A characterization of well-orders

by

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Abstract. Let $\langle X, \leqslant\rangle$ be a linear order. X is *cushioned* if, for every $x \in X$ and every order-isomorphism $f\colon X \to X$, $x \leqslant f(x)$. We prove the following results. THEOREM. *If X is cushioned, then X is a well-order or a densely-ordered sum of well-orders.* COROLLARY. *X is a well-order iff X is both cushioned and scattered.* COROLLARY. *X is a well-order iff X is hereditarily cushioned.* THEOREM. *If $|X| \leqslant \omega$, X is a well-order iff X is cushioned; however, assuming the* GCH, *for any $\varkappa > \omega$ there are both dense and non-dense cushioned orders of power $\varkappa$ which are not well-orders.*

0. Introduction. Let $\langle X, \leqslant\rangle$ be a linear order; if, for every $x \in X$ and every order-isomorphism $f\colon X \to X$, $x \leqslant f(x)$, we shall say that $\langle X, \leqslant\rangle$ is *cushioned.* Obviously every well-order is cushioned, and it is natural to ask whether the converse ts also true. The answer, as one might guess, is "no": there is a dense subset, X, of ihe real line, of power 2^{ω}, such that the only order-isomorphism of X into itself is the identity (see 3.2). In this note we characterize cushioned linear orders and use this characterization to derive two new characterizations of well-orders (1).

0.0 THEOREM. *If a linear order is cushioned, then it is either a well-order or a densely ordered sum of well-orders.*

0.1. COROLLARY. *A linear order is a well-order iff it is both cushioned and scattered (i.e., no infinite subset is dense — in — itself).*

0.2. COROLLARY. *A linear order is a well-order iff it is hereditarily cushioned.*

1. Notation. We adhere to the convention that an ordinal number *is* the set of smaller ordinals, and that a cardinal number *is* an initial ordinal; these will be denoted by lower-case Greek letters. According to context, α may also designate the order-type of the well-order $\langle \alpha, \in\rangle$. Indeed, if A is any set with some linear order on A understood, we use A also to denote the order-type of A; A^* denotes the inverse order-type obtained by "turning A around." We use the standard notation for open and closed intervals, e.g., $(x, y) = \{z\colon x < z < y\}$, and $[x, \to) = \{y\colon y \geqslant x\}$.

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(1) I am grateful to the excellent referee for bringing to my attention a serious oversight in the original statement of 4.0, for suggestions leading to significant improvements in the readability of 3.1–3.4, and for his efforts in attempting to track down the reference mentioned in the footnote to 3.3.

$[Z]$ and $[W]$ will denote two generalized order-types, as follows: $[Z]$ is any order-type of the form $\alpha^*+\beta$, and $[W]$ is any order-type of the form $\sum\{\alpha_n^*: n\in\omega\}$, where α and β are limit ordinals. (As usual, if $\langle A, \leqslant_A\rangle$ and $\langle B_a, \leqslant_a\rangle$ are linear orders (for $a\in A$), then $\sum\{B_a: a\in A\}$ is the order-type of $\bigcup\{\{a\}\times B_a: a\in A\}$ ordered so that $\langle a, b\rangle\leqslant\langle a', b'\rangle$ iff $a\leqslant_A a'$, or $a=a'$ and $b\leqslant_a b'$.)

Finally, Q denotes the set of rational numbers (with the customary order).

2. The proofs. We begin by assigning to each linear order, X, an ordinal rank $r(X)$. Suppose that R is a convex equivalence relation on X, i.e., an equivalence relation with convex equivalence classes. For $x\in X$, let $R(x)=\{y\in X: xRy\}$; since R is convex, $X/R=\{R(x): x\in X\}$ is the image of X under an order-homomorphism whereby if $R(x)\neq R(y)$, then $R(x)\leqslant R(y)$ iff $x\leqslant y$. We define from R a new equivalence relation, $R^{\#}$, on X as follows. If $x, y\in X$ with $x\leqslant y$, put $xR^{\#}y$ iff there is a finite family $\{x_0, \ldots, x_n\}\subseteq X$ such that $x=x_0\leqslant x_1\leqslant\ldots\leqslant x_n=y$, $[x_i, x_{i+1}]/R$ is a well-order if $i<n$ is even, and $[x_i, x_{i+1}]/R$ is an inverted well-order if $i<n$ is odd; and if $y<x$, put $xR^{\#}y$ iff $yR^{\#}x$. Clearly $R\subseteq R^{\#}$, and $R^{\#}$ is a convex equivalence relation on X.

