(a) \( r(X_a) = r(Y_a) = \alpha; \)
(b) \( X_a \) is scattered, and \( Y_a \) is not;
(c) \( |Y_a| = |\alpha + \omega|; \) and
(d) \( |X_a| = |\alpha + \omega| \) unless \( \alpha = 0 \), in which case \( |X_a| = 1 \).

Proof. Let \( Z \) be the integers (with the usual order). Given \( X_a \), let \( X_{a+1} = Z \times X_a \), ordered lexicographically, and let \( Y_a = Q \times X_a \), also ordered lexicographically.

If \( \alpha \) is a limit ordinal, let \( X_\alpha = \{ f: \alpha + 1 \rightarrow Z: f \) is continuous (with respect to order topologies on \( \alpha + 1 \) and \( Z \) and \( f(\alpha) = 0 \); if \( f, g \in X_\alpha \) with \( f \neq g \), let \( \beta = \max \{ \xi < \alpha: f(\xi) \neq g(\xi) \} \), and write \( f < g \) if \( f(\beta) < g(\beta) \). It is easily checked that \( X_\alpha \) has the desired properties.

On contractible fans

by

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Abstract. The purpose of this paper is to give a characterization of weakly confluent-contractible fans. After giving several definitions, it is shown that such a fan must be pair-wise smooth, must contain no zero-end, and lastly must contain no P-point. It is then shown that a fan which satisfies these three properties must be monotone-contractible. This implies the fan is weakly confluent-contractible in as much as monotone functions are always weakly confluent. Hence these properties also yield a characterization of monotone-contractible fans.

Introduction. Several mathematicians (see [1], [4], [5], [7]) in recent years have studied the contractibility of dendroids. We will use the term dendroid to designate a compact metric continuum which is arc-wise connected and is hereditarily unicoherent. A ramification point of a dendroid is a point which is the intersection of three or more arcs. K. Borsuk [2] has described simple types of dendroids, containing only one ramification point, which are called fans. The ramification point is called the top of the fan.

A topological space \( X \) is contractible if there exists a continuous map \( F: [0, 1] \times X \rightarrow X \) such that \( F(0, p) = p \), for each point \( p \) of \( X \); and there is a point \( g \) in \( X \) such that \( F(1, p) = g \) for each point \( p \) of \( X \). The map \( F \) is called a contraction of \( X \).

Figure 1 in the Appendix is a contractible dendroid \( A \) with the surprising property that for each choice of a contraction \( F \), there must be a time \( t \) in \([0, 1]\) for which \( F(t \times A) \) is a noncontractible sub-dendroid of \( A \). In order to restrict the spaces it was decided to place a stronger requirement on the maps involved. The property chosen was first defined by A. Lelek [9], that of weak-confluence of the maps. It was found that for dendroids, even with weakly-confluent maps, examples of the type found in Figure 1 are still admissible. The investigation was further restricted to the case of fans. It will be shown that a fan is weakly-confluent-contractible if and only if it is confluent-contractible, if and only if it is monotone-contractible.

A continuous map is said to be monotone if the pre-image of each continuum lying in its image is itself a continuum. A contraction \( F \) on a space \( X \) is a monotone contraction provided that for each time \( t \) in \([0, 1]\), the map \( F \) restricted to \( \{t\} \times X \) is monotone.
A continuous map is confluent if, for each continuum $K$ lying in its image, it is true that every component of the pre-image of $K$ is mapped onto $K$. A contraction $F$ on a space $X$ is a confluent contraction if $F$ restricted to $(r) \times X$ is confluent for each $t$ in $[0, 1]$.

A continuous map is said to be weakly-confluent if, for each continuum $K$ lying in its image, it is true that at least one component of the pre-image of $K$ is mapped onto $K$. Given a space $X$ and a contraction $F$ of $X$, we will say that $F$ is a weekly-confluent contraction provided that for each time $t$ in $[0, 1]$, the map $F$ restricted to $(r) \times X$ is weakly-confluent.

The main result of this paper is a characterization of those fans which admit a weakly-confluent contraction. The following definitions will be used throughout the paper.

**Definition.** Let $X$ be a dendroid and let $r$ be a point in $X$. Suppose there are two sequences $(r_1, n), (r_2, n)$ $(n = 1, 2, 3, \ldots)$ of points of $X$, each converging to $r$. We say that the former sequence dominates the latter sequence provided that whenever there exists a point $s$ in $X$ and a sequence $(s_1, n)$ converging to $s$, with the property that the arcs $(r_1, s_1, (s_1, n))$ converge to the arc $(r_2, s)$, then it follows that there also exists a sequence $(n(s_2, n))$ converging to $s$ such that the arcs $(r_2, n(s_2, n))$ converge to $(r, s)$ set-wise.

**Definition.** We say that a dendroid is pairwise-smooth provided that whenever a pair of sequences converge to a common point, then one of the pair dominates the other. Figures 2 and 3 in the Appendix illustrate fans which are not pairwise-smooth.

**Definition.** We say that a dendroid $X$ contains a zig-zag if there are distinct points $a, b$ belonging to $X$ and a sequence of arcs $(a_1, b_1, c_1, d_1), n = 1, 2, \ldots$ (with endpoints $a_1, d_1$ and interior points $b_1, c_1$ in the order indicated) converging to the arc $(a, b)$ in such a way that $(a_1, n = 1, 2, \ldots)$ converge to $a$, while $(b_1, n = 1, 2, \ldots)$ and $(c_1, n = 1, 2, \ldots)$ each converge to $b$. Figures 4 and 5 in the Appendix show some examples of a zig-zag.

The $P$-point defined next is a slight modification of R. Bennett's O-point.

**Definition.** Let $X$ be a dendroid and let $b$ be a point of $X$. We call $b$ a $P$-point if there is a sequence of points in $X$ $(b_1, n = 1, 2, \ldots)$ converging to $b$ such that $L_a[b, b_n]$ is not equal to $b$, and such that if $(b_n, x_n)$ is the arc irreducible between $a$ and $b_n$, then it follows that $(x_n, n = 1, 2, \ldots)$ converge to $b$. A simple example of a $P$-point is given in Figure 6 of the Appendix.

