \times \{1\}$, we may isotop the attaching map of h (and afterwards that of h') to the product of the attaching map of the restricted handle by $\mathrm{id}_{[0,1]}$. Thus we can assume without loss of generality that $h = (h \cap \partial M) \times [0, 1]$ and $h' = (h' \cap \partial M) \times [0, 1]$ and the smooth structure of $[-1, 1] \times [0, 1]$ coming from U is induced by the embedding in R^2 pictured below:



Let $\varphi\colon [-1, 1] \times [0, 1] \to [0, 1] \times [0, 1]$ be a homeomorphism such that

$$\varphi(-1, 1) = (0, 0),$$
$$\varphi(0, 0) = (\tfrac{1}{2}, 0),$$
$$\varphi(-1, 0) = (\tfrac{1}{4}, 0),$$
$$\varphi(t, 1) = (t, 1),$$
$$\varphi(1, t) = (1, t) \quad \text{for} \quad 0 \leqslant t \leqslant 1,$$

and φ is a diffeomorphism except at $(-1, 0)$. The map $\mathrm{id}_N \times \varphi$ can be extended by identity to a diffeomorphism (ignoring for a moment the point $(-1, 0)$) $\psi\colon U \cup h \cup h' \to N \times [0, 1] \times [0, 1] \cup h \cup h'$. By the assumption we have a diffeomorphism $f\colon M_q \cap \partial M \to (M_q \cup h \cup h') \cap \partial M$ which is isotopic to the inclusion $M_q \cap \partial M \to (M_q \cup h \cup h') \cap \partial M$ through an isotopy H_s, $s \in [0, 1]$. We can take H_s equal to the identity on an open set containing the closure of the complement of $U \cap \partial M$. Define an isotopy of the inclusion $U \to U \cup h \cup h'$ by

$$\tilde{H}_s = \psi^{-1}(H_s \times \mathrm{id}_{[0,1]})(\psi | U).$$

Since $\tilde{H}_s$ is equal to an identity near $N \times \{1\} \times \{0\}$, the new isotopy is smooth (even at $(-1, 0)$) and it is the identity in a neighbourhood of $N \times [-1, 1] \times \{0\} \cup N \times \{-1\} \times [0, 1]$, which allows us to extend $\tilde{H}_s$ smoothly to M_q by the identity. This gives the required isotopy.

Remark. Theorem 1 may be used to detect all types of complementary pairs of handles, type b) being dual to the case a) of the theorem.

3. The chain complexes associated with an m-decomposition. We shall now describe an algebraic object arising from an m-decomposition and containing all the information needed to resolve our minimalization problem. The object is a homomorphism of chain complexes of based free groups. Homology groups of the chain complexes are isomorphic to $H_*(\partial M)$ and $H_* M$, respectively (integer coefficients). The bases correspond to handles of the decomposition. Three operations:

— introduction of a complementary pair of handles,
— cancellation of a complementary pair,
— change of the attaching map of a handle by an isotopy

introduce some useful operations on the chain complexes and on the homomorphism (compare Lemmas 4 and 5 below). All the statements in this section have well-known analogues in Morse theory (cf. [1], [2], [3], [10]) and their proofs, which only need easy alterations (if any), will be omitted.

An m-decomposition is called *nice* if there is a sequence of submanifolds

$$\emptyset = M_{-1} \subset \tilde{M}_0 \subset M_0 \subset \tilde{M}_1 \subset M_1 \subset \ldots \subset \tilde{M}_n \subset M_n = M$$

such that

1) $\tilde{M}_i = M_{i-1} \cup h_1 \cup \ldots \cup h_r \cup h'_1 \cup \ldots \cup h'_s$, where $h_1, \ldots, h_r$ are handles of index i on $M_{i-1} - (\partial M \cap M_{i-1})$ and $h'_1, \ldots, h'_s$ are handles of index $(i, -1)$ on $(M_{i-1}, \partial M \cap M_{i-1})$,

2) $M_i = \tilde{M}_i \cup h''_1 \cup \ldots \cup h''_t$, where $h''_1, \ldots, h''_t$ are handles of index $(i, 1)$ on $(\tilde{M}_i, \partial M \cap \tilde{M}_i)$,

3) for any two handles of the same index the attaching maps have disjoint images.

Lemma 3. *For any compact manifold M there is a nice m-decomposition of M. If M and $\partial M \neq \emptyset$ are simply connected,* $\dim M \geqslant 6$, *then there is a nice m-decomposition with no handles of indices* $0, 1, n, n-1, (1, \pm 1), (n-2, \pm 1)$.

Consider a nice m-decomposition of M and choose orientations for D^i, D^{n-i-1}, D^1 in any boundary handle $D^i \times D^{n-i-1} \times D^1$ and for D^i, D^{n-i} in each interior handle $D^i \times D^{n-i}$. In the free groups

$$R_i = H_i(M_i \cap \partial M, M_{i-1} \cap \partial M),$$
$$S'_i = H_i(\tilde{M}_i \cup (\partial M \cap M_i), \tilde{M}_{i-1} \cup (\partial M \cap M_{i-1}))$$
$$T_i = H_i(\tilde{M}_i \cup (\partial M \cap M_i), \tilde{M}_{i-1} \cup (\partial M \cap M_i))$$

the oriented discs determine bases which we call preferred bases. The connecting homomorphism of the triple $M_i \cap \partial M$, $M_{i-1} \cap \partial M$, $M_{i-2} \cap \partial M$ is a homomorphism

$\partial_i^R\colon R_i \to R_{i-1}$ and $R = \{R_i, \partial_i^R\}$ is a chain complex. Similarly, we get complexes $S' = \{S_i', \partial_i'\}$, $T = \{T_i, \partial_i^T\}$ and inclusions induce the exact sequence

$$(*) \qquad 0 \to R \to S' \to T \to 0\,.$$

The CW-complex associated with an m-decomposition of M has the homotopy type of M. Since the image of $\tilde{M}_i \cup (\partial M \cap M_i)$ under the natural projection is the ith skeleton of the CW-complex, we have isomorphisms

$$H_*(R) = H_*(\partial M)\,,$$
$$H_*(S') = H_*(M)\,,$$
$$H_*(T) = H_*(M, \partial M)\,.$$

Let $D_i^{\pm}$ denote the free abelian group generated by the set of handles of index $(i, \pm 1)$, and let C_i be the group freely generated by handles of index i.

We have isomorphisms

$$R_i = D_i^- \oplus D_i^+\,,$$
$$S_i' = C_i \oplus D_i^- \oplus D_i^+ \oplus D_{i-1}^+\,,$$
$$T_i = D_{i-1}^+ \oplus C_i\,,$$

where generators corresponding to C_i are given by the left discs of interior handles, those from D_i^-, D_i^+ by the left discs of restricted handles and those in D_{i-1}^+ by the a-discs of boundary handles. When we use these identifications, the matrix of $\partial'\colon S_i' \to S_{i-1}'$ has the form

$$\begin{bmatrix} X_i & 0 & 0 & A_i \\ Y_i & \multicolumn{2}{c}{\boxed{\partial_i^R}} & B_i \\ 0 & & & \mathrm{id}_{D_{i-1}^+} \\ 0 & 0 & 0 & \partial_{i-1}^+ \end{bmatrix}$$

i.e., for any $c_i \in C_i$, $d_i^- \in D_i^-$, $d_i^+ \in D_i^+$, $d_{i-1}^+ \in D_{i-1}^+$ we have

$$\partial'(c_i + d_i^- + d_i^+ + d_{i-1}^+)$$
$$= X_i(c_i) + Y_i(c_i) + \partial_i^R(d_i^- + d_i^+) + A_i(d_{i-1}^+) + B_i(d_{i-1}^+) + d_{i-1}^+ + \partial_{i-1}^+(d_{i-1}^+)\,,$$