Now let R_0 be the identity relation on X, and, for each ordinal $\alpha>0$, let $R_\alpha=(\bigcup\{R_\xi^{\#}: \xi<\alpha\})$; then each R_α is a convex equivalence relation on X, and $R_\alpha\subseteq R_\beta$ whenever $\alpha<\beta$. Define $r(X)=\inf\{\alpha: R_\alpha=R_{\alpha+1}\}$; clearly $r(X)$ exists, since it must in fact be less than $|X|^+$. Note also that if $\alpha=r(X)$, then either $|X/R_\alpha|=1$, (i.e., $R_\alpha=X\times X$), or X/R_α is a dense linear order; and that X is scattered iff $|X/R_\alpha|=1$. We say that X is *elementary* if X is scattered and $r(X)\leqslant 1$.

2.0. Lemma. *Let X be elementary, and let $x\in X$ be arbitrary. Then the order-type of $[x, \to)$ can be represented as $\alpha+1+\sum\{T_n: n\in\omega\}$, where each T_n is of some type $[Z]+1$, or as $\alpha+1+([Z]+1)+\ldots+([Z]+1)+[W]$ (for some finite number of terms $[Z]+1$), or as an initial segment of either of these generic forms. T is a possible order-type for $(\leftarrow, x]$ iff T^* is a possible type for $[x, \to)$.*

The proof of 2.0 is straightforward but tedious and is omitted. (It would be possible to formulate this result without reference to $[Z]$ and $[W]$, but it is useful to think of these generalized types as being pictorially "like" Z and "like" ω, respectively.)

2.1. Lemma. *If X is elementary and cushioned, then X is a well-order.*

Proof. Since no inverse (infinite) well-order is cushioned, it is clear from 2.0 that if X is not a well-order, then it is of the form $[W]^*$; thus, it suffices to show that X has a first element. If not, let $\langle x_n: n\in\omega\rangle$ be a strictly decreasing sequence co-initial in X, and, for $n>m$, let $\alpha(n, m)$ be the order-type of the segment $[x_n, x_m)$. If there were a strictly increasing sequence $\langle n(i): i\in\omega\rangle$ such that

$$\alpha(n(i+2), n(i+1))\geqslant\alpha(n(i+1), n(i))\quad\text{for each } i\in\omega,$$

then clearly X would not be cushioned; thus, there must be a strictly increasing sequence $\langle n(i); i\in\omega\rangle$ such that

$$\alpha(n(2i+1), n(2i))>\alpha(n(2i+3), n(2i+2))\quad\text{for all } i\in\omega,$$

which is impossible. Thus, X has a first element and is therefore a well-order.

Proof of Theorem 0.0. Suppose that X is a cushioned linear order; then $r(X)\leqslant 1$. For if not, there must be points $x, y\in X$ such that $R_1(x)$ and $R_1(y)$ are adjacent in X/R_1; but by Lemma 2.1, $R_1(x)$ and $R_1(y)$ are well-ordered subsets of X, which implies that xR_1y, i.e., that $R_1(x)=R_1(y)$, a contradiction. If, however, X is cushioned, and $r(X)\leqslant 1$, then either X is scattered, in which case $|X/R_1|=1$ and X is a well-order, or X is not scattered, in which case X is a densely ordered sum of well-orders.

Proof of Corollary 0.1. Immediate from Theorem 0.0.

Proof of Corollary 0.2. Immediate from Corollary 0.1 and the observation that every dense linear order, and hence every non-scattered linear order, contains an infinite, decreasing sequence and is therefore not hereditarily cushioned.