We will show that a fan is weakly-confluent contractible if and only if it is pairwise smooth, contains no zig-zag, and contains no $P$-point.

The following notation will be used:

- $R_C = \text{Closure}$
- $H = \text{Open ball of radius } \ldots \text{ centered at } \ldots$
- $[a, b] = \text{Arc with endpoints } a, b, \text{ the order does not matter unless otherwise indicated,}$

Given a fan $X$ with point $c$, the weak cut point order (with respect to $c$) is defined on $X$ by $p < c$ if $p$ belongs to $(c, q]$ and $p < q$ if $p < q$ but $p$ is distinct from $q$.

Given a fan $X$ with a partial order $\leq$ defined on $X$ a metric $q$ on $X$ is radially convex provided that $p < q < r$ implies $q(p, q) < q(p, r)$.

A partial order $\leq$ on $X$ is closed if the set $\{a, b | a \leq b\}$ is closed in $X \times X$.

**Chapter 1.** This section contains some basic results which will be needed to obtain the main theorem.

**Lemma 1.1.** Let $X$ be a dendroid. Let $\{x_n\}_{n=1}^\infty$, $(r_n)_{n=1}^\infty$ be sequences of points of $X$ converging to $x_0$, $r_0$, respectively. Let $b$ be a point of $X$ and let $\leq$ denote the weak-cut-point order, with respect to $b$ defined on $L_a[x_n, r_n]$. There exists a subsequence $\{x_{n_j}(j)\}$ and sequence $\{b_j(j)\}$ \(\leq b\), converging to $b$, with $b_j(j)$ contained in $[x_{n_j}(j), r_{n_j}(j)]$ such that $L_a[x_{n_j}(j), b_j(j)] \leq b$ (if $x_{n_j} \leq b$).

**Proof.** We may as well assume that $x_0$ is distinct from $b$, that for each $n = 1, 2, \ldots$, $x_n$ is not contained in $C(B(1, b))$, and that $b$ belongs to $L_a[x_n, r_n]$. For each $j = 1, 2, \ldots$, there exists a subarc $[x_{n_j}, b_{n_j}]$ of $[x_n, r_n]$ which is irreducible between $x_n$ and $C(B(1, b))$, for each $n$ greater than say $N_j$. Also, since $b$ does not belong to $L_a[x_{n_j}, b_{n_j}]$, it follows that $L_a[x_{n_j}, b_{n_j}] \leq b$ (respectively, $\geq b$). Hence for each $n$ greater than say $M_j$ it must be true that $[x_{n_j}, b_{n_j}]$ is in the $1/j$ neighborhood of $C(B(1, b))$, respectively, of $B(0)$. Choose $n(1) \leq \ldots \leq n(j) \leq \ldots$, with $n(j) > M_j$ such that $[x_{n_j}(j), b_{n_j}(j)]$ is in the $1/j$ neighborhood of $C(B(1, b))$ respectively, of $B(0)$. Note that $b_{n_j}(j)$ is contained in $C(B(1, b))$, for each $j$. Thus $b_{n_j}(j) \leq b$ converges to $b$ and it is evident that $L_a[x_{n_j}(j), b_{n_j}(j)] \leq b$ (respectively, $\geq b$).

**Lemma 1.2.** Let $X$ be a dendroid and $(x_n)_{n=1}^\infty$, $(r_n)_{n=1}^\infty$ be sequences of points of $X$, converging to $x_0$, $r_0$, respectively such that the arcs $[x_n, r_n]$ are pairwise disjoint or else $x_n = x_0$ for all $n$ while $[x_n, r_n]$ are pairwise disjoint, and such that $L_a[x_n, r_n] = [x_n, r_n]$. Define $y \leq x$ if $y$ is contained in $[x_n, x]$ for some $n = 0, 1, 2, \ldots$. If $X$ contains no zig-zags, then $\bigcup_{n=0}^\infty [x_n, r_n]$ admits a radially convex metric, with respect to $\leq$.

**Proof.** It is known (see [3] and [10]) that the result follows if it can be shown that
whenever a sequence of points \( \{y_n\}_{n=1}^\infty \) contained in \( \bigcup \{[x_n, r_n]\} \) converges to a point \( y_0 \), then it follows that \( \text{Ls}(L(y_n)) \) is included in \( L(y_0) \). Suppose the lemma is false. There then exist points \( y_n \) in \([x_n, r_n] \), say, \( \{y_n\}_{n=1}^\infty \) converging to \( y_0 \) in \([x_0, r_0] \) and there exist points \( p \) in \([x_0, y_0] \) with \( \{p_n\}_{n=1}^\infty \) converging to a point \( p \) in \([y_0, r_0] \). We may as well suppose that \( p = \text{max} \text{Ls}[x_0, r_0] \). Let \( q \) be \( \min \text{Ls}[x_0, r_0] \) and note that \( q \leq y_0 < p \).

By taking a subsequence and relabeling indices we can assume that there exist points \( q_n \) in \([p_n, r_n] \), \( \{q_n\}_{n=1}^\infty \) converging to \( q \). If \( \beta \) denotes \( \max \text{Ls}[x_0, q_0] \) we find similarly, points \( p_n \) in \([x_n, q_0] \), \( \{p_n\}_{n=1}^\infty \) converging to \( p \). Note that \( q \leq y_0 < p \leq \beta \).

