6. Now we can prove the main theorem of the paper.

THEOREM. *The cohomological dimension* $\dim_K R$ *of the ordered set of real numbers* R *is equal to three for any commutative ring* K.

Proof. It is sufficient to prove that the sequence $0 \to T_3 \xrightarrow{d_3} T_2$ does not split. Suppose the converse, thus there exists a map $g\colon T_2 \to T_3$ such that $gd_3 = 1_{T_3}$.

Let us remark that a non-zero map $R(.,a)\otimes R(b,.) \to R(.,a')\otimes R(b',.)$ exists if and only if $a \leqslant a'$ and $b' \leqslant b$, moreover each map $R(.,a)\otimes R(b,.) \to \bigoplus_\lambda R(.,a_\lambda)\otimes R(b_\lambda,.)$ factorizes through a finite direct sum.

We denote by $\pi_p\colon T_3 \to \bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$ a projection map thus

$$\pi_p g v_{p,n}\colon R(.,p)\otimes R(s_n(p),.) \to \bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$$

and $\pi_p g v_{p,n} = 0$ because $p > r_n(p)$ for all n. Similarly we show that $\pi_p g v_{n,p} = 0$. There exists a countable set $A \subset R$ such that all maps $gv_{m,n}$ factorizes through $\bigoplus_{p\in A} \bigoplus_n$. We fix a 2-irrational element $p \in R\setminus A$, then $\pi_p g v_{m,n} = 0$ for all m, n and by the above equalities we get

$$\begin{aligned}\pi_p t_{p,n} = \pi_p g d_3 t_{p,n} &= \pi_p g(v_{n,p,n} - v_{n+1,p,n+1} - v + v_{k_n(p),n}) \\ &= \pi_p g v_{n,p,n} - \pi_p g v_{n+1,p,n+1} \\ &= g_n - g_{n+1}\end{aligned}$$

where $g_n = \pi_p g v_{n,p,n}$, $n = 0, 1, \ldots$ and $v = v_{p,n}$ or $v_{n,p}$. Maps $t_n = \pi_p t_{p,n}$ are structural injections of a direct sum $\bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$ and let π_n be a corresponding projection onto nth summand. Then we have $g_n = t_n + g_{n+1}$ and since g_0 factorizes through a finite sum then there exists such $k \geqslant 0$ that $\pi_n g_0 = 0$ for $n > k$. Since

$$g_0 = t_0 + g_1 = t_0 + t_1 + g_2 = \ldots = t_0 + \ldots + t_{k+1} + g_{k+2}$$

then $\pi_{k+1} g_0 = \pi_{k+1} t_{k+1} = 1$ because

$$\pi_{k+1} g_{k+2}\colon R(.,r_{k+2}(p))\otimes R(s_{k+2}(p),.) \to R(.,r_{k+1}(p))\otimes R(s_{k+1}(p),.)$$

is a zero map. We get a contradiction thus d_3 does not split.

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The space of maximal convex sets

by

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Abstract. This paper is devoted to the study of *hypercones* (also called "semispaces") — maximal convex sets missing a point in a vector space of arbitrary dimension over a totally ordered field. It is shown how the hypercones can be organized into a compact Hausdorff space with an intrinsic system of "convex" sets. Relationships of "convexity" to topology on the hypercone space are studied. Also metrizability, separatility, and other topological properties of the hypercone space are characterized in terms of the underlying ordered field and the dimension of the given vector space.

1. Introduction. Maximal convex sets missing a given point have arisen naturally in connection with separation properties of convex sets. In a series of lectures in 1951, T. S. Motzkin [15] described such sets in detail for 3-dimensional Euclidean space and used them in some separation theorems. Köthe [12] also makes use of the maximal convex sets missing a point — which he calls *hypercones* — in his proof of the geometric form of the Hahn–Banach Theorem. Related ideas are attributed by Kelley, Namioka, et al. [10] to Stone and Kakutani. In 1955 Preston Hammer [4] independently noted some of the elementary properties of hypercones in real vector spaces. (Although Hammer's term "semispace" is often used, Köthe's term "hypercone" seems better. There is a larger class of convex sets deserving to be called "semispaces", and the term "semispace" is inappropriate in situations as in [7] where spaces more general than vector spaces are treated.)

The first deeper results on hypercones were given in [11] by V. L. Klee, who gave (among other results) a complete characterization of hypercones in real vector spaces. In the setting of topological vector spaces, a certain class of hypercones compatible with the topological structure was studied by C. E. Moore [14]. Using his separation results, Moore achieved a new characterization of reflexive Banach spaces.

In this paper the structure of hypercones will again be investigated — not individually, but rather as a space. It will be shown that the collection of hypercones can be made into a totally disconnected compact Hausdorff space which can be endowed in a natural way with a notion of "convexity". The questions of metrizability, separability and first countability of the hypercone space will be investigated. In addition, we shall study the basic similarities and dissimilarities of its convexity structure with ordinary convexity.

The construction is not limited to the real field and will be given for vector spaces over an arbitrary totally ordered field. If the construction here seems reminiscent of the construction of the Stone space of a Boolean algebra or the maximal ideal space of a vector lattice or Banach algebra, this is no accident. All are special cases of a general procedure which is applicable in a wide range of situations (see [7]).

Since every hypercone corresponds to a total order on the underlying vector space and vice versa, the results here may also be regarded as a contribution to the study of ordering on vector spaces.

2. Basic topological properties of the structure space. Throughout this paper V will denote a vector space over a totally ordered field F. (See [3] for examples.) A subset K of V is *convex* provided $\lambda x+(1-\lambda)y$ is in K whenever x and y are in K and λ is a scalar in F with $0\leqslant\lambda\leqslant 1$. Many of the rudimentary properties of convex sets in real vector spaces are valid over any ordered field. Some of these are listed below.