where

$$X_i\colon C_i \to C_{i-1}\,,$$
$$Y_i\colon C_i \to D_{i-1}^-\,,$$
$$A_i\colon D_{i-1}^+ \to C_{i-1}\,,$$
$$B_i\colon D_{i-1}^+ \to D_{i-1}^-$$

are some homomorphisms determined by attaching maps of handles and ∂_{i-1}^+ is the composition of ∂_{i-1}^R with the projection on D_{i-2}^+. The maps B_i, A_{i-1} reflect how the handles of index $(i-1, +1)$ are attached to $\tilde{M}_{i-1}$. Since the addition of handles of index $(q, 1)$ to a manifold is always a trivial operation from the point of view of homotopy, one can replace the complex S' by a homotopy equivalent

chain complex $S = \{S_i = H_i(M_i, M_{i-1}), \partial_i^S\}$, where ∂_i^S is the connecting homomorphism of the triple (M_i, M_{i-1}, M_{i-2}). The groups S_i are isomorphic to $C_i \oplus D_i^-$ and

$$\partial_i^S(c_i + d_i^-) = X_i(c_i) + Y_i(c_i) + (\mathrm{id}_{D_{i-1}^-} - A_i - B_i)(\partial_i^R(d_i^-))\,.$$

The chain maps

$$\eta\colon S \to S'\colon\ c_i + d_i^- \mapsto c_i + d_i^- - \partial_i^+(d_i^-)$$

and

$$\psi\colon S' \to S\colon\ c_i + d_i^- + d_i^+ + d_{i-1}^+ \mapsto c_i + d_i^- - A_{i+1}(d_i^+) - B_{i+1}(d_i^+)$$

(the latter induced by $(\tilde{M}_i \cup (\partial M \cap M_i), \tilde{M}_{i-1} \cup (\partial M \cap M_{i-1})) \subset (M_i, M_{i-1})$) are homotopy equivalences, because $\psi \circ \eta = \mathrm{id}$ and the homotopy $\Delta\colon \mathrm{id} \sim \eta \circ \psi$ is given by $\Delta_i\colon S_i' \to S_{i+1}'\colon c_i + d_i^- + d_i^+ + d_{i-1}^+ \mapsto d_i^+$. All this is verified by straightforward and rather tedious computations, using the equality $2\partial_i^+ + \partial_i^+ A_{i+1} = 0$, which in turn is an immediate consequence of the rule $\partial' \circ \partial' = 0$.

The exact sequence $(*)$ can now be completed to a homotopy commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & R & \to & S' & \to & T & \to & 0 \\ & & \| & & \downarrow\psi & & \| & & \\ & & R & \underset{\varphi}{\to} & S & \to & T & & \end{array}$$

where φ is the composition of the inclusion of R into S' with ψ. The homology of S is isomorphic to $H_*(M)$ and the homomorphism induced by φ in homology is the homomorphism induced by the inclusion $\partial M \subset M$. Note that $\varphi | D_i^-$ is the inclusion. A generator h of D_i^+ goes by φ to an element represented by $h \cap M_i$. Thus, if $\varphi(h) = \sum_\alpha a_\alpha g_\alpha$, where $\{g_\alpha\}$ is the preferred base of $C_i \oplus D_i^-$, then a_α is the intersection number of the boundary of the a-disc of h with the right sphere of g_α (when orientations are chosen properly).

The complexes R, S and the chain map $\varphi\colon R \to S$ depend only on the m-decomposition under consideration and the orientations chosen. By an abuse of language, we shall not distinguish φ from the m-decomposition. The following two lemmas describe some operations induced by changes of m-decompositions. Given a preferred generator $r \in R$ and an element $s \in \mathrm{Im}\,\partial$, there is an isotopy of the attaching map of the handle corresponding to r such that, in the new decomposition $\bar{\varphi}$, we have $\bar{\varphi}(\bar{r}) = \varphi(r) + s$ (cf. [7]). Thus we get (for $\dim M > 6$):

LEMMA 4. *Any map chain-homotopic to φ is determined by an m-decomposition.*

Let $\{a_1, \ldots, a_k\}$ be a preferred base of R_i (or S_i). We say that a_1 can be added to a_2 if there is an m-decomposition and an automorphism $A = \{A_i\}$ of R (or S) such that A_j have the identity matrices for $j \neq i$ and the matrix of A_i corresponds to the passing from the given base to the base $\{a_1 + a_2, a_2, \ldots, a_k\}$, the automorphism induced by a change of the m-decomposition (cf. [3], [11]). Additions of handles are realized by alterations of attaching maps by isotopies and the result is just a change of the preferred base.

LEMMA 5. *Any two preferred generators of R_i can be added if $2\leqslant i\leqslant \dim M-3$. Two elements of a preferred base of S_i can be added if either both belong to $C_i\subset S_i$ or both belong to $D_i^-\subset S_i$ and $2\leqslant i\leqslant \dim M-2$. Thus, when $2\leqslant i\leqslant \dim M-3$, any base of R_i (and also C_i, D_i^- for $2\leqslant i\leqslant \dim M-2$) is the preferred base of an m-decomposition.*

4. Computation of νM on the chain level. For the rest of the paper, fix a compact manifold M of dimension $n>6$ and assume that M and ∂M are simply connected. We shall determine νM, the minimum over all m-functions F on M of the number of points which are critical for F or for $F|\partial M$.

It is a simple exercise on the collar neighbourhood theorem to show that, given a Morse function $\tilde{F}$ on M (constant and maximal on ∂M) and a Morse function f on ∂M, there is an m-function F with the following properties:

(i) F restricted to ∂M is equal to f up to a constant,

(ii) the gradient of F points outwards at any point of ∂M,

(iii) F is equal to $\tilde{F}$ outside a neighbourhood of ∂M and it has not critical points in that neighbourhood.

It follows from Lemma 2 that up to unimportant changes an m-function F is the result of this construction if and only if $d_i^- F = 0$ for all i, i.e., if and only if F has no boundary critical points of index $(i, -1)$. Corresponding m-decompositions are those with $D_i^- = 0$ for all i. Any such a decomposition induces a handle decomposition of M and a decomposition of ∂M. An m-decomposition with $D^- = 0$ may not be minimal, but starting from it we shall obtain a minimal one. One way to do this is decreasing the number of handles with the help of the following lemma, a partial converse of Proposition 1.

LEMMA 6. *Let $2\leqslant i\leqslant n-3$ and $x\in D_i^+$, $y\in C_i$. If x, y are preferred generators and $\varphi x = y$, then the handle determining y can be removed from the decomposition changing the index of x to $(i, -1)$. Algebraically, R is unchanged and in S the free summand generated by y collapses and the summand generated by x passes from D_i^+ to D_i^-.*

Proof. The equality $\varphi x = y$ means that the intersection number of the a-disc of x and the right sphere of y is 1. Since all assumptions of the Whitney lemma are fulfilled, there is an isotopy of the attaching map of x making the intersection one-point and transversal. The lemma follows from Theorem 1(a).

The following lemma is essentially known (cf. [7]).