Now \( \beta \) is contained in \([q, r_0] \) which is included in \( \text{Ls}[x_0, r_0] \) and, applying Lemma 1.1 we find (without loss of generality) points \( p_n \) in \([x_n, r_0] \), with \( \{p_n\}_{n=1}^\infty \) converging to \( p \) such that \( \text{Ls}[p_n, q_0] \leq \beta \). But \( \text{Ls}[p_n, q_0] \geq q \) of minimality of \( q \), hence \( \text{Ls}[p_n, q_0] = [p, q] \). Also, since \( q \in \text{Ls}[x_n, p_n] \) it follows from Lemma 1.1 and the maximality of \( \beta \) that there exist points \( q_n \) in \([x_n, p_n] \), \( \{q_n\}_{n=1}^\infty \) converging to \( q \), such that \( \text{Ls}[q_n, p_n] = [p, q] \). The set \( \text{Ls}[q_n, p_n] = [p, q] \) because \( p \) is maximal and for almost all \( n \), \( p_n \) is greater than or equal to \( x_n \), while \( q \leq \text{min} \text{Ls}[p_n, r_n] \). By relabeling indices on the appropriate subsequences, we obtain the fact that \( \{[x_n, q_n] \} \) converges to \([p, q] \) in the manner required to form a zig-zag. This contradicts the hypothesis and the lemma must be true.

**Lemma 1.3.** If \( X \) is a contractible dendroid and \( \{x_n\}_{n=1}^\infty \), \( \{b_n\}_{n=1}^\infty \) are sequences of points of \( X \) which converge to \( a_0, b_0 \) respectively, then \( \text{Ls}[x_n, b_n] \) is hereditarily locally connected. (This is an unpublished result of Charatonik.)

**Proof.** Suppose \( F \) is a contraction of \( X \) with \( F([1] \times X) = x \). For each \( n \) let \( [x_n, b_n] \) be irreducible between \( x \) and \([a_n, b_n] \). Now

\[
\text{Ls}[x_n, b_n] = \text{Ls}([x_n, a_n] \cup [x_n, b_n]) \subseteq \text{Ls}([x_n, a_n] \cup \text{Ls}[x_n, b_n]) \\
= \text{Ls}([x_n, a_n] \cup \text{Ls}[x_n, b_n]) \subseteq \text{Ls}[x_n, a_n] \cup \text{Ls}[x_n, b_n] \\
= \text{Ls}([0, 1] \times \{a_n\}) \cup \text{Ls}([0, 1] \times \{b_n\}) \\
= \text{Ls}([0, 1] \times \{a_n\}) \cup \text{Ls}([0, 1] \times \{b_n\}) \\
= \text{Ls}([0, 1] \times \{a_n\}) \cup \text{Ls}([0, 1] \times \{b_n\}) \\
= \text{Ls}([0, 1] \times \{a_n\}) \cup \text{Ls}([0, 1] \times \{b_n\})
\]

Now \( \text{Ls}([0, 1] \times \{a_n\}) \cup \text{Ls}([0, 1] \times \{b_n\}) \) is locally connected and hereditarily unicoherent and is thus hereditarily locally connected. The set \( \text{Ls}[x_n, b_n] \) therefore inherits the latter property.

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**Chapter 2.** We are now prepared to show the necessity of each of the three conditions — pairwise smooth, no zig-zag, no \( P \)-point — in order that a fan be weakly-confluent contractible.

**Theorem 2.1.** A contractible dendroid does not contain a zig-zag.

**Proof.** The zig-zag is a special case of a continuum of type \( \sigma \) defined by Lex G. Oversteegen. See Theorem 2.1 of his paper *Non-contractibility of Continua* in Bull. Acad. Polon. Sci. (to appear 1978).

**Lemma 2.2.** If a contractible fan contains a \( P \)-point, then that point must be the top of the fan.

**Proof.** Let \( X \) be a contractible fan with endpoints \( \{a_n\}_{n=1}^\infty \), top \( c \), and let \( x \) be a \( P \)-point of \( X \) distinct from \( c \). We wish to obtain a contradiction. There is a sequence of points \( \{x_n\}_{n=1}^\infty \) converging to \( x \) for which the points \( x_n \) lie on distinct arcs \( [c, x_n] \), such that if \( x_n \) is irreducible between \( x_n \) and \( \text{Ls}[x, x_n] \), then the sequence of points \( \{x_n\}_{n=1}^\infty \) converges to \( x \). It follows that both \( x \) and \( c \) belong to \( \text{Ls}[x, x_n] \) and we know, as a result of Lemma 1.3, that \( \text{Ls}[x, x_n] \) is hereditarily locally connected. It connected. Therefore, it is also true that \( \text{Cl}(\bigcup \{c, x_n\}) \) is locally connected. Since \( x \) belongs to \( \bigcup \{c, x_n\} \), there is a relatively open neighborhood \( U \) of \( x \) lying in \( \bigcup \{c, x_n\} \).

\( \text{Cl}(\bigcup \{c, x_n\}) \) such that \( U \) is connected and does not contain \( c \). However, \( U \) must contain \( x_n \), for almost all \( n \), and hence must contain the arcs \([x_n, x]\) for almost all \( n \). This implies that \( y_n \) lies on \([c, x]\) for almost all \( n \). The arc \([c, x]\) would then be a proper subcontinuum of \([x_n, x]\), joining \( x_n \) to \( \text{Ls}[x, x_n] \) for almost all \( n \), contrary to the choice of the points \( y_n \). This contradiction establishes the lemma.

**Theorem 2.3.** If a fan is weakly-confluent contractible, then it does not contain a \( P \)-point.

**Proof.** Let \( X \) be a fan with top \( c \), endpoints \( \{a_n\}_{n=1}^\infty \), and let \( F \) be a weakly-confluent contraction of \( X \). If \( X \) contains a \( P \)-point, then \( c \) must be a \( P \)-point (Lemma 2.2). There is then a sequence of points \( \{x_n\}_{n=1}^\infty \) converging to \( c \), such that \( x_n \) is contained in \([c, x_m] \) for distinct endpoints \( \{c, x_n\}_{n=1}^\infty \), and possessing the property that \( \text{Ls}(c) \) contains at least one point, say \( y \), different from \( c \).