(1) The empty set $\emptyset$ and the whole space are convex.

(2) Any intersection of convex sets is convex.

(3) The union of any chain (i.e., totally ordered by inclusion) of convex sets is convex.

Any subset S of V is contained in a smallest convex set called the *convex hull* of S and denoted by $\mathrm{conv}(S)$. The following useful formula is easily established over any ordered field.

(4) If A and B are convex, then

$$\mathrm{conv}(A\cup B) = \bigcup\{\mathrm{conv}(a, b)\colon a\in A, b\in B\}.$$

From (1), (3), and Zorn's Lemma, it follows that maximal convex subsets of $V\sim\{0\}$ exist.

Following Köthe [12] we shall call such sets *hypercones*. The collection of all hypercones in V will be denoted by $\Sigma(V)$ and called the *structure space* of V. This space may be topologized by the so-called "Zariski topology" in which a subbase of closed sets is given by all families of the form

$$A^* := \{H\in\Sigma(V)\colon A\subseteq H\}$$

where A is any arbitrary subset of V. Those *sets of the form $x^* := \{x\}^*$ will be called *principal *sets*. Since for any family (A_i) $i\in I$ of subset of V

$$\bigcap\{A_i^*\colon i\in I\} = (\bigcup\{A_i\colon i\in I\})^*, \tag{5}$$

every *set is an intersection of principal *sets. Thus the principal *sets are also a subbase of closed sets for the Zariski topology.

THEOREM 1. *The structure space $\Sigma(V)$ is a totally disconnected compact Hausdorff space.*

Before giving the proof, we record for future use three easy facts [4], [11], [12] about hypercones.

(6) Any hypercone H is a *cone over* O in the sense that $\lambda h\in H$ whenever $\lambda>0$ and $h\in H$.

(7) A convex set H is a hypercone iff $O\notin H$ and for each $x\neq 0$ in V, exactly one of x and $-x$ belongs to H.

(8) If H is a hypercone, then $-H := \{-h\colon h\in H\}$ is also a hypercone and $V\sim H = -H\cup\{O\}$ is convex.

Proof of Theorem 1. If H and K are distinct hypercones, then there is an $x\in H$ which is not in K. Thus $-x\in K$ by (7). Also by (7) the *sets x^* and $(-x)^*$ partition $\Sigma(V)$, so they are open as well as closed. Thus x^* and $(-x)^*$ are disjoint open sets containing H and K, respectively, establishing Hausdorff separation. It is also clear that no connected set containing H can contain K. Thus the component of H in $\Sigma(V)$ contains only H, so $\Sigma(V)$ is totally disconnected.

We now show that $\Sigma(V)$ is compact. Since the principal *sets from a subbase of closed sets in $\Sigma(V)$, it suffices by Alexander's Subbase Theorem [9] to prove that any family of principal *sets with the finite intersection property has nonempty intersection. Let $\{x^*\colon x\in A\}$ be such a family. The set A is of course some subset of V. If $O\in\mathrm{conv}(A)$, then $O\in\mathrm{conv}(E)$ for some finite subset E of A. But by assumption there is some hypercone H in the finite intersection

$$E^* = \bigcap\{x^*\colon x\in E\}.$$

But then $\mathrm{conv}(E)\subseteq H$ since H is convex and contains E. As $O\notin H$, this is a contradiction. Whence O cannot be in $\mathrm{conv}(A)$, so $\mathrm{conv}(A)$ can be extended by Zorn's Lemma to some hypercone K. Then since $A\subseteq\mathrm{conv}(A)\subseteq K$, we get

$$K\in A^* = \bigcap\{x^*\colon x\in A\}$$

as desired. ∎

It will be convenient to have an alternate description of the topology on $\Sigma(V)$. In general, if X is any set, identifying each subset of X with its characteristic function allows the power set $\mathrm{Pow}(X)$ to be considered as a product of two point discrete spaces $\{0, 1\}$ indexed by X. In the product topology thereby induced on $\mathrm{Pow}(X)$, the families of subsets of X of the form

$$[I; E] := \{A\subseteq X\colon I\subseteq A \text{ and } E\cap A = \emptyset\}$$

where I and E are finite subsets of X, constitute a base of open sets. For this reason, this topology has been called the *inclusion-exclusion topology* on $\mathrm{Pow}(X)$ [6], [7], [8].

PROPOSITION 2. *The Zariski topology on $\Sigma(V)$ coincides with the inclusion-exclusion topology on $\Sigma(V)$ as a subspace of $\mathrm{Pow}(V)$. A base of open sets in $\Sigma(V)$ is given by the collection of all *sets E^* where E is finite.*

Proof. It follows from (7) that the principal *sets are open as well as closed and in fact form a subbase for the open sets as well as for the closed sets in $\Sigma(V)$. The second assertion of the proposition follows at once from this observation. Also by (7) one sees easily that the restriction of the open set $[I; E]$ in $\mathrm{Pow}(V)$ to $\Sigma(V)$ is just $(I \cup -E)^*$. Thus on $\Sigma(V)$ the Zariski topology and the inclusion-exclusion topology have the same basic open sets. ∎

3. A closer look at hypercones. Before presenting the development over arbitrary ordered fields, it may be helpful to review the real case. In the real line, there are only two hypercones — namely, the upper and lower half lines. The corresponding structure space is thus a 2-point discrete space.