LEMMA 7. *Let X be a simply connected manifold of dimension $n>5$. Using the operations of the alteration of attaching maps of handles by isotopies, addition and cancellation of complementary pairs of handles, one can transform any non-minimal handle decomposition of X to a handle decomposition with the same number of handles and at least one complementary pair. In this way any handle decomposition can be transformed into a minimal one.*

Proof. Just as in the case of m-decompositions, the handle decomposition determines the based chain complex $\{C_i, \partial_i\}$ with the homology isomorphic to the homology of X. If the decomposition is non-minimal, then there is a base element $h\in \operatorname{Im}\partial_{q+1}$, as follows from the characterization of minimal Morse functions on simply connected manifolds [11] (cf. also [7]). If $2\leqslant q\leqslant n-2$, then any base of C_q can be obtained from the preferred one by using additions of handles and reorienting some handles, see e.g. [3]. Thus, after some rearrangement, we can find preferred generators such that $\partial g = h$ and the Whitney lemma implies that it is a complementary pair. When $h\in C_1$, one must apply the more complicated procedure from [7], which results in replacing h by a handle of index 3. In this way we may replace all handles of index 1 by handles of index 3 and handles of index $n-1$ by handles of index $n-3$, and this provides the reduction of the general case to the one already dealt with. The complementary pair can be cancelled and we can continue this process till we get a minimal handle decomposition.

For a homomorphism $\varphi\colon G\to G'$ of free, finitely generated abelian groups denote by $r(\varphi)$ the maximal rank of a direct summand of G' contained in $\operatorname{Im}\varphi$. For a matrix A with integer coefficients denote by $r(A)$ its rank. Thus $r(\varphi)$ is the maximal value of $r(A)$, where A ranges over all matrices of φ, which in turn is the maximal number k such that some matrix of φ has the $k\times k$ identity matrix as a direct summand. Therefore $r(\varphi) = R(G')-R(G'/\operatorname{Im}\varphi)$, where $R(X)$ denotes the minimal number of generators of an abelian group X. The following theorem expresses νM by μM, $\mu(\partial M)$ and $r(\varphi)$.

THEOREM 2. *If M and ∂M are simply connected and $\dim M>6$, then $\nu M = \mu M + \mu(\partial M)-\varkappa M$, where $\varkappa M$ is the maximum of $r(\varphi)$ over all m-decompositions with $D^- = 0$ which induce minimal handle decompositions of M and ∂M.*

Proof. Given an m-decomposition realizing $\varkappa M$, we may cancel $\varkappa M$ interior handles with the help of Lemma 6. We do not bother about handles of indices $1, n-1$, $(1, \pm 1)$, $(n-2, \pm 1)$, because handles of index 1 cannot appear in a minimal handle decomposition of M or ∂M [11]. Similarly, no handles of indices equal to n can appear in a minimal handle decomposition of M corresponding to a Morse function maximal on ∂M. In any minimal handle decomposition of M or ∂M there is exactly one handle of index 0 and $\varphi_0\colon R_0\to S_0$ is an isomorphism. Since there is no handle of index 1, the connectivity of M implies that the index of the boundary handle is $(0, 1)$. The sum of the two handles is the sum of two discs glued together along a common subdisc in boundaries, hence still a disc. This shows that any pair of handles consisting of a handle of index 0 and another of index $(0, 1)$ may be replaced by one handle of index $(0, -1)$, which is an extension of Lemma 6 to handles of index 0. Thus $\nu M\leqslant \mu M+\mu(\partial M)-\varkappa M$. In order to show the opposite inequality, consider a minimal m-decomposition of M. If the decomposition induced on the boundary is not a minimal one, we can rearrange it to find a complementary pair of handles (Lemma 7) and make the signs of their indices equal, possibly adding one interior handle. By the cancellation of the two complementary handles (Theorem 1(b)) we decrease the number of handles. Therefore the restriction to the boundary applied to a minimal

m-decomposition gives a minimal handle decomposition. Now, by the procedure of Proposition 1, make all indices of boundary handles equal to $(q, +1)$. To do this we must add a number, say p, of interior handles. This alteration introduces the identity matrix of rank p as a direct summand of φ, because φ is given on a boundary handle of index $(q, +1)$ by the intersection numbers of the a-disc of the handle with right spheres of interior handles. Therefore $r(\varphi) = p$. If the induced handle decomposition of M is not minimal, then by Lemma 7 we may cancel a number of complementary pairs to get S' which is minimal. If there are s complementary pairs, then for the resulting m-decomposition φ' we have

$$r(\varphi)-r(\varphi') = 2s-R(S/\mathrm{Im}\,\varphi)+R(S'/\mathrm{Im}\,\varphi')\leqslant 2s,$$

since the natural map $S/\mathrm{Im}\,\varphi \to S'/\mathrm{Im}\,\varphi'$ is onto. The decomposition φ' belongs to the class considered in the definition of $\varkappa M$, whence $\varkappa M\geqslant r(\varphi')\geqslant p-2s$. But $\nu M+p = \mu(\partial M)+\mu M+2s$, what in turn gives

$$\nu M+\varkappa M\geqslant \nu M+p-2s = \mu M+\mu(\partial M).$$

The proof is complete.

5. Determination of $\varkappa M$. We shall now describe $\varkappa M$ (hence νM as well) in terms of homology. First we shall construct some homology operations which are used (besides homology groups and the homomorphism induced by the inclusion $i\colon \partial M\to M$) in the description of $\varkappa M$.

Let $A = \{A_q, \partial_q\}$, $B = \{B_q, \delta_q\}$ be chain complexes of free abelian groups and let $\varphi\colon A\to B$ be a chain map. Choose a splitting $B = \ker\delta \oplus \mathrm{coim}\,\delta$. Equivalently, we can fix a base $\beta = \{b_i\colon i\in I\}$ of B extending a base $\beta' = \{b_i\colon i\in I'\subset I\}$ of $\ker\delta$. It determines the projection

$$P_\beta\colon B\to \ker\delta\colon P_\beta\big(\sum_{i\in I}\lambda_i b_i\big) = \sum_{i\in I'}\lambda_i b_i.$$

For any such base we can define an operation Θ_β which leads from tor H_*A (the torsion part of H_*A) to the family of subsets of H_*B. Let $\mathrm{ord}(x)$ denote the order of x. For $x\in \mathrm{tor}\,H_qA$ and $\xi\in x$ we have $\mathrm{ord}(x)\xi\in \mathrm{Im}\,\delta$; thus the set

$$P_\beta\varphi\partial_q^{-1}(\mathrm{ord}(x)\xi)+\mathrm{ord}(x)\ker\delta_q$$

is non-empty and contained in $\ker\delta$. It generates a set $\Theta_\beta(x)$ of homology classes which is a coset of $\mathrm{ord}(x)H_{q+1}B+\varphi_*(H_{q+1}A)$. For example id: $A\to A$ induces $\Theta_\beta(x) \equiv H_{q+1}A$.

Suppose that $\varphi'\colon A'\to B'$ is another chain map and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B\\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g}\\ A' & \xrightarrow[\varphi']{} & B' \end{array}$$

commutes up to chain homotopy. If f, g are homotopy equivalences, then we say that $(f, g)\colon \varphi\to\varphi'$ is a homotopy equivalence.

PROPOSITION 2. *The operation Θ_β^{γ} is well defined and the family $\{\Theta_\beta\}$ depends only on the homotopy type of chain maps. More precisely, for any homotopy equivalence $(f, g)\colon \varphi\to\varphi'$, the operation $\Theta(y) = g_*\Theta_\beta(f_*^{-1}(y))$ belongs to the family of operations generated by φ'.*

Proof. If $\xi, \xi'\in x$ so that $\xi-\xi' = \partial\alpha$ and $\eta\in\partial^{-1}(\mathrm{ord}(x)\xi)$, $\eta'\in\partial^{-1}(\mathrm{ord}(x)\xi')$, then there is an $\alpha'\in\ker\partial$ such that $\eta-\eta' = \mathrm{ord}(x)\alpha+\alpha'$. Thus $P_\beta\varphi(\eta-\eta')$ $= \mathrm{ord}(x)P_\beta\varphi(\alpha)+\varphi(\alpha')\in \mathrm{ord}(x)\ker\delta+\varphi(\ker\partial)$; hence ξ and ξ' define the same sets.