With the appropriate choice of subsequence \( \{y_n\}_{n=1}^\infty \), one can find a sequence of points \( \{y_n\}_{n=1}^\infty \) converging to \( y \) such that \( y_n \) belongs to \([c, x_n]\) (\( n = 1, 2, \ldots \)). For each \( n \) let \( t_n \) be the greatest value of \( t \) in \([0, 1]\) for which \( F(c, x_n) \) belongs to \([y_n, (c, x_n)]\). We may assume that the sequence \( \{t_n\}_{n=1}^\infty \) converges, if not one uses a subsequence which does converge. Let \( t \) be the limit of the sequence \( \{t_n\}_{n=1}^\infty \). Now

\[ F(t, c) = F(\text{Ls}(t, c)) = \text{Ls}(F(t, c)) = \text{Ls}(y) = y. \]
Let \( U \) be an open neighborhood of \( y \) which is small enough that it does not contain \( c \) and let
\[
K = (Cl \{ \bigcup_{j=1}^{\infty} [c, F(t_j, c_j(j))] \}) - U.
\]

Let \( M \) be the component of \( K \) which contains \( c \) and note that \( M \) is a continuum which is not locally connected. (If \( M \) were locally connected, then it would be impossible for \( c \) to be a P-point.) Since \( c \) does not belong to \( F_0^{-1}(M) \), each component of \( F_0^{-1}(M) \) is an arc or a point. However, these are locally connected and thus cannot be mapped onto \( M \). This is contradictory to the assumption that \( F \) is weakly-confluent. Therefore \( X \) must contain no P-point.

**Theorem 2.4.** A weakly-confluent contractible fan is pairwise smooth.

**Proof.** Let \( X \) be a weakly-confluent contractible fan with top \( c \), endpoints \( \{ e_a \}_{a=1}^{\infty} \), and let \( F \) be a contraction of \( X \). Suppose that \( X \) is not pairwise smooth. Then there is a point \( r \) belonging to \( X \) and sequences \( \{ r(t, n) \}, \{ r(t, n) \} \) for \( n = 1, 2, \ldots \), each converging to \( r \) a point \( s \) and sequence \( \{ s(t, n) \} \) converging to \( s \) such that \( \lim [r(t, n), s(t, n)] = [s, z] \) and a point \( q, \) sequence \( \{ q(t, n) \} \) converging to \( q \) such that \( \lim [r(t, n), q(t, n)] = [s, z] \).

Because \( c \) is not a P-point (Theorem 2.3), we may choose the points \( s, r, q, t \) to lie in (that order) on an arc \( [c, e_\beta] \in \{ c \} \) for some \( \beta \in A \). Let \( e_\beta(1, n), e_\beta(2, n) \) be the endpoints of the arcs on which lie (respectively) the points \( r(t, n), r(t, n) \). We may assume that the points \( s(t, n), q(t, n) \) also belong to the arcs \( [c, e_\beta(1, n)], [c, e_\beta(2, n)] \) respectively.

It is important to note that the points \( s(t, n), q(t, n) \) may be chosen to lie on the arcs \( [c, r(t, n)], [c, r(t, n)] \) (respectively). For example, if the points \( s(t, n) \) belong to \( [c, e_\beta(1, n)] \), and if \( [c, r(t, n)] = [s, z] \), there are then points \( s(t, n) \) in \( [c, r(t, n)] \) converging to \( s \) such that \( \lim [r(t, n), s(t, n)] = [s, z] \).

Actually there is a subsequence, but we relabel the indices. We have \( \lim [r(t, n), s(t, n)] \leq z \) by choice of \( z \) and \( \geq r \) by using Lemma 1.1. It is evident that \( z \leq q \), or else \( (r(t, n)) \) dominates \( (r(t, n)) \) since we could put a radially convex metric \( q \) on the set
\[
\{ [r, z] \cup \bigcup_{n=1}^{\infty} [r(t, n), s(t, n)] \}.
\]

(by Lemma 1.2) and \( q \leq z \) would enable us to choose a sequence \( \{ q(t, n) \} \) converging to \( q \) such that \( \lim [r(t, n), q(t, n)] = [r, q] \), the points \( q(t, n) \) lying on \( [r(t, n), s(t, n)] \) at the obvious correct distance \( q(r, q) \) from \( r(t, n) \).

But with \( z \leq q \), we can put a radially convex metric \( q \) on the set
\[
\{ \bigcup_{n=1}^{\infty} [r, s(t, n)] \} \cup [c, z]
\]
such that for each $j = 0, 1, 2, ..., n$, the sequence $(p(m,j))_{m=1,2,...}$ converges to the point $p_j$ say; and such that for $j = 1, 2, ..., n$, the sequence

$$(p(m,j-1), p(m,j))_{m=1,2,...}$$

of arcs converges to the arc $[p_{j-1}, p_j]$, with the additional features:

(a) $[p_{j-1}, p_j]$ is properly contained in $[p_{j-1}, p_j]$ for $j = 1, 2, ..., n - 1$;

(b) $p_{j-1} = p_j = p$; and finally,

(c) $[p(m,n), p(m)]$ is properly contained in $[p_{n-1}, p_n]$.

We call the point $p_{n-1}$ the top of the partial $n$-hook and the point $p_n$ the bottom of the partial $n$-hook.

Note. It follows from the definition of a partial $n$-hook that for a given $n$: either, for each partial $n$-hook $[a, b]$, $a < b$ (if $n$ even) or, for each partial $n$-hook $[a, b]$, $b < a$ (if $n$ odd). This is because in order to satisfy the portion of the definition concerning proper containment we must have $p_0 < p_1, p_1 > p_2, p_2 < p_3$, etc.; that is, $p_{j-1} < p_j$ if $j$ is odd, while $p_{j-1} > p_j$ if $j$ is even.

(Recall that our partial order $\leq$ is defined as $p \leq q$ provided $p$ weakly cuts $q$ from $c$)

**Lemma 3.1.** If a pair of partial $n$-hooks intersect, then their tops must coincide.

**Proof.** The proof is handled by induction on $n$. For details see [6].

**Lemma 3.2.** Let $\varepsilon$ be a positive real number and suppose that $X$ contains no partial $k$-hook for $k = 2, 3, ...,$ of diameter less than $\varepsilon$. If for a fixed $k$ one chooses a sequence $(\{p_{n-1}(0), p_n(0)\})_{n=1,2,...}$ of partial $k$-hooks such that $(p_{n-1}(0)) \to p_{n-1}(0)$,

$$\lim_n [p_{n-1}(0), p_n(0)] = [p_{n-1}(0), p_n(0)]$$

and the latter set is also a partial $k$-hook.