To construct a hypercone in the plane, one starts with an open halfplane with O in its boundary and then adjoins half of the bounding line. The structure space $\Sigma(R^2)$ may be visualized topologically as a circle in which each point has been split in half. Namely, with each hypercone H associates the point on the unit circle given by the unit normal to the boundary of H. Now split this point in two, one for each of the halves of the bounding line which H may contain. One may unwrap this "splintered circle" simply by pulling apart the two halves of some "split point" until the whole space lies straight. In this way one can see that $\Sigma(R^2)$ is homeomorphic to the space Y consisting of the top and bottom sides of the unit square where Y has the order topology coming from the lexicographic order restricted to Y. In fact, for any ordered field F, one can see that $\Sigma(F^2)$ carries a similar order topology by describing the hypercones in F^2 in terms of their intersections with a line missing O.

The process is analogous in R^3. A hypercone is formed by taking an open half-space and adjoining a hypercone in the bounding plane. Hence $\Sigma(R^3)$ looks like a sphere in which each point has been replaced by a copy of the "splintered circle". For d dimensions this procedure yields a natural map from the structure space $\Sigma(R^d)$ to the sphere S^{d-1} such that the inverse image of each point is $\Sigma(R^{d-1})$. Topologically this map is continuous. Indeed, for any $p \neq 0$ the inverse image of the open hemisphere

$$\{x \in S^{d-1}\colon \langle p, x\rangle < 0\}$$

is the union of all open *sets E^* where E is a finite subset of R^d with $p \in \mathrm{int\,conv}(E)$. Since $\Sigma(R^d)$ is compact and S^{d-1} is Hausdorff, the map is a quotient mapping. However, it is not a fibration in any usual sense because S^{d-1} is connected but $\Sigma(R^d)$ is totally disconnected.

The classification of hypercones by Klee [11] essentially formalizes and extends the method of construction above to arbitrary real vector spaces. In general, suppose Φ is a family of linear functionals from V to F. A total ordering $\prec$ of Φ is *admissible* if for each $x \neq 0$ in V there is a first (with respect to $\prec$) member φ_x of Φ such that $\varphi_x(x) \neq 0$. If $\prec$ is admissible, one easily sees that the set

$$H(\Phi, \prec) = \{x \in V\colon x \neq 0 \text{ and } \varphi_x(x) > 0\}$$

is a hypercone in V. In [11] Klee proved that every hypercone in a real vector space arises this way. However, his result is true only for the real field.

PROPOSITION 3. *Suppose F is an ordered field other than the reals and that $\dim_F V \geq 2$. Then there is a hypercone in V which does not have the form $H(\Phi, \prec)$ for any admissibly ordered family of functionals Φ on V.*

Proof. Clearly any hypercone in a subspace W of V can be extended to a hypercone in V. If the extension has a representation as $H(\Phi, \prec)$, then by restricting the functionals in Φ to V and keeping the same ordering, one obtains a Klee representation of the original hypercone in W. Hence it suffices to prove the proposition in the case V is 2-dimensional.

Now any hypercone $H(\Phi, \prec)$ on V must contain a line. Indeed, there will be points x and y in $H(\Phi, \prec)$ such that $\varphi_x \prec \varphi_y$. For any $\lambda \in F$, it follows that φ_x is the first functional in Φ which does not vanish at $x+\lambda y$ and that

$$\varphi_x(x+\lambda y) = \varphi_x(x) > 0 .$$

Thus the proposition will be proved if we exhibit in $F \times F$ a hypercone which contains no line.

Let Z be any bounded set of positive elements of F such that for any $z \in Z$ there is a $z' \in Z$ with $z < z'$. Then the set

$$J(Z) = \{(x, y)\colon \text{for some } \zeta \in Z,\ y < \eta x \text{ for all } \eta \in Z \text{ with } \eta > \zeta\}$$

is a convex set not containing $(0, 0)$. Given $(x, y) \neq (0, 0)$ and $\zeta < \eta < \theta$ in Z, the three relations

$$y \leq \zeta x, \quad y \geq \eta x, \quad y \leq \theta x$$

cannot all hold. From this it follows easily that either

$$(x, y) \in J(Z) \quad \text{or} \quad (-x, -y) \in J(Z),$$

so $J(Z)$ is a hypercone by (7). Since Z is bounded, $J(Z)$ can contain no vertical line. But suppose $J(Z)$ contains a line whose equation is

$$y = \sigma x + \tau .$$

Then for any $x \in F$ and all sufficiently large η in Z, we have

$$\sigma x + \tau < \eta x . \tag{9}$$

Taking $x = 0$ yields $\tau < 0$. If $\sigma < \zeta$ for some $\zeta \in Z$, then taking $x = \tau(\zeta - \sigma)^{-1} < 0$ and dividing (9) by x yields for all large $\eta > \zeta$ the contradiction $\zeta > \eta$. Thus σ is an upper bound on Z. If $\sigma > \beta$, taking $x = \tau(\beta - \sigma)^{-1} > 0$ and dividing (9) by x yields $\beta < \eta$ for all large η in Z. Thus $\sigma = \sup Z$.

As is well-known, the real field R is the only ordered field that is (relatively) complete with respect to its order. Thus since $F \neq R$, there is a bounded set Z of positive elements in F with no supremum in F. Thus $J(Z)$ contains no line for such a Z. ∎

Remark. For a nonarchimeadean ordered field F, one can also choose the set Z to be the copy of the positive integers contained in F.

4. Convexity in the structure space. It is possible to endow the structure space $\Sigma(V)$ with a system of "convex" sets whose properties to some extent parallel those of ordinary convex sets. This will be done via the notion of alignment as developed in [6], [7], [8]. Although we shall require a portion of the general theory, the emphasis will be kept on properties of the structure space.

DEFINITION. A family $\mathscr{K}$ of hypercones in $\Sigma(V)$ is *convex* provided, for any finite collection $H_1, \ldots, H_n$ of hypercones in $\mathscr{K}$, every hypercone which contains their intersection $H_1 \cap \ldots \cap H_n$ is also in $\mathscr{K}$.