Now we must show that for any β there is a base γ of B' such that $g_*\Theta_\beta(x)$ $= \Theta_\gamma(f_*x)$ for any $x\in\mathrm{tor}\,H_*A$. Let $\xi\in x\in H_*A$. If $\eta\in\partial^{-1}(\mathrm{ord}(x)\xi)$, then $f(\eta)\in(\partial')^{-1}(\mathrm{ord}(x)f(\xi))$. Denote by $\tilde g$ a homotopy inverse to g and let $g\tilde g = \mathrm{id}_{B'}+$ $+\delta'\Delta'+\Delta'\delta'$ and $\tilde g g = \mathrm{id}_B+\Delta\delta+\delta\Delta$, where $\Delta_q\colon B_q\to B_{q+1}$ and $\Delta_q'\colon B_q'\to B_{q+1}'$. Let $P_\gamma = gP_\beta\tilde g-\delta'\Delta'\colon B'\to\ker\delta'$. It is easily verified that $P_\gamma|\ker\delta' = \mathrm{id}$, hence it induces a splitting $B' = \ker\delta'\oplus\ker P_\gamma$. Modulo $\mathrm{Im}\,\delta'$ we have

$$\begin{aligned} gP_\beta\varphi(\eta)-P_\gamma\varphi' f(\eta) &\equiv (gP_\beta-P_\gamma g)\varphi(\eta) = (gP_\beta\Delta\delta+gP_\beta\delta\Delta-\delta'\Delta' g)\varphi(\eta)\\ &\equiv gP_\beta\Delta\delta\varphi(\eta). \end{aligned}$$

But

$$gP_\beta\Delta\delta\varphi(\eta) = gP_\beta\Delta\varphi(\mathrm{ord}(x)\xi) = \mathrm{ord}(x)\big(gP_\beta\Delta\varphi(\xi)\big)\in\mathrm{ord}(x)\ker\delta'.$$

This yields the inclusion $g_*\Theta_\beta(x)\subset\Theta_\gamma(f_*x)$ and the reverse inclusion follows in the same way.

For any two m-decompositions φ, φ' of a manifold M the identity map induces a homotopy equivalence $\varphi\to\varphi'$. Thus the family of homology operations induced by an m-decomposition depends only on the manifold. More generally, we have

COROLLARY. *The family $\{\Theta_\beta\}$ of homology operations defined by an m-decomposition of a manifold M depends only on the homotopy type of the pair $(M, \partial M)$.*

For the rest of the paper we assume that all groups under consideration are finitely generated and abelian. In any chain complex $\{C_q, \partial_q\}$, C_q will be a free abelian, finitely generated group and $\bigoplus_q C_q$ will be finitely generated. Note that complexes arising from m-decompositions of compact manifolds satisfy these conditions.

Denote by $((x_1, \ldots, x_n))$ the subgroup generated by the set $\{x_1, \ldots, x_n\}$. By a base of a group G we mean a set $\{b_1, \ldots, b_r\}\subset G$ such that $G = ((b_1))\oplus\ldots\oplus((b_r))$ and $\mathrm{ord}(b_i)|\mathrm{ord}(b_{i+1})$ for $i = 1, \ldots, r-1$. The number r, equal to the minimal number of generators of G, as well as the sequence of orders are determined by G.

Any set of generators $\beta\subset G$ induces a free resolution $0\to F_1\to F_0\to G\to 0$ of G, where F_0 is the group freely generated by β, π is the natural homomorphism induced by the inclusion $\beta\subset G$ and $\delta\colon F_1\to F_0$ is the inclusion of $\ker\pi$. We shall call a free resolution minimal if it is the result of this construction for a minimal set of generators. Easily, minimal resolutions are those with F_0 of minimal rank.

The free part of G gives no elements in F_1. If β is a base of G, then we have G

split into the free part and the torsion part, and when we restrict ourselves to the torsion part, then for the choice of the base in F_1 the matrix of δ is diagonal. The diagonal entries must form the sequence of orders of torsion base elements of G. Such based resolutions will by called canonical.

If G is graded, we consider only homogeneous generators. The corresponding gradation of $F_1 \oplus F_0$ is given by

$$(F_0)_q = ((\beta \cap G_q)),\ (F_1)_q = \delta^{-1}(F_0)_{q-1}\,.$$

In this way we get the based chain complex $\{F_0 \oplus F_1, \delta\}$. Given a group homomorphism $f\colon G \to G'$ and free resolutions of G and G', there is a set of covering homomorphisms, any of them provided by the commutative diagram

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{f_1} & F_1' \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\delta'} \\ F_0 & \xrightarrow{f_0} & F_0' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ G & \xrightarrow{f} & G' \end{array}$$

There is a 1-1 correspondence between the possible choices of f_0 and the matrices of f when the resolutions are canonical. In the graded case we assume that f_0, f_1 preserve the gradation. Note that if we add to f_0+f_1 an arbitrary gradation preserving map $\psi\colon F_1 \to F_0'$, then the resulting map is still a chain map and the set of maps of the form $f_0+f_1+\psi$ such that $\delta' f_1 = f_0\delta$ and $f\pi = \pi' f_0$ is the set of chain maps covering f.

We shall denote the functor $\mathrm{Hom}(\cdot, Z)$ by $*$ and call A^* (resp. f^*) dual to A (resp. f). Applying this functor to the diagram above, we get — under the assumption that G, G' are finite — the homomorphism of free resolutions

$$\begin{array}{ccc} \mathrm{Ext}(G,Z) & \xleftarrow{\mathrm{Ext}(f)} & \mathrm{Ext}(G',Z) \\ \uparrow & & \uparrow \\ F_1^* & \xleftarrow{\quad f_1^*\quad} & (F_1')^* \\ {\scriptstyle\delta^*}\uparrow & & \uparrow{\scriptstyle(\delta')^*} \\ F_0^* & \xleftarrow{\quad f_0^*\quad} & (F_0')^* \\ \uparrow & & \uparrow \\ 0 & \longleftarrow & 0 \end{array}$$

Since the matrices of δ, δ' are diagonal, each entry on the diagonal dividing the following one, the bases of F_1 and F_1' induce bases of $\mathrm{Ext}(G, Z)$ and $\mathrm{Ext}(G', Z)$, which we call *dual bases*.

We shall now describe a construction which associates with any chain complex with based homology a homotopy equivalent chain complex of minimal rank and with any chain map a set of chain maps between the resulting complexes. In fact,

this construction will use only homology groups and the family of homology operations induced by f. Applied to a handle decomposition of M, this construction will give a standard model for a minimal handle decomposition of M. In the relative case we obtain a set of models of the map of handle decompositions induced by $i\colon \partial M \to M$.

Let C be a chain complex of free, finitely generated groups with based homology and $0 \to F_1 \xrightarrow{\delta} F_0 \xrightarrow{\pi} H_* C$ the canonical resolution of $H_* C$. Let $C^0 = F_0 \oplus F_1$ and $\delta^0\colon F_0 \oplus F_1 \to F_0 \oplus F_1$ be given by $\delta^0 | F_0 = 0$, $\delta^0 | F_1 = \delta$. We obtain a chain complex $\{C^0, \delta^0\} = \{F_0 \oplus F_1, \delta^0\}$ with the gradation and bases coming from $H_* C$. This complex is called the *homology model of* C.

For a chain map $f\colon C \to C'$ define a chain map $\Gamma\colon C^0 \to C'^0$ in the following way: For any base element $b \in \mathrm{tor}\, H_* C$ choose an element x_b such that $\pi'(x_b) \in \Theta_\beta(b) \subset H_* C'$. Any base element $b^{(1)} \in F_1$ corresponds to a base element $b \in \mathrm{tor}\, H_* C$, the 1-1 correspondence being given by the condition $\delta(b^{(1)}) \in (F_0)_q \cap \pi^{-1}(b)$. We define a map $\psi\colon F_1 \to F_0'$ putting $\psi b^{(1)} = x_b$, and assume that it preserves the gradation. Let

$$\Gamma = \begin{bmatrix} f_0 & \psi \\ 0 & f_1 \end{bmatrix}\colon F_0 \oplus F_1 \to F_0' \oplus F_1',$$

where $f_0 \oplus f_1$ is a map induced by f.