**Proof.** One need only show the lemma is true when $k = 2$, since by definition, each sequence of partial $n$-hooks (for $n > 2$) is embedded in a sequence of partial 2-hooks and with the lemma true for $k = 2$, we can put a radially convex metric $d$ on the closure of the sequence of partial 2-hooks, which is then inherited by the sequence of partial $n$-hooks. Using the radially convex metric $d$, it is easy to show that the lemma then holds for $k = n$. We proceed by showing that if the lemma fails for $k = 2$ then one can find a zig-zag lying inside the fan, contrary to our general hypothesis. See [6] for further details.

**Lemma 3.3.** Let $\varepsilon$ be a positive real number and suppose that $X$ contains no partial $k$-hook for $k = 2, 3, ...,$ of diameter less than $\varepsilon$. Then there exists a positive real number $\delta$, called the nesting diameter of $X$ such that for each partial $k$-hook, the diameter of $[p_{n-1}, p_n]$ is at least $\delta$ greater than the diameter of $[p_{n-1}, p_n]$ for $k = 2, 3, ...$ (using the same notation as in the definition).

**Proof.** If no such $\delta$ exists, then it is possible to consider two cases:

Case I. For some integer $k$ greater than 1 there exists a sequence

$$\{a_i, b_i\} \to [a_1, b_1]$$

of partial $k$-hooks such that the difference in diameter between $[a_1, b_1]$ and $[a_{n-1}, b_{n-1}]$ is less than $1/2$. Now by taking subsequences and relabeling, we may assume (in view of Lemma 3.2) that

$$\lim_n [a_i, b_i] = [a_1, b_1]$$

are distinct points (using the $\varepsilon$ hypothesis). Now if $k = 2$, it follows that the point $c$ is a $P$-point. For $k > 2$, we may “diagonalize” the three double-sequences

$$\lim [p(i, m, j), p(i, m, j + 1)] = [p(i, m, j), p(i, m, j + 1)]$$

(whence $[p(i, m, j), p(i, m, j + 1)] = [p(i, m, j), p(i, m, j + 1)]$ for $j = k - 3$, $k - 2$, $k - 1$, $l = 1, 2, ..., n$)

We may now suppose after some relabelling that

$$\lim [p(i, m, k - 3), p(i, m, k - 2)] = [p(i, m, k - 3), p(i, m, k - 2)]$$

and

$$\lim [p(i, m, k - 2), p(i, m, k - 1)] = [p(i, m, k - 2), p(i, m, k - 1)]$$

where $p_{n-1}(0) = \lim_n p_{n-1}(0)$ and $p_{n-2}(0) = \lim_n p_{n-2}(0)$. However, because of our assumption on the diameters of $[p_{n-2}(0), p_{n-1}(0)]$, $(p_{n-2}(0), p_{n-1}(0))$ having difference less than $1/2$, it is evident that $p_{n-2}(0)$ is identical to $p_{n-1}(0)$. Also, by using Lemma 1.2 to put a radially convex metric on $C \{ \bigcup_{i=1}^{n} p(i, m, k - 2) \}$, we may find points $z_i$ belonging to $[p(i, m, k - 3), p(i, m, k - 2)]$ with $z_i \to p(i, m, k - 3)$. The arcs

$$[z_i, p(i, m, k - 2), p(i, m, k - 1), a_1, b_1]$$

then form a zig-zag, so this case cannot occur.

Case II. There exists a sequence $(p(m,j), p(m,j))_{m=1,2,...}$ of partial $t$-hooks such that the difference in diameter between $[p_{n-2}(0), p_{n-1}(0)]$ and $[p_{n-1}(0), p_{n-1}(0)]$ is less than $1/2$. Using processes similar to those of Case I, we obtain sequences:

$$\lim [p(i, m, j - 1), p(i, m, j)] = [p(i, m, j - 1), p(i, m, j)]$$

such that for each $j$, the sequence

$$\lim [p(i, m, j - 1), p(i, m, j)]$$

converges to $[p(i, m, j - 1), p(i, m, j)]$ where $p(i, m, j) = \lim_n p(i, m, j)$ (possible by Lemma 3.2).
(Note that for \( i \neq j \), each partial \( i \)-hook is contained in a partial \( j \)-hook from which we obtain the points \( p_{2j}(i) \).)

We have:

\[
p_{0}(0) < p_{0}(0) < p_{0}(0) < \ldots < p_{2k}(0) < \ldots < p_{2k-1}(0) < \ldots < p_{3}(0) < p_{3}(0) .
\]

It follows from the \( i \)-hypothesis that \( \{p_{2i}(0), r \} \) while \( \{p_{2i+1}(0)\} \) \( \in \mathcal{S} \) such that \( s \), \( r \)
are distinct points (at least \( \varepsilon \) apart). But, without loss of generality, it follows that the
"diagonal" sequences

\[
\{p(i, m_i, i-3)\}, \quad \{p(i, m_i, i-2)\}, \quad \{p(i, m_i, i-1)\}, \quad \{p(i, m_i, i)\}, \quad i = 3, 4, \ldots,
\]

each converge to \( s \), \( r \), \( s \), \( r \) respectively, and the arcs formed by these four sequences
yield a zig-zag.

Hence, the lemma is true.

**Corollary 3.4.** Let \( \varepsilon \) be a positive real number. If the fan \( X \) contains no partial
\( k \)-hook for \( k = 1, 2, \ldots \) of diameter less than \( \varepsilon \), then there exists a positive integer \( n \)
such that for each \( n \geq n \), \( X \) contains no partial \( k \)-hook.