The collection $\mathfrak{C}$ of all such convex families satisfies properties (1), (2), and (3) and hence is an *alignment* on $\Sigma(V)$ in the sense of [6]. Evidently every *set is $\mathfrak{C}$-convex, and it can be shown [7] that $\mathfrak{C}$ is the smallest alignment on $\Sigma(V)$ containing the *sets. Every family $\mathscr{F}$ of hypercones is included in a smallest convex family — the $\mathfrak{C}$-*hull* of $\mathscr{F}$ — which will be denoted $\mathfrak{C}(\mathscr{F})$. If $\mathscr{F}$ is a finite family, say $\mathscr{F} = \{H_1, \ldots, H_n\}$, then the $\mathfrak{C}$-hull of $\mathscr{F}$ is called a $\mathfrak{C}$-*polytope*, and directly from the definitions we have

$$\mathfrak{C}(\mathscr{F}) = (H_1 \cap \ldots \cap H_n)^*,$$

a *set. Evidently every *set is both closed and $\mathfrak{C}$-convex. A principal objective of this section is to show the converse, thereby establishing a fundamental connection between the topology and convexity on $\Sigma(V)$. Another goal is to prove the analogue of (4) for the alignment $\mathfrak{C}$. This will permit a simplification in the description of $\mathfrak{C}$-convex families.

LEMMA 4. *In the vector space V, let A be a convex cone over O which misses O, and let B be a convex set missing O. If $A \cap B = \emptyset$, there is a hypercone H at O with $A \subseteq H$ and $H \cap B = \emptyset$.*

Proof. If O belongs to $\operatorname{conv}(-B \cup A)$, then by (4) there are elements a in A, b in B, and λ in F with $0 \leqslant \lambda \leqslant 1$ such that $0 = \lambda a + (1-\lambda)(-b)$. Since O is in neither A nor B, λ can be neither 1 nor 0. Thus the above equation can be solved for b, resulting in an expression of b as a positive multiple of a. This implies that b is in the cone A, contrary to $A \cap B = \emptyset$. Hence $O \notin \operatorname{conv}(-B \cup A)$, so there is a maximal convex set H containing $\operatorname{conv}(-B \cup A)$ but not O. Clearly $A \subseteq H$. Since $-B \subseteq H$, it follows from (7) that $H \cap B = \emptyset$. ∎

LEMMA 5. *Let $\mathscr{A}$ be a nonempty closed subset of $\Sigma(V)$ and let $A = \bigcap \mathscr{A}$. Then* $\operatorname{cl} \mathfrak{C}(\mathscr{A}) = A^*$.

Proof. The closed convex family A^* obviously includes $\mathscr{A}$ and hence $\operatorname{cl} \mathfrak{C}(\mathscr{A})$. To show the reverse inclusion, consider any $H \in A^*$. Then $-H \cap A = \emptyset$ by (7). For any finite dimensional subspace L of V, define

$$\mathscr{A}_L = \{P \cap L \colon P \in \mathscr{A}\}.$$

As we have seen, $\Sigma(V)$ is compact and carries the inclusion-exclusion topology. Thus $\mathscr{A}$ is also compact in this topology. As intersection is easily seen to be continuous in the inclusion-exclusion topology (see p. 43 of [6]), it follows that $\mathscr{A}_L$ is a compact family of ordinary convex sets in $\operatorname{Pow}(V)$. Now

$$(-H \cap L) \cap (\bigcap \mathscr{A}_L) = -H \cap L \cap A = \emptyset.$$

Since L is finite dimensional and hence has finite Helly number (see p. 33 of [2]), the extension of Helly's Theorem to compact families in [8] applies. Whence some finite number of sets from $\mathscr{A}_L$ have empty intersection with $-H \cap L$. Say,

$$(-H \cap L) \cap P_1 \cap \ldots \cap P_m = \emptyset$$

where each P_i is in $\mathscr{A}$.

Now the cone $P_1 \cap \ldots \cap P_m$ is disjoint from the convex set $-H \cap L$ and hence is contained by Lemma 4 in a hypercone P_L with

$$-H \cap L \cap P_L = \emptyset. \tag{10}$$

From the construction it is clear that $P_L \in \mathfrak{C}(\mathscr{A})$.

Since $\Sigma(V)$ is compact, the net (P_L), indexed by the finite dimensional subspaces, has a subnet which converges to some hypercone $Q \in \operatorname{cl} \mathfrak{C}(\mathscr{A})$. By continuity of intersection in the inclusion-exclusion topology, in the limit (10) becomes

$$-H \cap V \cap Q = \emptyset.$$

Thus $-H$ is disjoint from Q, so by (7) we get $Q \subseteq H$. But the maximality of Q then implies $Q = H$ whence $H \in \operatorname{cl} \mathfrak{C}(\mathscr{A})$ as desired. ∎

THEOREM 6. *Every closed $\mathfrak{C}$-convex family in $\Sigma(V)$ is a *set.*

Proof. If $\mathscr{K}$ is closed and convex, then $\mathscr{K} = \operatorname{cl} \mathfrak{C}(\mathscr{K})$. Thus by Lemma 5 we have $\mathscr{K} = (\bigcap \mathscr{K})^*$. ∎

THEOREM 7. *If $\mathscr{A}$ is a closed family in $\Sigma(V)$, then the closure of $\mathfrak{C}(\mathscr{A})$ is again convex.*

Proof. As in Lemma 5, $\operatorname{cl} \mathfrak{C}(\mathscr{A}) = A^*$ which as a *set is clearly convex. ∎

These two results are by no means consequences of a general theory of convexity, as the examples in [7] and [6] show, but are measures of the good behaviour of the structure space convexity. As we shall see later, however, the closure requirement in Theorem 7 is indispensible: there are $\mathfrak{C}$-convex families whose closures are not $\mathfrak{C}$-convex. So in this sense the alignment $\mathfrak{C}$ is not as well behaved as the alignment of ordinary convex sets.