The condition $f_0\delta = \delta' f_1$ is equivalent to

$$\begin{bmatrix} f_0 & \psi \\ 0 & f_1 \end{bmatrix}\begin{bmatrix} 0 & \delta \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \delta' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} f_0 & \psi \\ 0 & f_1 \end{bmatrix}$$

and therefore it says that Γ is a chain map. We shall call Γ a homology model of f. Note that while $\{C^0, \delta^0\}$ is unique for a complex C with based homology, there are many homology models of f. In the sequel we shall use the notation introduced here: the homology model of C will be denoted by $\{C^0, \delta^0\}$, F_0, F_1 will always come from a free resolution $F_1 \xrightarrow{\delta} F_0 \xrightarrow{\pi} G$ and (f_0, f_1) will be a map of resolutions covering $f\colon G \to G'$.

LEMMA 8. *Any chain complex is homotopy equivalent to its homology model and any chain map is homotopy equivalent to its homology model.*

Proof. Choose a base of C inducing a decomposition $C = \ker\partial \oplus \mathrm{coim}\,\partial$. Define the map $\alpha\colon C \to C^0$ first on $\ker\partial$, sending any base element b to an element of F_0 representing the homology class $[b]$. For any base element $c \in \mathrm{coim}\,\partial$ we have $\alpha\partial c = \delta\xi$, where $\xi \in F_1$. If we put $\alpha c = \xi$, then the resulting map is a homotopy equivalence since it is a chain map inducing an isomorphism on homology.

To prove the second part of the lemma it is enough to check the commutativity, up to homotopy, of the diagram

$$\text{(A)}\qquad \begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ C^0 & \xrightarrow[\Gamma]{} & C'^0 \end{array}$$

where α' is defined in the same way as α. As can easily be seen from the definitions of α, α' and Γ, for any base element $b \in \ker\partial$ elements $\Gamma\alpha(b)$ and $\alpha' f(b)$ differ by $\delta'(\xi_b)$, where $\xi_b \in F_1'$. Thus the homotopy defined on the base by

$$\tilde{\Delta}(b) = \begin{cases} \xi_b & \text{for} \quad b \in \ker\partial\,, \\ 0 & \text{for} \quad b \in \operatorname{coim}\partial \end{cases}$$

makes diagram (A) commutative after restriction to $\ker\partial$.

In the rest of the proof the free part F of $H_* C$ plays no important role: if $C = F \oplus \tilde{C}$, where $\tilde{C}$ is a subcomplex with finite homology groups, then the homotopy defined above is final on F. Thus we shall exclude unimportant details by the assumption that $H_* C$ is finite. We now specify the base of C (hence the map α as well) to be a base $\gamma = \gamma_0 \cup \gamma_1 \cup \gamma_2$ such that

1) $\gamma_0 \subset \ker\partial - \operatorname{im}\partial$ induces a base $\bar{\gamma}$ of $H_* C$ such that any element of $\bar{\gamma}$ is uniquely represented by an element of γ_0,

2) the subgroup generated by γ_1 is disjoint from $\ker\partial$,

3) $\gamma_2 \subset \operatorname{Im}\partial$,

4) the matrix of ∂ is diagonal.

The existence of such bases may be proved by elementary considerations of matrices of ∂_i and induction on i (cf. [3]). Let D be the subcomplex generated by $\gamma_2 \cup \partial^{-1}\gamma_2$. It is an acyclic complex and therefore we may suppose that $\alpha | D = 0$. We have the decomposition $C = D \oplus D'$, where D' is generated as a group by $\gamma_0 = \{b_1, \ldots, b_k\}$ and those elements of γ_1 which miss D, say $\{b_1^{(1)}, \ldots, b_k^{(1)}\}$. The matrix of $\partial_{D'} = \partial | ((b_1^{(1)}, \ldots, b_k^{(1)}))$ is diagonal, and diagonal entries $a_1, \ldots, a_k$ give the orders of elements of the base $\bar{\gamma} \subset H_* C$. Assume also that α' has been defined with the help of the base β of C', which intervened in the construction of the given homology model Γ. Let $g_0 = P_\beta \circ (f | \operatorname{coim}\partial)$, $g_1 = \operatorname{pr}_{\operatorname{coim}\partial'} \circ (f | \operatorname{coim}\partial)$.

Denote by $\{e_1, \ldots, e_k\}$, $\{e_1^{(1)}, \ldots, e_k^{(1)}\}$ the given bases of F_0, F_1 and by (α_{ij}), $(\bar{\alpha}_{ij})$ the matrices of $\alpha | \ker\partial_{D'}$, $\alpha | \operatorname{coker}\partial_{D'}$, respectively. The diagonal matrix of $\delta\colon F_1 \to F_0$ is equal (permuting bases if necessary) to the matrix of $\partial_{D'}$, since in both matrices the diagonal $\{a_1, \ldots, a_k\}$ is the sequence of orders of base elements in $H_* C$, which is unique. The fact that α is a chain map is expressed by the equality $\alpha_{ij}\alpha_j = \bar{\alpha}_{ij}\alpha_i$. We shall check that the homology class represented by $\psi\alpha(b_i^{(1)})$ falls into $\Theta_\beta([b_i])$. If $\xi_j \in D'$ represents $[e_j] \in H_* C$, then for some $\beta_j \in \ker\partial'$ we have

$$\begin{aligned} \pi\psi\alpha(b_i^{(1)}) &= \pi\Big(\sum_j \bar{\alpha}_{ji}\psi(e_j^{(1)})\Big) \ni \sum_j \bar{\alpha}_{ji}(P_\beta f \partial^{-1} a_j \xi_j + a_j \beta_j) \\ &= \sum_j P_\beta f \partial^{-1} \alpha_{ji} a_i \xi_j + \sum_j \alpha_{ji} a_i \beta_j \\ &= P_\beta f \partial^{-1} a_i\Big(\sum_j \alpha_{ji}\xi_j\Big) + a_i\Big(\sum_j \alpha_{ji}\beta_j\Big) \in X \in \Theta_\beta[e_i]\,, \end{aligned}$$

by the definition of Θ_β and the fact that $\sum_j \alpha_{ji}\xi_j$ represents $[b_i]$.