**Proof.** Let the nesting diameter of \( X \) be \( \delta \). Since \( X \) is compact, we may suppose that
diameter \( X = 1 \). Choose \( m = 1 \) to be greater than \( 1/\delta \). If \( \{p_{n-1}, p_{n}\} \) is
a partial \( m \)-hook lying in \( X \), then \( \{p_{n-1}, p_{n}\} \) is properly contained in a partial
(\( m = 1 \)-hook) which is properly contained in a partial (\( m = 2 \)-hook) which is
properly contained in a partial 1-hook. By virtue of the property of the nesting
diameter, it follows that the diameter of \( \{p_{n-1}, p_{n}\} \) must be less than \( \varepsilon / 2 \). This being
impossible, we have an upper bound \( n \) as desired.

**Definition.** Let \( X \) be a fan which satisfies the conditions of Corollary 3.4
(as well as satisfying the hypothesis of this chapter; namely, pairwise smooth, no
zig-zag, and no \( P \)-point). We say that \( X \) is an \((\varepsilon, n)\)-fan.

**Lemma 3.5.** Let \( X \) be an \((\varepsilon, n)\)-fan. Let \( k \) be a positive integer less than or equal to \( n \)
and let \( p_{k} \) be the top of a partial \( k \)-hook. We claim that the union of these
partial \( k \)-hooks which contain the point \( p_{k} \) forms a closed set and is also a partial
\( k \)-hook.

**Proof.** Each summand of the union has the point \( p_{k} \) at its top, by virtue of
Lemma 3.1. If \( k \) is even (respectively, \( k \) odd), then the bottom of each summand
belongs to \( \langle \varepsilon, p_{k} \rangle \) (respectively, \( \langle \varepsilon, p_{k} \rangle \)) for some \( \varepsilon \in \mathcal{A} \)
and there is then a point \( q \) which is the infimum (respectively, supremum) of these bottom points.
If we approach this limit point with a countable sequence \( \{p(i, k)\}_{i=1}^{m} \) of the bottom points,
then we have:

\[
(a) \quad \{p(i, k)\}_{i=1}^{m} \rightarrow q,
\]

\[
(b) \quad \{p(i, k-1)\}_{i=1}^{m} \rightarrow p_{k-1} \quad \text{implies}
\]

\[
(c) \quad \{\{p(i, k), p(i, k-1)\}_{i=1}^{m} \rightarrow [p_{k-1}, q] .
\]

It follows from Lemma 3.2 that the set \( \{p_{k-1}, q\} \) is also a partial \( k \)-hook. This
set is thus contained in the union under consideration, but also contains the union
by choice of \( q \). The union is therefore equal to the partial \( k \)-hook \( \{p_{k-1}, q\} \) and is
closed.

**Definition.** A set of the form \( \{p_{k-1}, q\} \) is called a \( k \)-hook. We drop the adjective
"partial" since such a \( k \)-hook is complete in the sense that it does not properly
lie in another partial \( k \)-hook. It should be noted, however, that each \( k \)-hook also
satisfies the definition of a partial \( k \)-hook (Lemma 3.5), and every lemma or corollary
we prove concerning a partial \( k \)-hook is also true for a \( k \)-hook. Now, for any pair
of \( k \)-hooks, either the two are identical or else they do not intersect, in view of
Lemma 3.1. We now refine this statement.

**Lemma 3.6.** Let \( X \) be an \((\varepsilon, n)\)-fan. There exists a positive real number \( \delta \) such
that for each \( k \leq n \), and for each \( q \in A \), for each pair of \( k \)-hooks lying on the arc \([c, e] \)
It is true that their \( \tau \)-neighborhoods are mutually disjoint.

**Proof.** If the lemma fails, then we may choose a \( k \)-hook \( \{p_{k-1}, p_{k}\} \) on an arc
\([c, e] \) with a sequence \( \{p(i, k-1), p(i, k)\}_{i=1}^{m} \) of \( k \)-hooks on \([c, e] \), each one of
which is mutually disjoint from \( \{p_{k-1}, p_{k}\} \) with the property that

\[
d(p_{k-1}, p(i, k)) < (1/\delta)
\]

depending upon whether the \( k \)-hooks converge to \( \{p_{k-1}, p_{k}\} \) from "above" or from
"below." We may assume that \( (p(i, k))_{i=1}^{m} \) converges to \( p(0, k) \) say, and that
\( (p(i, k-1))_{i=1}^{m} \) converges to \( p(0, k-1) \). From Lemma 3.2 we know that
\( \{p(i, k-1), p(i, k)\}_{i=1}^{m} \) converges to \( \{p(0, k-1), p(0, k)\} \) and that the latter set
is at least a partial \( k \)-hook (of diameter at least \( \varepsilon \)). Now the point \( p(0, k-1) \) (respectively,
\( p(0, k) \)) must lie outside the arc \( \{p_{k-1}, p_{k}\} \), but it follows from (a) that

\[
p(0, k) = p_{k-1}
\]

(respectively, \( p(0, k-1) = p_{k} \)),

which is contrary to Lemma 3.1. It therefore must be possible to find the desired
number \( \delta \).

**Lemma 3.7.** Let \( X \) be an \((\varepsilon, n)\)-fan. There exists a positive real number \( \tau \) such
that for each partial \( n \)-hook lying in \( X \), it is true that for each \( k < n \), and for each
partial \( k \)-hook, neither the top nor the bottom of that \( k \)-hook may lie inside the
\( \tau \)-neighborhood of the \( n \)-hook.