The next result is the analogue of property (4) which is known in abstract settings as "join-hull commutativity" [1], [16] or "well-fittedness" [5]. Because it forces a very strong resemblance to the geometry of ordinary convex sets, it has been a popular axiom in abstract theories of convexity, particularly among the followers of Prenowitz.

THEOREM 8. *If $\mathscr{S}$ and $\mathscr{T}$ are $\mathfrak{C}$-convex subsets of $\Sigma(V)$, then*

$$\mathfrak{C}(\mathscr{S} \cup \mathscr{T}) = \bigcup \{S \cap T)^*: S \in \mathscr{S} \text{ and } T \in \mathscr{T}\}.$$

Proof. Since $\mathfrak{C}(\mathscr{S} \cap \mathscr{T})$ clearly includes the union above, it suffices to show the reverse inclusion. If $H \in \mathfrak{C}(\mathscr{S} \cup \mathscr{T})$, then from the definition of convexity it is clear that

$$P_1 \cap \ldots \cap P_m \cap Q_1 \cap \ldots \cap Q_n \subseteq H$$

for some finite number of sets $P_i \in \mathscr{S}$ and $Q_i \in \mathscr{T}$.

Thus the cone $Q_1 \cap \ldots \cap Q_n$ is disjoint from the convex set $-H \cap P_1 \cap \ldots \cap P_m$ and hence can be extended by Lemma 4 to a hypercone T with

$$-H \cap P_1 \cap \ldots \cap P_m \cap T = \emptyset .$$

But then the cone $P_1 \cap \ldots \cap P_m$ is disjoint from the convex set $-H \cap T$ and hence can be extended to a hypercone S disjoint from $-H \cap T$. That is, $S \cap T \subseteq H$. From the construction, it is clear that $S \in \mathscr{S}$ and $T \in \mathscr{T}$ since $\mathscr{S}$ and $\mathscr{T}$ are convex. ■

An immediate consequence of the above result is that $\mathfrak{C}$-convexity, like ordinary convexity, can be defined purely in terms of segments.

COROLLARY 9. *A family $\mathscr{K} \subseteq \Sigma(V)$ is convex iff for each pair of hypercones P and Q in $\mathscr{K}$, every hypercone containing $P \cap Q$ is also in $\mathscr{K}$.*

Proof. To show $\mathscr{K}$ is convex, one must show, for any n hypercones $H_1, \ldots, H_n$ in $\mathscr{K}$, that $\mathscr{K}$ contains

$$(H_1 \cap \ldots \cap H_n)^* = \mathfrak{C}((H_1 \cap \ldots \cap H_{n-1})^* \cup \{H_n\}) .$$

By an induction using Theorem 8, it follows that the condition in the corollary is sufficient for convexity. ■

To conclude this section, we give a proof of a basic separation theorem for $\mathfrak{C}$-convex sets. In any alignment a convex set whose complement is also convex is called a *hemispace*. In the ordinary alignment on V all open (or closed) halfspaces are hemispaces as are all hypercones. In $\Sigma(V)$ the principal *sets are hemispaces since by (7) the complement of x^* is $(-x)^*$.

SEPARATION AXIOM S_4. Given two disjoint convex sets, there is a hemispace containing one which is disjoint from the other [6].

THEOREM 10. *The alignment of $\mathfrak{C}$-convex families on $\Sigma(V)$ satisfies* S_4.

Proof. By Theorem I. 9 of [6], if any pair of disjoint polytopes can be separated by a hemispace, then axiom S_4 holds for all disjoint pairs of convex sets. Since $\mathfrak{C}$-polytopes are *sets, it suffices to show that any two disjoint *sets A^* and B^* can be separated.

Without loss of generality, we may assume that A is the intersection of all hypercones in A^* and similarly for B, so A and B are convex cones over O. Now if $A \cap (-B) = \emptyset$, then by Lemma 4 there is a hypercone H with $A \subseteq H$ and $-B \cap H = \emptyset$. But then H is a hypercone in $A^* \cap B^*$ which was supposed to be empty. Thus there is some x in $A \cap (-B)$, so the hemispace x^* contains A^* but is disjoint from B^*. ■

5. Continuity of structure space convexity. When confronted with an alignment on a topological space, it is natural to ask about the continuity of the convex hull operator. Within the framework developed in [6], the upper semicontinuity of the hull operator is equivalent to a "regularity" condition on the convex sets. As we shall see in Theorem 12, the structure space alignment satisfies a strong form of this condition. The lower semicontinuity is equivalent to the validity of the so-called "Blaschke Selection Theorem": that the compact convex sets are closed in the Vietoris topology on the space of all compact subsets (see p. 64 of [2]). This does not hold for convexity in $\Sigma(V)$. For lower semicontinuity implies that the closures of convex sets are convex (see Proposition III.7 of [6]), and as shown in Theorem 13 below the closure of a $\mathfrak{C}$-convex family need not be $\mathfrak{C}$-convex.

It is convenient to give first an embedding principle that will be useful in the sequel.

If V and W are isomorphic vector spaces over F, then their structure spaces $\Sigma(V)$ and $\Sigma(W)$ are homeomorphic and isomorphic in the obvious sense. Thus for any cardinal n we may denote by $\Sigma(F, n)$ a canonical representative of the structure space of a vector space over F of algebraic dimension n.