Since the set $\Theta_\beta(x)$ is a coset of $\operatorname{Im} f_* + \operatorname{ord}(x) H_* C'$, the elements $\psi\alpha(b_i^{(1)})$ and $\alpha' g_0(b_i^{(1)})$ differ by $r_i + a_i r_i' + \delta'\xi_i$, where $r_i \in f_0(\ker\delta^0)$, r_i' is an element of $\ker\delta'^0$ and $\xi_i \in F_1'$. Since in the definition of α we could take the values on $\operatorname{coim}\partial$ up to $\ker\delta^0$, we can redefine α by adding to $\alpha(b_i^{(1)})$ an element $s_i \in \ker\delta^0$ such that $f_0(s_i) = r_i$. Finally, for any $c \in \gamma_2$ there exists a $\bar{c} \in \gamma_1$ such that $\partial\bar{c} = c$. The homotopy $\Delta\colon \Gamma \circ \alpha \sim \alpha' \circ f$ is given by

$$\begin{aligned} \Delta(b_i) &= r_i' + \tilde{\Delta}(b_i) && \text{if} \quad b_i \in \gamma_0\,, \\ \Delta(b_i^{(1)}) &= \xi_i && \text{if} \quad b_i^{(1)} \in \gamma_1 \cap D'\,, \\ \Delta(c) &= \psi\alpha(\bar{c}) - \alpha' g_0(\bar{c}) + \tilde{\Delta}(c) && \text{if} \quad c \in \gamma_2,\ \bar{c} \in D \cap \partial^{-1}(c)\,, \\ \Delta(x) &= 0 && \text{if} \quad x \in \gamma_1 \cap D\,. \end{aligned}$$

It is easy to check that Δ has the required properties on $\gamma_0 \cup \gamma_2$. It remains to verify the equality $\Gamma\alpha - \alpha' f = \delta'\Delta + \Delta\partial$ on γ_1. We have

$$\begin{aligned} \delta'(\Gamma\alpha - \alpha' f) &= f_0\delta\alpha - \alpha'\partial f = f_0\alpha\partial - \alpha'(f | \ker\partial)\partial \\ &= (\Delta\partial + \delta'\Delta)\partial = \delta'\Delta\partial = \delta'(\Delta\partial + \delta'\Delta)\,. \end{aligned}$$

Decompose C'^0 as $F_0' \oplus F_1'$. The $F_0' = \ker\delta'$ components in $\Gamma\alpha - \alpha' f$ and $\Delta\partial + \delta'\Delta$ are equal by the definition of Δ. Since δ' is a monomorphism on F_1', the equality above gives $\Gamma\alpha - \alpha' f = \Delta\partial + \delta'\Delta$ on γ_1. The proof is complete.

An acyclic complex is always a finite sum of complexes of the form $\ldots \to 0 \to X \to \overset{\partial}{\to} X \to 0 \to \ldots$, where $\partial = \mathrm{id}_X$.

Two chain complexes will be called equivalent if there are acyclic complexes E, E' such that $\operatorname{rank} E_q = \operatorname{rank} E_q'$ for each q and an isomorphism $w\colon C \oplus E \to C' \oplus E'$. Any chain map homotopic to $C \hookrightarrow C \oplus E \overset{w}{\to} C' \oplus E' \to C'$ is called an *equivalence*.

Two chain maps $\varphi^i\colon C^i \to D^i$, $i = 1, 2$ are called *equivalent* if there are equivalences $f\colon C^1 \to C^2$ and $g\colon D^1 \to D^2$ such that the diagram

$$\begin{array}{ccc} C^1 & \xrightarrow{\varphi^1} & D^1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ C^2 & \xrightarrow[\varphi^2]{} & D^2 \end{array}$$

commutes up to homotopy. Note that it is an equivalence relation. It imitates passing from a minimal m-decomposition to another minimal one by stabilization, homotopy and destabilization. Compare the following example:

Let $D_q = D_{q-1} = Z$, $\partial_q = 5\mathrm{id}$, $D_i = 0$ for $i \neq q$, $q-1$, $C_{q-1} = Z$, $C_i = 0$ for $i \neq q-1$, $\varphi_{q-1}\colon C_{q-1} \to D_{q-1}$ is the multiplication by 3. By stabilization and homotopy we get

$$\begin{array}{ccc} & & D_q \oplus Z \\ & & \downarrow{\scriptstyle \partial_q \oplus \mathrm{id}_Z} \\ C_{q-1} & \xrightarrow{\binom{3}{2}} & D_{q-1} \oplus Z \end{array}$$

The base $\{\langle 3, 2\rangle, \langle 5, 3\rangle\}$ induces a splitting of $D_{q-1} \oplus Z$ such that the summand generated by $\langle 5, 3\rangle$ is contained in $\operatorname{Im}(\partial_q + \mathrm{id})$. After destabilization $\varphi_{q-1}\colon C_{q-1} \to$

$\to D'_{q-1} = (\langle\langle 3, 2\rangle\rangle)$ becomes an isomorphism. To get it we must pass through the stabilization.

LEMMA 9. *Given two minimal resolutions of a graded group, there is an equivalence of the resolutions inducing identity on the group. Any two homology models of a given chain map are equivalent.*

Proof. By the theory developed in [8] (especially Th. 2, p. 52) we know that any homotopy equivalence $f\colon C \to C'$ is stably homotopic to an isomorphism, i.e. there exist acyclic complexes E, E' such that the map $f \oplus 0\colon C \oplus E \to C' \oplus E'$ is homotopic to an isomorphism. Thus any homotopy equivalence $f\colon C \to C'$ is an equivalence provided $\operatorname{rank} C_q = \operatorname{rank} C'_q$ for any q. This implies that the first part of the lemma holds and also that homotopy equivalent chain maps between minimal resolutions are equivalent. But two homology models of the same map are homotopy equivalent by Lemma 8.

Now the homology model construction applied to an m-decomposition with $D^- = 0$ (or equivalently, to the chain map induced by $i\colon \partial M \to M$ between chain complexes given by handle decompositions of ∂M and M) will give us a model of an m-decomposition minimal on ∂M and M. We shall show that the set of maps equivalent to such a model is the very set of m-decompositions considered in Theorem 2.

Let $\varphi\colon R \to S$ be an m-decomposition with $D^- = 0$. The decomposition gives handle decompositions of ∂M and M such that R and S are their associated chain complexes and φ is induced by $i\colon \partial M \to M$. Consider an arbitrary homology model Γ of φ. Define $\mathfrak{A}(M)$ to be the set of all chain maps equivalent to Γ.

In the definition of homology models we do not use the map φ directly, but only the family $\{\Theta_\beta\}$ and $i_*\colon H_*\partial M \to H_*M$ induced by it, and these are the same for all m-decompositions of M. By Lemma 9 any two homology models of φ are equivalent, and thus $\mathfrak{A}(M)$ does depend only on M.

THEOREM 3. *We have* $\varkappa M = \max\{r(A)\colon A \in \mathfrak{A}(M)\}$ *for any simply connected manifold M of dimension greater than 6, such that ∂M is simply connected. Consequently,* $\nu M = \mu M + \mu(\partial M) - \varkappa M$ *depends only on* $i_*\colon H_*\partial M \to H_*M$ *and the family* $\{\Theta_\beta\}$ *of homology operations.*

Proof. By definition, $\varkappa M$ is the maximum of $r(\varphi)$ over chain maps $\varphi\colon R \to S$ induced by the inclusion $\partial M \to M$ and minimal handle decompositions of ∂M and M. We claim that $\mathfrak{A}(M)$ is the set of such maps. From Section 3 we see that with our assumptions on M any change of bases of R and S may be achieved by changes of decompositions and acyclic complexes can be cancelled. The addition of an acyclic complex to R or S is always possible and a chain homotopy of φ is realized by Lemma 4. Thus any map equivalent to φ is induced by a handle decomposition and therefore if one element of $\mathfrak{A}(M)$ can be realized as φ, then so can any other. By Lemma 8 each φ is homotopy equivalent to its homology model and if R, S are minimal, then the homotopy equivalence is an equivalence. Thus each φ induced by

minimal handle decompositions of ∂M and M belongs to $\mathfrak{A}(M)$. This establishes our claim and implies the theorem.

COROLLARY. *For simply connected manifolds with simply connected boundaries, of dimension greater than 6, the number νM is an invariant of relative homotopy type.*

In fact, for such manifolds the Morse numbers as well as $\varkappa M$ depend only on the relative homotopy type. In the non-simply connected case one can prove by methods of [9], [4] that νM is an invariant of relative simple homotopy type.

We shall now show that Theorem 3 yields some explicit computations. First we shall prove a lemma.