3. A Special Case. We retain the notation of Section 2.

3.0. Theorem. *If X is a cushioned linear order, and $|X/R_1|\leqslant\omega$, then X is a well-order. (In particular, a countable, cushioned linear order is a well-order.)*

Proof. By Corollary 0.1, it suffices to show that X is scattered. Suppose not; then, as is well known, X/R_1 is isomorphic to one of Q, $Q\cap[0, 1]$, $Q\cap[0, 1)$, or $Q\cap(0, 1]$, none of which is cushioned, so we must have $r(x)=1$. It follows that there are non-zero ordinals α_p ($p\in Q$) such that some convex subset of X is isomorphic to $Y=\sum\{\alpha_p: p\in Q\}$. Let $A=\{\alpha_p: p\in Q\}$, and, for $\alpha\in A$, let $Q(\alpha)=\{p\in Q: \alpha_p\geqslant\alpha\}$. We may without loss of generality assume that $Q(\alpha)$ is dense in Q for each $\alpha\in A$. (For if not, there are a $\beta_0\in A$ and an open interval $J_0\subseteq Q$ such that $\alpha_p<\beta_0$ for each $p\in J_0$; but then either $Q(\alpha_p)$ is dense in J_0 for each $p\in J_0$, or there are a $\beta_1<\beta_0$ and an open interval $J_1\subseteq J_0$ such that $\alpha_p<\beta_1$ for each $p\in J_1$. Strictly decreasing sequences of ordinals being finite, this reduction must terminate after a finite number of steps, at which point — after suitable relabling — we have the desired Y.) But then Y is not cushioned, for we can construct a "bad" order-isomorphism of Y as follows.

Enumerate $Q=\{p(n): n\in\omega\}$. By induction on n, choose distinct $q(n)\in Q$ so that $q(0)<p(0)$, $\alpha_{p(n)}\leqslant\alpha_{q(n)}$ for each $n\in\omega$, and, for $n<m<\omega$, $q(n)<q(m)$ iff $p(n)<p(m)$. Then $f: Y\to Y: \langle p(n), \xi\rangle\mapsto\langle q(n), \xi\rangle$ is the desired function.

This completes the proof, since X is cushioned iff every convex subset of X is cushioned.

As the following results show, however, it is at least consistent with the usual axioms of set theory (with choice) that $|X/R_1|\leqslant\omega$ be the only such special case.

3.1. Lemma. *Let $\varkappa$ be an infinite cardinal such that $2^{\underset{\smile}{\varkappa}}=\varkappa$. (By definition, $2^{\underset{\smile}{\varkappa}}=\sup\{2^\lambda: \lambda<\varkappa\}$.) Then there is a complete, dense, linear order, $\langle X, \leqslant\rangle$, such that*

(1) $|X|=2^\varkappa$;

(2) *if $x, y\in X$, and $x<y$, then $|(x, y)|=2^\varkappa$; and*

(3) *X has a dense subset, D, such that $|D|=\varkappa$. (That is, D is dense in the order topology on X.)*

Outline of the proof. Let $Y = {}^{\varkappa}2$, the set of all functions from $\varkappa$ to $2(=\{0,1\})$, and let R be the relation on Y that identifies adjacent points in the lexicographic ordering of Y; that is,

$$\begin{aligned} R = \{\langle x, y\rangle \in Y\times Y:\ & x = y \vee \\ & \vee \exists \alpha\in\varkappa\Big(\big[\big(x(\alpha) = 0 \wedge y(\alpha) = 1 \wedge \forall \beta\in\varkappa\setminus(\alpha+1)\big(x(\beta) = 1\wedge y(\beta) = 0\big)\big)\vee \\ & \vee \big(x(\alpha) = 1 \wedge y(\alpha) = 0 \wedge \forall\beta\in\varkappa\setminus(\alpha+1)\big(x(\beta) = 0\wedge y(\beta) = 1\big)\big)\big]\wedge \\ & \wedge \forall\beta\in\alpha\big(x(\beta) = y(\beta)\big)\Big)\}. \end{aligned}$$

Let $X = Y/R$, and let $\leqslant$ be the ordering on X induced (as in Section 2) by the lexicographic ordering on Y. Finally, let D consist of those points of X that derive from adjacent pairs in Y: $D = \{R(y): y\in Y\wedge |R(y)| = 2\}$. It is straightforward to verify that $\langle X, \leqslant\rangle$ satisfies all requirements except possibly that $|D| = \varkappa$, which follows from the easy observation that $|D| = 2^{\underset{\smile}{\varkappa}}$.