PROPOSITION 11. *If $n < \dim_F V$, then every nonvoid open set in $\Sigma(V)$ contains a copy of $\Sigma(F, n)$. If V is infinite dimensional, this is true even for $n = \dim_F V$.*

Proof. By Proposition 2, it suffices to prove this for any nonvoid open set of the form E^* where $E \subseteq V$ is finite. If H is any hypercone in E^*, then $E \subseteq H$, so $\operatorname{conv}(E) \subseteq H$. Whence $O \notin \operatorname{conv}(E)$. Hence there is a linear functional φ on V such that $\varphi(x) > 0$ for all $x \in \operatorname{conv}(E)$. (See [17] or p. 113 of [6].)

Under the hypotheses on n, it is clear that the null space of φ contains a linear subspace L of V of dimension n. Choose a convex set M in V maximal with respect to the conditions

$$M \cap L = \emptyset \quad \text{and} \quad \operatorname{conv}(E) \subseteq M .$$

Using (7) it is easy to check for any hypercone J in $\Sigma(L)$ that $J \cup M$ is a hypercone in $\Sigma(V)$. This procedure identifies $\Sigma(L) \cong \Sigma(F, n)$ with the *set M^* in $\Sigma(V)$ in a way which preserves both topological and convex structure. By choice of M, we also have $M^* \subseteq E^*$ as desired. ■

The following result is the promised theorem on the regularity of the alignment $\mathfrak{C}$.

THEOREM 12. *Let $\mathscr{K}$ be a closed convex family in $\Sigma(V)$ and $\mathscr{U}$ an open family with $\mathscr{K} \subseteq \mathscr{U}$. Then there is a closed and open convex family $\mathscr{N}$ in $\Sigma(V)$ such that $\mathscr{K} \subseteq \mathscr{N} \subseteq \mathscr{U}$.*

Proof. By Theorem 6 $\mathscr{K} = K^*$ for some set K. If H is a hypercone not in K^*, then there is a point $x \in K$ such that $x \notin H$. By (7) $-x \in H$, so the open sets $(-x)^*$ with $x \in K$ cover the complement of $\mathscr{K}$ and in particular the complement of $\mathscr{U}$.

Since $\mathscr{U}$ is open and $\Sigma(V)$ is compact, there is a finite subcover given by, say, $(-x_1)^*$, $(-x_2)^*, \ldots, (-x_n)^*$. Now taking

$$\mathscr{N} := \{x_1, \ldots, x_n\}^* = \bigcap_{i=1}^{n} (x_i)^*$$

yields the required neighborhood of $\mathscr{K}$. ■

THEOREM 13. *If* $\dim V \geqslant 2$, *then* $\Sigma(V)$ *contains a convex set whose closure is not convex.*

Proof. By Proposition 11 it suffices to give the construction in the case $V = F \times F$. For each $r>0$ in F, define the cone

$$C_r = \{(x, y): x<0 \text{ and } ry<x\}.$$

As r decreases, the cone C_r shrinks so the corresponding *sets C_r^* increase. Thus

$$\mathscr{K} := \bigcup \{C_r^*: r>0\}$$

is the nested union of convex families and hence convex. Define points $p = (0, 1)$ and $q = (1, 1)$. For any $s \in F$ with $0<s<1$, the point $v_s := (s, 1)$ lies on the segment from p to q whereas $-v_s \in C_r$ for all $r>s$.

Thus no hypercone containing p and q can lie in any C_r^*. Thus $\{p, q\}^*$ is an open family disjoint from $\mathscr{K}$ and hence from $\mathrm{cl}\,\mathscr{K}$. But $\{p, q\}^*$ is not empty, since the segment from p to q is a convex set missing zero which can be extended to a hypercone containing p and q. Thus $\mathrm{cl}\,\mathscr{K}$ is not all of $\Sigma(V)$.

Now the hypercone

$$H = \{(x, y): x<0 \text{ or both } x = 0 \text{ and } y>0\}$$

includes the cone C_1 and hence belongs to $\mathscr{K}$. We will show that $-H \in \mathrm{cl}\,\mathscr{K}$.

Given a finite subset E of $-H$, let E' consist of all points (x, y) in E with $y>0$. By definition of H, each of the ratios x/y where $(x, y) \in E'$ is strictly positive, so the minimum such ratio is also strictly positive.

Let r be a positive element of F strictly less than this minimum. Take $r = 1$ if $E' = \emptyset$. We claim $ry<x$ for all $(x, y) \in E$. This holds by choice of r if $y>0$. If $y<0$, this holds trivially since $x \geqslant 0$ for all $(x, y) \in E \subseteq -H$. It follows that $ry<x$ for any (x, y) in $K := \mathrm{conv}(E \cup C_r)$. Hence $(0, 0) \notin K$, so K may be extended to a hypercone. This hypercone belongs to C_r^* and hence to $\mathscr{K}$ and also belongs to E^*. Thus every basic neighborhood of $-H$ in $\Sigma(V)$ contains an element of $\mathscr{K}$, so $-H \in \mathrm{cl}\,\mathscr{K}$ as desired.

But $\mathfrak{C}(-H, H) = \emptyset^* = \Sigma(V)$, so since $\mathrm{cl}\,\mathscr{K}$ is not all of $\Sigma(V)$, it cannot be convex. ■

Visualizing the above construction in the "splintered circle" model of $\Sigma(F^2)$, one sees that the set $\mathscr{K}$ corresponds to a semicircle, closed at one end but open at the other. In taking the closure, one obtains a pair of "antipodal" hypercones whose $\mathfrak{C}$-hull is the entire circle. Using the full strength of Proposition 11, one obtains the following stronger corollary.