**Proof.** Let \( \{p_{n-1}, p_{n}\} \) be a fixed partial \( n \)-hook, with top \( p_{n} \) and
bottom \( p_{n} \). If no \( \tau \) works for this particular case, then there exists a sequence
\( \{q(i, k-1), q(i, k)\}_{i=1}^{m} \) of partial \( k \)-hooks (for some \( k < n \)), which converges
by choice to say, \( \{q(0, k-1), q(0, k)\} \), a partial \( k \)-hook itself (Lemma 3.2) and, with either
\( q(0, k-1) \) or \( q(0, k) \) belonging to \( \{p_{n-1}, p_{n}\} \). But \( \{p_{n-1}, p_{n}\} \) is contained in a partial
\( k \)-hook whose top lies outside of \( \{p_{n-1}, p_{n}\} \), by definition. Since \( \{q(0, k-1), q(0, k)\} \)
intersects this partial \( k \)-hook, Lemma 3.1 implies that the top \( q(0, k-1) \) also must
lie outside of \( \{p_{n-1}, p_{n}\} \). We are led then to the case where the bottom \( q(0, k) \) belongs
to \([p_{k+1}, p_k]\). The partial \(n\)-hook \([p_{k-1}, p_k]\) is contained in a partial \((k+1)\)-hook \([p_k, p_{k+1}]\) by definition, with, say, the sequence \(\{l(m, k), p(m, k+1)\}\) of points belonging to \([p_{k+1}, p_k]\) in the usual way. As in the proof of Lemma 3.1, we may assume that there is a sequence \(\{q(m)\}_{n=1}^{\infty}\) of points belonging to

\[p(m, k), p(m, k+1)\]

for each \(m\), which converges to \(q(0, k)\). There is also a sequence \(\{q(0, m, k)\}_{n=1}^{\infty}\) of points, given by the definition of partial \(k\)-hooks, which converges to \(q(0, k)\). It can be shown that neither of the pair \(\{q(m)\}_{n=1}^{\infty}, \{q(0, m, k)\}_{n=1}^{\infty}\) dominates the other (using methods similar to those in the proof of Lemma 3.1), which is contrary to the pairwise smoothness of the fan \(X\). We can therefore find an positive real number \(\tau\) such that for each \(k<n\), the top or bottom of no partial \(k\)-hook lies within \(\tau\) of \([p_{k-1}, p_k]\). Moreover, the minimum value of \(\tau\) we will need to choose as we let the partial \(n\)-hook vary, will be greater than zero. If it is equal to zero, we find a sequence \(\{H_n\}_{n=1}^{\infty}\) of partial \(n\)-hooks requiring values of \(\tau\) say, \(\tau_n\), where \(\tau_n\) converges to zero. But then \(\lim H_n\), which is itself a partial \(n\)-hook (Lemma 3.2), will require a choice of \(\tau = 0\) (using a diagonalization process as done previously), which is contrary to the proof, just completed, that each given partial \(n\)-hook admits a positive value of \(\tau\). The lemma is therefore proved as stated.

**Lemma 3.8.** Let \(\mathcal{X}\) be an \((n, m)\)-fan with \(n \geq 1\). There exists a map \(F\) from \([0, 1] \times X\) into \(X\) such that for each point \(p\) of \(X\) we have \(F(0, p) = p\) and \(F([1] \times X)\) is an \((n-1, m)\)-fan and, moreover, for each time \(t\) in \([0, 1]\), the map \(F\) restricted to \([t] \times X\) is monotone.

**Proof.** Can be found in [6].

**Lemma 3.9.** Let \(X\) be a fan which is pairwise smooth, contains no zig-zag, and no \(P\)-point. There exists a map \(F\) from \([0, 1] \times X\) into \(X\) such that for each \(x\) in \(X\), \(F(0, x) = x\) and \(F([1] \times X)\) contains no partial \(k\)-hooks except \(k = 1\). Moreover, \(F\) may be chosen so that \(F\) restricted to \([t] \times X\) is monotone for each \(t\) belonging to \([0, 1]\).

**Proof.** In view of Corollary 3.4, we may assume that for \(k = 1, 2, \ldots\), the fan \(X\) contains no partial \(k\)-hook of diameter less than \(1/k\). For some \(n \geq 1\) we assume that we have a map \(F\) so that the image \(F([1/n] \times X)\) contains no partial \(k\)-hook for each \(k \geq n\). That is to say \(F([1/n] \times X)\) is a \((1/n, n)\)-fan. We may then apply Lemma 3.8 to obtain a \((1/(n-1), n-1)\)-hook during the time between \(t = 1/n\) and \(t = 1/(n-1)\). By composing these maps in the appropriate fashion and setting \(F(0, X)\) to be the identity mapping, we obtain the desired result.

**Theorem 3.10.** Let \(X\) be a fan which is pairwise smooth, contains no zig-zag and no \(P\)-point. Then \(X\) is monotone contractible.

**Proof.** In view of Lemma 3.9, we may assume that for \(n > 1\) it is true that \(X\) contains no \(n\)-hook. Let \(\leq\) denote the weak cut point order with respect to the top \(c\) of \(X\). We claim that \(X\) admits a metric \(q\) which is radially convex with respect to \(\leq\). It follows from [3] and [10] that this will be the case provided we show:

If a sequence \(\{x_n\}_{n=1}^{\infty}\) of points of \(X\) converges to a point \(p_0\) in \(X\), then it follows that \(\operatorname{LS}(L(p_n))\) is contained in \(L(p_0)\). Let us say that \(p_n\) belongs to \([c, e_0]\) for \(n = 0, 1, 2, \ldots\). Let \(x_0 = \max\{e_0, e_0(n)\}\). We may as well assume that there are points \(x_n\) belonging to \([c, c_0]\) such that \(\{x_n\}_{n=1}^{\infty}\) converges to \(x_0\). If \(L\{x_n, e_0(n)\}\) contains a point different from \(x_0\), then \(x_0\) is the least such point. It follows that \(\{x_n, e_0(n)\}\) is a 2-hook, contrary to our above assumption. If, on the other hand, \(L\{x_n, e_0(n)\} = x_0\), then for almost all \(n\), \(p_n\) belongs to the arc \([c, x_0]\). Since \(L\{x_n, e_0(n)\} = x_0\), Lemma 1.2 shows us that \(L\{c, p_n\}\) is contained in \([c, x_0]\), which was to be shown. It is now easy to see that \(X\) is monotone contractible.

**Appendix.** This section is devoted to a discussion of various examples. The first one we wish to consider is a dendroid which has the property that, although it is contractible — no matter which choice of a contraction is made, there must be a time at which the image is a noncontractible sub-dendroid.

![Fig. 1](image-url)

The dendroid \(A\) consists of the triod \(abc\) with center \(d\), together with a sequence of arcs \([c, d, c_1, e_1, e_2, c_2, f_2, g_2]\) where \(n = 1, 2, \ldots\), converging to the triod in the manner indicated.