3.2. THEOREM. *Let $\varkappa\geqslant\omega$ be such that $2^{\underset{\smile}{\varkappa}} = \varkappa$. Then there is a dense linear order of power $2^{\varkappa}$ which admits no non-trivial order-isomorphism into itself.*

Proof. Let $\langle X, \leqslant\rangle$ be the linear order given by Lemma 3.1, with dense subset D of power $\varkappa$. View X as a topological space with the order topology. X has a base of power $\varkappa$ (consisting of open intervals with endpoints in D), so every subspace of X has a dense subset of power at most $\varkappa$. Let $\mathscr{F}$ be the set of 1-1, order-preserving functions from X into X, and, for $f\in\mathscr{F}$, let $B(f) = \{x\in X: f \text{ is discontinuous at } x\}$. If $D(f)$ is a dense subset of $X\setminus B(f)$ of power $\varkappa$, then $f\upharpoonright(X\setminus B(f))$ is completely determined by $f\upharpoonright D(f)$, so there are only $|X|^{|D(f)|}\leqslant 2^{\varkappa}$ possibilities for $f\upharpoonright(X\setminus B(f))$. Moreover, each $f\in\mathscr{F}$ is monotone, so $|B(f)|\leqslant\varkappa$, and there are at most $2^{\varkappa}$ possibilities for $f\upharpoonright B(f)$. Finally, X has $2^{\varkappa}$ subsets of power $\varkappa$, so $|\mathscr{F}| = 2^{\varkappa}$; let $\mathscr{F} = \{f_\alpha: \alpha\in 2^{\varkappa}\}$ and let $\{V_\alpha: \alpha\in 2^{\varkappa}\}$ be an enumeration of the non-empty, open intervals of X.

Let $\{p_\alpha: \alpha\in 2^{\varkappa}\}$ be an enumeration of $2^{\varkappa}\times 2^{\varkappa}$, and let $\pi_0: 2^{\varkappa}\to 2^{\varkappa}$ and $\pi_1: 2^{\varkappa}\to 2^{\varkappa}$ be such that $p_\alpha = \langle\pi_0(\alpha), \pi_1(\alpha)\rangle$ for each $\alpha\in 2^{\varkappa}$. By recursion on $\alpha\in 2^{\varkappa}$ choose points x_α and y_α in X so that for each $\alpha\in 2^{\varkappa}$

(i) $x_\alpha\in V_{\pi_1(\alpha)}\setminus(\{x_\beta: \beta<\alpha\}\cup\{y_\beta: \beta<\alpha\})$;
(ii) $y_\alpha = f_{\pi_0(\alpha)}(x_\alpha)\notin\{x_\beta: \beta<\alpha\}\cup\{y_\beta: \beta<\alpha\}$; and
(iii) if $f_{\pi_0(\alpha)}\upharpoonright V_{\pi_1(\alpha)}$ is not the identity on $V_{\pi_1(\alpha)}$, then $y_\alpha\neq x_\alpha$.

Condition (2) of Lemma 3.1 and the fact that $f_{\pi_0(\alpha)}$ is 1-1 ensure that (i) and (ii) are always satisfiable. The satisfiability of (iii) follows from (2) of Lemma 3.1 and the fact that $f_{\pi_0(\alpha)}$ is monotone. (E.g., taking $f = f_{\pi_0(\alpha)}$ and $V = V_{\pi_1(\alpha)}$, if $x\in V$ and $f(x)>x$, choose $y\in V\cap(x, f(x))$, and note that $f(z)>z$ for all $z\in(x, y)\subseteq V$.)