COROLLARY 14. *If* $\dim_F V \geqslant 3$, *then every open set in* $\Sigma(V)$ *contains a* $\mathfrak{C}$-*convex set whose closure is not* $\mathfrak{C}$-*convex.*

6. Metrizability and G_δ points. In the last two sections, specific properties of the underlying ordered field played no role, the arguments being the same in the general case as over the real field. However, in examining more special topological properties of the structure space, the nature of the underlying field becomes an essential factor. In this section we shall determine criteria for the metrizability, first countability, and separability of $\Sigma(V)$. We begin with a characterization of G_δ points in $\Sigma(V)$.

LEMMA 15. *A hypercone H in $\Sigma(V)$ is a G_δ point in $\Sigma(V)$ iff, for some countable subset C of V, H consists of all positive multiples of points in* $\mathrm{conv}(C)$.

Proof. By Proposition 2 the families E^* where E is finite form a base of open sets in $\Sigma(V)$, so H is a G_δ iff there is a sequence (E_n) of finite sets such that H is the only hypercone in

$$\bigcap_{n=1}^{\infty} (E_n)^*.$$

That is, H is the only hypercone containing the countable set $C = E_1 \cup E_2 \cup \ldots$ Let K be the smallest cone over O containing C. If $x \neq 0$ is not in K, then applying Lemma 4 with $A = K$ and $B = \{x\}$, we obtain a hypercone missing x but containing K and hence C. Thus K is the intersection of the hypercones containing C, so $K = H$ iff H is the only hypercone containing C. Since K clearly consists of all positive multiples of points in $\mathrm{conv}(C)$, the lemma is proved. ■

If the dimension of V is uncountable, the criterion in the lemma can never be satisfied. Indeed, for any countable set C with $O \notin \mathrm{conv}(C)$, there is a point p not in the linear span of C. Thus O belongs to neither of the sets $\mathrm{conv}(C \cup p)$ and $\mathrm{conv}(C \cup -p)$ which may be expanded to two necessarily distinct hypercones containing C.

THEOREM 16. *If* $\dim V \geqslant 2$, *then* $\Sigma(V)$ *is metrizable if and only if V is a countable set.*

Proof. If V is countable, then V has only countably many finite subsets. Hence the open families E^*, where E is a finite subset of V, form a countable base for $\Sigma(V)$. Whence $\Sigma(V)$ is metrizable [9].

Suppose $\Sigma(V)$ is metrizable. Then each point of $\Sigma(V)$ is a G_δ, so the dimension of V must be countable. To show the denumerability of V, it thus suffices to prove that the scalar field is countable. Since $\Sigma(V)$ is metrizable and compact, every base of open sets contains a countable subfamily which also forms a base. Thus there is a countable collection $\{E_n: n = 1, 2, \ldots\}$ of finite subsets of $V \sim \{0\}$ such that $\{E_n^*: n = 1, 2, \ldots\}$ is a base of open sets in $\Sigma(V)$. Let

$$C = E_1 \cup E_2 \cup \ldots$$

Now in defiance of our desire, suppose F is uncountable. Then every line in V is also uncountable. If L is any line in V not passing through O, then the lines

$L_x = \{rx: r \in F\}$ where x is in L have only O in common and are uncountable in number. Since C is countable, there is some L_p that contains no point of C. Extend p to a vector space basis for V. Since V is of countable dimension, this basis can be enumerated: $p, b_1, b_2, \ldots$ Let Φ_i be the co-ordinate functional corresponding to b_i, and define:

$$K = \{x \in V: \text{ for some } n,\ \Phi_n(x)>0 \text{ and } \Phi_i(x) = 0 \text{ for all } i<n\}.$$

Let $H^+ = K \cup \{rp: r>0\}$ and $H^- = K \cup \{rp: r<0\}$. Then H^+ and H^- are hypercones. Furthermore, H^+ and H^- contain exactly the same points of C. Thus it is impossible to separate H^+ and H^- by open sets from the basis $\{E_n^*: n = 1, 2, \ldots\}$. This is a contradiction as the topology on $\Sigma(V)$ is Hausdorff. ■

The next result is a characterization of those structure spaces which satisfy the first axiom of countability. An ordered field F is called *completely sequential* [8] provided every nonempty subset S of F contains a sequence $s_n \in S$ such that for each $s \in S$ there is an n with $s \leqslant s_n$. Equivalently, F is completely sequential iff the order completion of F (which is not a field) is first countable in the order topology. Clearly any ordered field with a countable order dense subset is completely sequential, but there are completely sequential fields that are not order separable [8].

THEOREM 17. *If* $\dim V \geqslant 2$, *then* $\Sigma(V)$ *is first countable if and only if F is completely sequential and the dimension of V is at most countable.*

Proof. Suppose F is completely sequential and $\dim V$ is countable. By the main theorem of [8], these hypotheses on F and V imply that any hemispace in V is the convex hull of a countable set. In particular, every hypercone in V satisfies the G_δ criterion of Lemma 15 since by (8) all hypercones are hemispaces. But in a compact Hausdorff space, any G_δ point has a countable neighborhood base. Thus first countability follows for $\Sigma(V)$.

In showing the converse, we may rule out the case that $\dim V$ is uncountable since no point of $\Sigma(V)$ is a G_δ point in that case. Suppose now that F is not completely sequential. We shall exhibit a non-G_δ point in $\Sigma(F^2)$. As $\Sigma(F^2)$ can be embedded in $\Sigma(V)$ by Proposition 11, this will finish the proof.

If F is not completely sequential, there is a bounded set Z of positive elements of F with no cofinal increasing subsequence. Consider the hypercone $J(Z)$ as defined in the proof of Proposition 3.