It is possible to contract \(A\) to the point \(a\) by the following informal recipe.
Step 1. While keeping the points $a, b, c$ fixed in place you must push the point $d$ along the arc $[db]$ all the way up to $b$, contracting the points in front of $d$ ($[cd, b]$) into $b$, and stretching out those behind $d$ ($[b, d'), [c, d']$). At the same time and in the same manner you must move the points $(d_k)$ for $n = 1, 2, \ldots, (d_k)_{n=1,2,\ldots}$ and $(f_n)_{n=1,2,\ldots}$ up to the points $(b_n), (b_k), (y_n)$ respectively.

Step 2. Keep the arc $[d_n, e_n]$ together as a single point (for each $n$) and move this point down from $b_n$ to $e_n$. At the same time you keep the arc $[e_n, f_n]$ together as a single point (for each $n$) and move it from $e_n$ down to $f_n$. This forces the arc $[d, b]$ to remain together as a single point and to move from $b$ down to $d$. The arc $[a, d]$ is now stretched out to cover $[a, d_n, b_n, e_n]$ while the arcs $[e_n, f_n], [e_n, c], [e_n, c]$ slip back to their original positions. The arc $[a, d]$ covers $[a, b, d]$ by going up to $b$ and folding back over itself to $d$.

At the end of this step the arcs $(f_n, g_n)$ have vanished.

Step 3. Push $d$ along the arc $[d, c]$, all the way out to $c$. At the same time you must move the points $e_n$ (now the image of $[d_n, e_n]$) out to $e_n$, and move the points $f_n$ (now the image of $[e_n, f_n]$) out to $e_n$. This action collapses the arcs $[d_n, b_n, e_n, f_n, g_n]$ down to the single point $c$ (for each $n$) and leaves the arcs $[a, d_n]$ stretched all the way along $[a, d_n, b_n, e_n, c]$. The arc $[ad]$ now is stretched up to $b$, folds back to $d$, and then out to $c$. The arcs $(c_n, f_n, g_n)$ have now vanished.

Step 4. Let the arcs $[a, d_n]$ snap back from $[a, d_n, b_n, e_n, c]$ to cover just $[a, d_n, b_n, e_n]$. This makes the arcs $(d_n, e_n, f_n, g_n)$ vanish.

Step 5. Let the arcs $[a, d_n]$ continue to reverse their stretching process so that they are back to covering just $[a, d_n]$. The arcs $(d, b), (d_n, b, c, e_n, f_n, g_n)$ have now vanished.

Step 6. Finally, contract $[a, d_n]$ down to $a$ (for each $n$), thus collapsing $(ad)$ to the single point $a$. The dendroid $A$ is therefore a contractible space. There follows a sketch of the proof that for each possible contraction $F$ of $A$, there exists a time $t$ in $[0, 1]$ such that $F(t \times A)$ is not contractible.

Proof of the above remark. Let the set $S$ consist of those times $t$ in $[0, 1]$ such that the arc $[b, d]$ intersects $L_s \{F(t \times A) \cap [b, d] \}$. Let the set $T$ include the times $t$ in $[0, 1]$ for which $b$ belongs to $L_s \{F(t \times A) \cap [b, d] \}$. Now $T$ is evidently a proper subset of $S$. For $t$ belonging to $S - T$, the set $F(t \times A)$ would look like:

The dendroid (see Fig. 2) is not contractible since any contraction of it would involve moving $d$ along $[ad]$ to $c$. Since the sequence $(d_n)$ converges to $d$, almost all of the points $d_n$ must slide up through $f_n, b_n$ before $d$ is moved to $c$ or else down through $(a)$ before $d$ is moved to $c$. But this it not possible because the point $d$ would then move to $b$ or $a$ without the points $(f_n)$ being able to follow (since the arcs $[b_n, f_n]$ are no longer in the space). Therefore, the remark above is correct.

The fan below was inspired by the example of F. Burton Jones in [7].

The fan (see Fig. 3) consists of a countable sequence of arcs $[e_n, b_n]$ converging to $[c, d]$, together with a countable sequence of 2-hooked arcs $[e_n, d_n, e_n]$. 

\[ \text{Fig. 2} \]

\[ \text{Fig. 3} \]

\[ \text{Fig. 4} \]
Out of the pair \((a_n), (b_n)\) each of which converge to \((a)\), it is not possible to find one which dominates the other.

In fact, the arcs \([a_n, d_n]\) converge to \([a, d]\) with no similar capability with respect to the sequence \((b_n)\) and the arcs \([c, b_n]\) converge to \([c, b]\) with no similar capability on the part of the sequence \((a_n)\). Hence this fan is not pairwise smooth.

This fan (see Fig. 4) consists of a countable sequence of 2-hooked arcs \([c, a_n]\) and a countable sequence of 3-hooked arcs \([c, b_n]\). Out of the pair \((a_n), (b_n)\), it is not possible to find one which dominates the other. The arcs \([a_n, d_n]\) converge to \([a, d]\) with nothing similar for \([b_n]\) and the arcs \([b_n, e_n]\) converge to \([a, e]\) with nothing similar for \([a_n]\).

Our next example is a fan which contains a zig-zag. The sequence
\[
\{(a_n, b_n, c_n, d_n)\}_{n=1}^\infty
\]
of arcs converges to the arc \([a, b]\) in the manner required by the definition of a zig-zag.

Another possible way in which a fan may contain a zig-zag is illustrated below.

Once again the sequence \((a_n, b_n, c_n, d_n)_{n=1}^\infty\) of arcs converges to the arc \([a, b]\) in the appropriate manner.

In the fan below, the point \(b\) is an example of a \(P\)-point. The set \(L_{[a,b]}\) is just the arc \([b,d]\). For \(n = 1, 2, \ldots\), the arc which is irreducible between the point \(b_n\) and the set \(L_{[a,b]}\) is simply the arc \([b_n,b]\).

This concludes our collection of examples.

References


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