Let $Y = \{x_\alpha: \alpha\in 2^{\varkappa}\}$. Clearly $|Y\cap(x, y)| = 2^{\varkappa}$ whenever $x, y\in X$ and $x<y$, so Y is a dense linear order of power $2^{\varkappa}$ (with respect to the order inherited from X). Suppose that f: $Y\to Y$ is 1-1 and order-preserving. X is complete, so f may be extended to a function g: $X\to X$ by setting

$$g(x) = \begin{cases} f(x), & \text{if } x\in Y,\\ \sup\{f(y): y\in Y\wedge y<x\}, & \text{if } x\in X\setminus Y,\end{cases}$$

where the supremum is taken with respect to the ordering on X. Clearly $g\in\mathscr{F}$, say $g = f_\alpha$. Moreover, if g is not the identity on X, there is a $\beta\in 2^{\varkappa}$ such that $g\upharpoonright V_\beta$ is not the identity on V_β. Then, choosing $\gamma\in 2^{\varkappa}$ so that $p_\gamma = \langle\alpha, \beta\rangle$, we see that $f(x_\gamma) = f_\alpha(x_\gamma) = y_\gamma\notin Y$, which is absurd. It follows that f is the identity on Y, so that there is indeed no non-trivial order-isomorphism of Y into itself.

3.3. Remarks. Of course, $2^{\underset{\smile}{\omega}} = \omega$, and in that case we may take X to be the real line [1]. For $\varkappa>\omega$ the condition is more restrictive. For instance, if $\varkappa = \lambda^+$, then $2^{\underset{\smile}{\varkappa}} = 2^{\lambda}$, so that $\varkappa = 2^{\underset{\smile}{\varkappa}}$ iff $2^{\lambda} = \lambda^+$. More generally, if the Generalized Continuum Hypothesis (GCH) holds, then $\varkappa = \sup\{\lambda^+: \lambda<\varkappa\} = \sup\{2^{\lambda}: \lambda<\varkappa\} = 2^{\underset{\smile}{\varkappa}}$ for all $\varkappa\geqslant\omega$.

3.4. THEOREM. *Assume the* GCH. *Then for any $\varkappa>\omega$ there is a dense, cushioned order of power $\varkappa$.*

Proof. If $\varkappa$ is a successor cardinal, the result follows from Lemma 3.2 and Remarks 3.3. If $\varkappa$ is a limit cardinal, let $\lambda = \operatorname{cf}\varkappa$, and write $\varkappa = \sup\{\varkappa_\alpha^+: \alpha\in\lambda\}$, where $\varkappa_\alpha<\varkappa_\beta$ whenever $\alpha<\beta<\lambda$. For each $\alpha\in\lambda$ let X_α be a dense, cushioned order of power $\varkappa_\alpha^+$ without endpoints, as constructed in the proof of Theorem 3.2. Observe that each $x\in X_\alpha$ has $\varkappa_\alpha^+$ predecessors in X_α. Let $X = \sum\{X_\alpha: \alpha\in\lambda\}$; $|X| = \varkappa$, X is densely ordered, and, if $x\in X_\alpha\subseteq X$, then x has $\varkappa_\alpha^+$ predecessors in X. Thus, if f: $X\to X$ is 1-1 and order-preserving, $x\in X_\alpha\subseteq X$, and $f(x)<x$, then f maps $\{y\in X_\alpha: y\leqslant x\}$ into a proper initial segment of itself, which is impossible; thus, X must be cushioned.

4. Remarks. Although the techniques of Lemma 3.2 and Theorem 3.4 produce only dense, cushioned orders, they can be combined with the following proposition to produce cushioned orders of rank 1 which are not well-orders.

4.0. PROPOSITION. *Let X be a dense, cushioned order, and let Y be any order such that Y/R_1 is isomorphic to X and $R_1(y)$ is a well-ordered subset of Y for each $y\in Y$; then Y is cushioned.*