LEMMA 10. *Let $f\colon G \to G'$ be a homomorphism of graded groups and let* $F_1 \xrightarrow{\delta} F_0 \xrightarrow{\pi} G$, $\tilde F_1 \xrightarrow{\tilde\delta} \tilde F_0 \xrightarrow{\tilde\pi} G'$ *be minimal resolutions. Suppose that β is a minimal set of generators of G' such that* $\{b_1, \dots, b_r\} = \beta \cap \operatorname{Im} f$. *Then for any chain map* $\tilde\Gamma\colon (F_0 \oplus F_1, \delta) \to (\tilde F_0 \oplus \tilde F_1, \tilde\delta)$ *covering f there exists a chain map* $\Gamma\colon (F_0 + F_1, \delta) \to (F'_0 + F'_1, \delta')$ *equivalent to $\tilde\Gamma$ and such that* $r(\Gamma | F_0) \geqslant r$.

Proof. Let $f[x_i] = b_i$ for some $x_i \in F_0$, $i = 1, \dots, r$. Consider the minimal prime summand of F_0 containing the set $\{x_1, \dots, x_r\}$ and choose a base $\{y_1, \dots, y_s\}$ of the summand. We have $s \leqslant r$ and $\{b_1, \dots, b_r\} \subset f\pi(\{y_1, \dots, y_s\})$. We now consider the new set of generators of G': $\beta' = (\beta - \{b_1, \dots b_r\}) \cup f\pi\{y_1, \dots, y_s\}$, which is obviously minimal (and therefore $s = r$). The minimal resolution $(F'_0 \oplus F'_1, \delta')$ of G' given by β' is equivalent to $(\tilde F_0 \oplus \tilde F_1, \tilde\delta)$. Composing $\tilde\Gamma$ with this equivalence, we get a map equivalent to $\tilde\Gamma$ and covering f. Since $r = s$ elements of the base β' belong to the image of the induced homomorphism $G \to G'$, for the resulting homomorphism $f_0\colon F_0 \to F'_0$ we have r base elements of F'_0 in $f_0\{y_1, \dots, y_r\} + \operatorname{im}\delta'$. Changing f_0 on $\{y_1, \dots, y_r\}$ by the appropriate homotopy, we obtain $f'_0\colon F_0 \to F'_0$ of rank r.

THEOREM 4. *Let M be a simply connected manifold with a simply connected boundary,* $\dim M > 6$. *Suppose that α, β are minimal sets (i.e., sets of minimal cardinality) of generators of H_*M and $H^*\partial M$, respectively. Then*

(i) $\varkappa M \geqslant \operatorname{card}(\alpha \cap \operatorname{Im} i_*) + \operatorname{card}(\beta \cap \operatorname{Im}(i^* | \operatorname{tor} H^*M))$,

(ii) *if $H_*\partial M$ has no torsion, then* $\varkappa M = \max\limits_{\alpha} \operatorname{card}(\alpha \cap \operatorname{Im} i_*)$,

(iii) *if H^*M has no torsion, then* $\varkappa M = \max\limits_{\beta} \operatorname{card}(\beta \cap \operatorname{Im} i^*)$.

Proof. (i) We must find in $\mathfrak{A}(M)$ a map of rank $\operatorname{card}(\alpha \cap \operatorname{Im} i_*) + \operatorname{card}(\beta \cap \operatorname{Im}(i^* | \operatorname{tor} H^*M))$. Since at this point we regard i^* restricted to torsions, we assume that β is a minimal set of generators of $\operatorname{tor} H^*\partial M$ only. We start with the minimal resolution $E_1 \xrightarrow{\partial} E_0 \to \operatorname{tor} H^*\partial M$ induced by β and define $F_1 = E_0^*$, $F_0 = E_1^* \oplus \operatorname{Hom}(H^*\partial M, Z)$, $\delta = \partial^* \oplus 0$. The Universal Coefficient Formula gives a map $F_0 \xrightarrow{\pi} H_*\partial M$ and $F_1 \xrightarrow{\delta} F_0 \xrightarrow{\pi} H_*\partial M$ becomes a minimal resolution of $H_*\partial M$. Let $\Gamma\colon (F_0 \oplus F_1, \delta) \to (F'_0 \oplus F'_1, \delta')$ be the map of minimal resolutions provided by Lemma 10 for $r = \operatorname{card}(\alpha \cap \operatorname{Im} i_*)$ and a homology model $\tilde{\Gamma}$: $(F_0 \oplus F_1, \delta) \to (\tilde{F}_0 + \tilde{F}_1, \tilde{\delta})$ of an m-decomposition of M. We have $\Gamma \in \mathfrak{A}(M)$ and $r(\Gamma | F_0) \geqslant \operatorname{card}(\alpha \cap \operatorname{Im} i_*)$. There is a free group E of rank $r(\Gamma | F_0)$ such that $F_0 = \bar{F}_0 \oplus E$, $F_0' = \bar{F}_0' \oplus E$ and $\Gamma | F_0 = f_0 + \mathrm{id}_E$. In the dual complexes this induces splittings $F_0^* = \bar{F} \oplus (\ker \delta^* + E^*)$ and $F_0'^* = \bar{F}' \oplus (\ker \delta'^* + E^*)$, so that

$$r(\Gamma^* | (\ker \delta^* + E^*)) = r(\Gamma | F_0)\,.$$

Consider the diagram

$$\begin{array}{ccc} \bar{F}' & \xrightarrow{\varphi_1} & \bar{F} \\ {\scriptstyle \delta'^*}\downarrow & \searrow^{\psi} & \downarrow{\scriptstyle \delta^*} \\ F_1'^* & \xrightarrow{\varphi_0} & E_0 \\ \downarrow & & \downarrow \\ G' & \longrightarrow & G \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

where φ_0, φ_1, ψ are induced by Γ. Obviously the columns are minimal resolutions and $\varphi_0 + \varphi_1 + \psi$ forms a chain map. The maps $F_0'^* \to \operatorname{tor} H^* M$, $E_0 \to \operatorname{tor} H^* \partial M$ induce epimorphisms $G' \to \operatorname{tor} H^* M$, $G \to \operatorname{tor} H^* \partial M$ and the commutative diagram

$$\begin{array}{ccc} G' & \xrightarrow{f} & G \\ \downarrow & & \downarrow \\ H^* M & \xrightarrow{i} & H^* \partial M \end{array}$$

The resolution $(F_0 \oplus F_1, \delta)$ has been chosen in such a way that β gives a base of $F_1^* = E_0$. It induces in turn a minimal set of generators $\tilde{\beta}$ of G such that $\operatorname{card}(\tilde{\beta} \cap \operatorname{Im} f) \geqslant \operatorname{card}(\beta \cap \operatorname{Im}(i^* | \operatorname{tor} H^* M))$. Now, applying Lemma 10 to f: $G' \to G$ and $\tilde{\beta}$, we get a minimal resolution $K_1 \xrightarrow{\eta} K_0 \to G$ and an equivalence ε: $(\bar{F} \oplus F_1, \delta^*) \to (K_1 \oplus K_0, \eta)$ as well as a chain map Ξ equivalent to $\varphi_0 + \varphi_1 + \psi$ such that

$$r(\Xi | F_1'^*) \geqslant \operatorname{card}(\beta \cap \operatorname{Im}(i^* | \operatorname{tor} H^* M))\,.$$

We construct a minimal resolvent of $H^* \partial M$ as follows: Add $\ker \delta^* + E^*$ to K_1, extend η by zero on $\ker \delta^*$ and put $\eta(x) = \Xi \delta'^*(x)$ for $x \in E^*$. Then define ψ': $\ker \delta'^* + E^* \to K_0$ by $\psi'(x) \in \pi^{-1}[p_{F_1^*} \Gamma^*(x)]$ for all elements of a base of $\ker \delta'^* + E^*$. Straightforward verification shows that the chain map

$$\Xi + p \circ (\Gamma | \ker \delta'^* + E^*) + \psi'\,,$$

where p is the projection $F_0^* \to \ker \delta^* + E^*$, is equivalent to Γ^*. Since its rank is at least $\operatorname{card}(\alpha \cap \operatorname{Im} i_*) + \operatorname{card}(\beta \cap \operatorname{Im}(i^* | \operatorname{tor} H^* M))$, inequality (i) follows from Theorem 3.