Proof. View Y as $\{\langle x, \xi\rangle: x\in X\wedge\xi\in\alpha_x\}$, ordered lexicographically, for some non-zero ordinals α_x. Suppose that f: $Y\to Y$ is an order-isomorphism, and define F: $X\to X$ by $F(x) = y$ if $f(\langle x, 0\rangle) = \langle y, \xi\rangle$ for some $\xi\in\alpha_y$. Then F is an order-isomorphism; for otherwise there are $x, y\in X$ with $x<y$ such that $F(y)\leqslant F(x)$ and $f(\langle x, 0\rangle)<f(\langle y, 0\rangle)$, i.e., there are a $z\in X$ and $\xi, \eta\in\alpha_z$ such that $\xi<\eta$, $f(\langle x, 0\rangle) = \langle z, \xi\rangle$, and $f(\langle y, 0\rangle) = \langle z, \eta\rangle$; but then f maps the non-scattered interval $[\langle x, 0\rangle, \langle y, 0\rangle)]$ order-isomorphically into the scattered interval $[\langle z, \xi\rangle, \langle z, \eta\rangle]$, which is impossible. If, now, $f(\langle x, \xi\rangle)<\langle x, \xi\rangle$ for some $\langle x, \xi\rangle\in Y$, then clearly $f(\langle x, 0\rangle)<\langle x, 0\rangle$, since α_x is a well-order; but then $F(x)<x$ in X, which, sinc X is cushioned, is impossible. Hence Y is cushioned.

Finally, we may note the following existence theorem.

4.1. PROPOSITION. *For each ordinal α there are linear orders X_α and Y_α such that*:

[1] I have been told that Sierpiński proved this case of Theorem 3.2 in the 1930's, but neither the referee nor I was able to find a reference.

(a) $r(X_\alpha) = r(Y_\alpha) = \alpha$;

(b) X_α *is scattered, and* Y_α *is not;*

(c) $|Y_\alpha| = |\alpha|+\omega$; *and*

(d) $|X_\alpha| = |\alpha|+\omega$ *unless* $\alpha = 0$, *in which case* $|X_\alpha| = 1$.

Proof. Let Z be the integers (with the usual order). Given X_α, let $X_{\alpha+1} = Z\times X_\alpha$, ordered lexicographically, and let $Y_\alpha = Q\times X_\alpha$, also ordered lexicographically. If α is a limit ordinal, let $X_\alpha = \{f\colon \alpha+1\to Z\colon f$ is continuous (with respect to order topologies on $\alpha+1$ and Z) and $f(\alpha) = 0\}$; if $f, g\in X_\alpha$ with $f\neq g$, let $\beta = \max\{\xi<\alpha\colon f(\xi)\neq g(\xi)\}$, and write $f<g$ if $f(\beta)<g(\beta)$. It is easily checked that X_α has the desired properties.

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On contractible fans

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Abstract. The purpose of this paper is to give a characterization of weakly confluent-contractible fans. After giving several definitions, it is shown that such a fan must be *pairwise smooth*, must contain no *ziz-zag*, and lastly must contain no *P-point*. It is then shown that a fan which satisfies these three properties must be monotone-contractible. This implies the fan is weakly confluent-contractible in as much as monotone functions are always weakly confluent. Hence these properties also yield a characterization of monotone contractible fans.

Introduction. Several mathematicians (see [1], [4], [5], [7]) in recent years have studied the contractibility of dendroids. We will use the term *dendroid* to designate a compact metric continuum which is arc-wise connected and is also hereditarily unicoherent. A *ramification point* of a dendroid is a point which is the intersection of three or more arcs. K. Borsuk [2] has described simple types of dendroids, containing only one ramification point, which are called *fans*. The ramification point is called the *top* of the fan.

A topological space X is *contractible* if there exists a continuous map $F\colon [0,1]\times X\to X$ such that $F(0,p)$ is p, for each point p of X; and there is a point q in X such that $F(1,p)$ is q for each point p of X. The map F is called a *contraction* of X.

Figure 1 in the Appendix is a contractible dendroid A with the surprising property that for each choice of a contraction F, there must be a time t in $[0,1]$ for which $F(t\times A)$ is a *noncontractible* sub-dendroid of A. In order to restrict the spaces it was decided to place a stronger requirement on the maps involved. The property chosen was first defined by A. Lelek [9], that of *weak-confluence* of the maps. It was found that for dendroids, even with weakly-confluent maps, examples of the type found in Figure 1 are still admissible. The investigation was further restricted to the case of fans. It will be shown that a fan is weakly-confluent contractible if and only if it is confluent contractible, if and only if it is monotone contractible.

A continuous map is said to be *monotone* if the pre-image of each continuum lying in its image is itself a continuum. A contraction F on a space X is a *monotone contraction* provided that for each time t in $[0,1]$, the map F restricted to $\{t\}\times X$ is monotone.