(ii) Provided $H^* \partial M$ is torsion free, any minimal resolution of it is isomorphic to $H_* \partial M$ with trivial differentials; thus any chain complex induced by a minimal handle decomposition of ∂M is of this form. Theorem 2 says that $\varkappa M$ is the maximal rank of φ: $R \to S$, where R and S are minimal. For φ realizing the maximum there is a base of S such that $\varkappa M$ base elements of S are contained in $\operatorname{Im} \varphi \subset \ker \partial$. Since the base of S determines a minimal set of generators of $H_* M$, we have the inequality $\varkappa M \leqslant \max \operatorname{card}(\alpha \cap \operatorname{Im} i_*)$. The converse inequality follows from (i).

(iii) This follows by using dual complexes and the same arguments as in (ii).

6. Other minimalization problems. Any manifold with a non-empty boundary allows a function without critical points [5]. One can ask what is the minimal number of critical points of a Morse function on ∂M extendable to an m-function of this kind.

THEOREM 5. *Let M be a simply connected manifold with a simply connected boundary, $\dim M > 6$. The minimal number of critical points of a Morse function on ∂M which extends to M without interior critical points is equal to $\mu(\partial M) + 2(\mu M - \varkappa M)$.*

Outline of the proof. The following geometrical argument shows that the number equals at most $\mu(\partial M) + 2(\mu M - \varkappa M)$. Consider a minimal m-function on M (by Th. 2, it has $\mu M - \varkappa M$ interior critical points). We can join all the interior critical points of f by an arc L. Moreover, assuming that any critical level contains only one critical point, we can find L which is a part of the boundary of a 2-disc D embedded in M such that critical points of $f | D$ are interior critical points of f and the disc intersects the boundary of M along an arc in ∂D disjoint with L. Deleting from M a carefully chosen collar of D, we get a submanifold M' diffeomorphic to M and $f | M'$ with no interior critical points. The collar can be chosen so that any interior critical point of f gives two boundary critical points of $f | M'$; hence the resulting m-function has $\mu(\partial M) + 2(\mu M - \varkappa M)$ critical points.

Let f be an m-function without interior critical points. As we are interested in the lower bound for $d^- f + d^+ f$, we can assume that $d^- f = \mu M$, because $d^- f$ can be decreased, by Lemma 7, if $d^- f > \mu M$. Changing all indices $(q, -1)$ to $(q, 1)$, we get an m-function with μM interior critical points and $d^- f + d^+ f$ boundary critical points, all the indices being of sign $+$. The complex induced by the restricted handle decomposition of the boundary may be given the form $R^0 \oplus R'$, where R^0 is minimal and R' is acyclic. By Theorem 2 at most $\varkappa M$ base elements lie in $\varphi(R^0)$. Since φ: $R^0 \oplus R' \to S$ is onto and $\varphi(\ker \partial^{R'}) \subset \operatorname{Im} \partial^S$, there is an epimorphism $\operatorname{coim} \partial^{R'} \to \operatorname{coim} \partial^S \oplus H_* S$. We have $\operatorname{rank} R' = 2 \operatorname{rank} \operatorname{coim} \partial^{R'}$ and $R(\operatorname{coim} \partial^S \oplus H_* S) = \operatorname{rank} S$ (be the minimality of S); hence $\operatorname{rank} R' \geqslant 2(\mu M - \varkappa M)$. This inequality gives in turn $d^- f + d^+ f = \operatorname{rank}(R^0 \oplus R') \geqslant \mu(\partial M) + 2(\mu M - \varkappa M)$.

Remark. A Morse function f on ∂M extends to an m-function without interior critical points if and only if the number of critical points of f is not less than $\mu(\partial M) + 2(\mu M - \varkappa M)$.

Another minimalization problem for simply connected manifolds of dimension greater than 6 which can be attacked by means of the theory developed in this paper is the question posed in [6]: given a Morse function f of ∂M, what is the minimal number of critical points of an m-function extending f? It should be possible to obtain similar results as in our case if one consider the chain complex induced by f in place of homology model R^0 in the definition of $\mathfrak{A}(M)$.

For non-simply connected manifolds the minimalization problems are much more difficult and even for Morse functions no satisfactory calculations are known. However, Theorems 2 and 3 are uneffective enough to have straightforward generalizations to that case. Our arguments ought to work if the homology groups of M and ∂M are replaced by the homology groups of universal covering considered as modules over the integer group rings $Z\pi_1 M$ and $Z\pi_1 \partial M$, at least when $i_*: \pi_1 \partial M \to \pi_1 M$ is an isomorphism.

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Special bases for compact metrizable spaces

by

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Abstract. Each compact metrizable space has a base $\mathcal{B}$ such that

(1) for every finite $\mathcal{A} \subseteq \mathcal{B}$, if any two members of $\{\bar{A}: A \in \mathcal{A}\}$ intersect then $\bigcap \mathcal{A} \neq \emptyset$; and

(2) if $\mathcal{R}$ is the ring generated by $\mathcal{B}$, then $\mathcal{R}$ consists of regularly open sets and $(\bigcap \mathcal{F})^- = \bigcap\{\bar{A}: A \in \mathcal{F}\}$ for every finite $\mathcal{F} \subseteq \mathcal{R}$.

This implies that every compact metrizable space is regularly supercompact. The construction of $\mathcal{B}$ is complicated but elementary.

0. Conventions and definitions. As usual, if X is a space, $^-$, ^{0} and ^{c} denote the closure operator, the interior operator and the complementation operator in X; if $\mathcal{F}$ is a family of subsets of X we write e.g. $\bar{\mathcal{F}}$ for $\{\bar{F}: F \in \mathcal{F}\}$.

If X is a space and $\mathcal{F}$ is a family of subsets, then $\mathcal{F}$ is called a *closed subbase* if it is a subbase for the closed sets, i.e. $\mathcal{F}^c$ is a subbase for the open sets, a *ring* if $F \cap G \in \mathcal{F}$ and $F \cup G \in \mathcal{F}$ for all $F, G \in \mathcal{F}$, *linked* if $F \cap G \neq \emptyset$ for any $F, G \in \mathcal{F}$ (not necessarily distinct), *binary* if every linked subfamily has nonempty intersection.

A space is called *supercompact* if it has a binary closed subbase, *regularly supercompact* if it has a binary closed subbase $\mathcal{S}$ such that the ring generated by $\mathcal{S}$ consists of regularly closed sets, *regularly Wallman* if it has a closed subbase which is a ring and which consists of regularly closed sets.

1. Introduction. The notion of supercompactness was introduced by de Groot in [dG]. It is a trivial consequence of Alexander's Subbase Lemma, [A], that every supercompact space is compact. An easy example of a compact T_1-space that is not supercompact was given by Verbeek, [V, II. 2.2(8)]. The question of whether all compact Hausdorff spaces are supercompact was settled in the negative by Bell, [B], this is a nontrivial result in spite of the fact that the answer was to be expected, [dG]. Subsequently van Douwen and van Mill showed that every infinite supercompact Hausdorff space has many nontrivial convergent sequences, [vDvM]; this gives a rich supply of compact Hausdorff spaces that are not supercompact.

This paper deals with de Groot's conjecture that all compact metrizable spaces are supercompact, [dG]. The first result is due to de Groot who proved that compact polyhedra are supercompact. An erroneous proof of de Groot's conjecture was