Suppose C is any countable subset of $J(Z)$. For each point $c = (c_1, c_2)$ in C, there is a $\zeta_c \in Z$ such that $c_2 < \eta c_1$ for all $\eta > \zeta_c$ in Z. Since Z has no countable cofinal subset, there is a $\beta \in Z$ such that $\beta > \zeta_c$ for all c in C. Thus $c_2 < \beta c_1$ for each (c_1, c_2) in C. It follows that if (x, y) is a positive multiple of a point in $\mathrm{conv}(C)$, then also

$$y < \beta x.$$

Clearly the point $p = (1/\beta, 1)$ does not satisfy this inequality. However, for any $\eta > \beta$ in Z, we have $1 < \eta/\beta$, so p belongs to $J(Z)$. Thus $J(Z)$ is not generated as a cone by any countable set, and therefore by Lemma 15 $J(Z)$ cannot be a G_δ point in $\Sigma(V)$. ■

The separability of $\Sigma(V)$ can also be characterized in terms of simple properties of V and F. However, it is also equivalent to the separability of the algebraic dual V' of V. (Given F with its order topology, V' has the topology of pointwise convergence as a space of functions from V into F.)

THEOREM 18. *Suppose* $\dim V \geqslant 2$. *Then the following are equivalent:*

(i) $\Sigma(V)$ *is separable,*

(ii) *the dual V' of V is separable,*

(iii) *F is order separable and $\dim V \leqslant \mathfrak{c}$, the cardinality of the reals.*

Proof. (i)$\to$(iii) Let $\mathscr{C}$ be a countable family of hypercones which are dense in $\Sigma(V)$. Choose a line L in V with parametric form $\{tp+q: t \in F\}$ for some nonzero p and q in V. For each $H \in \mathscr{C}$, define

$$\sigma(H) = \sup\{t \in F: tp+q \in H\}$$

where the supremum is to be taken in the order completion $\hat{F}$ of F. Then $D = \{\sigma(H): H \in \mathscr{C}\}$ is countable; we claim it is order dense in $\hat{F}$. Any open set in $\hat{F}$ contains an interval of the form $[a, b]$ where $a<b$ are in F. Now $\{-bp-q, ap+q\}^*$ is open in $\Sigma(V)$ and nonvoid. Hence it contains some $H \in \mathscr{C}$. For this H we have $a \leqslant \sigma(H) \leqslant b$ as desired. Now for each pair $c<d$ in D, choose some $t \in F$ with $c<t<d$. The set of all such t is then countable and order dense in F. Thus F is order separable.

Now let B be a basis for V. Associate with each $p \in B$, the subset

$$\mathscr{C}_p = \{H \in \mathscr{C}: p \in H\}.$$

If $q \in B$ is different from p, then the segment from p to $-q$ misses O and thus lies in a hypercone. Hence the open set $\{p, -q\}^*$ is nonvoid and thus contains some $H \in \mathscr{C}$. It is clear, then, that $H \in \mathscr{C}_p$ but $H \notin \mathscr{C}_q$. Therefore the map $p \to \mathscr{C}_p$ is an injection of B into $\mathrm{Pow}(\mathscr{C})$ which has cardinality $\mathfrak{c}$ since $\mathscr{C}$ is countable. Hence $\dim V \leqslant \mathfrak{c}$ as desired.

(iii)$\to$(ii) If $\dim V$ is finite, the separability of F immediately implies that of V'. In case $\dim V$ is infinite, we shall use a modification of the standard proof that the real-valued functions on the unit interval are separable.

For $\dim V$ infinite, choose a subset B of $[0, 1)$ which contains the rational points and has cardinality $|B| = \dim V$. We may identify V with the space of functions $f: B \to F$ which vanish on all but finitely many points. The dual V' is then the space of all functions $g: B \to F$ with the pairing

$$\langle g, f \rangle = \sum_{b \in B} g(b) f(b)$$

where the sum is defined, as it is only a finite sum if $f \in V$. Let D be a countable subset of F which is dense in F in the order topology. For each finite partition of B by rational points

$$0 = x_1 < x_2 < \ldots < x_n$$

and values $r_1, \ldots, r_n$ in D, consider the function $h: B \to F$ which assigns the constant value r_i to all points in $[x_i, x_{i+1})$ for $i = 1, \ldots, n$ (where we set $x_{n+1} = 1$). The set C of all such functions is clearly countable. It is also easily checked that C is dense in V' under the topology of pointwise convergence on B. But pointwise convergence on B implies pointwise convergence on V since B is a basis for V and the elements of V' are linear on V.

(ii)→(i) Given a countable dense subset C of V', choose for each $h \in C$ a hypercone H_h which contains the halfspace $\{x \in V: h(x) > 0\}$. The collection $\mathscr{C}$ of the hypercones so chosen is countable, and we claim it is dense in $\Sigma(V)$.

Consider any nonempty open set E^* where E is finite. Evidently $F^* \neq \emptyset$ iff $O \notin \mathrm{conv}(E)$, so by the basic separation theorem (see [17] or p. 113 of [6]) there is a linear functional g on V such that $g(x) \geqslant 1$ for each $x \in \mathrm{conv}(E)$. Since C is dense in V', there is an h in C such that $|g(e) - h(e)| < \frac{1}{2}$ for each $e \in E$. Thus $E^* \subseteq H_n$, so $H_h \in E^*$ as desired. ■

It should probably be remarked that both metrizability and first countability of $\Sigma(V)$ are strictly stronger than metrizability and first countability in V'. The dual of any finite dimensional vector space over an order separable field is metrizable, although the field need not be countable — take the reals, for example. If F is a first countable ordered field, the dual of any finite dimensional vector space over F, as a product of a finite number of copies of F, is also first countable. But such a field F need not be completely sequential. To construct such a field, start with any hyperreal residue class field K as in [3] (see pages 171–188). Such a field cannot be completely sequential. (In [3] compare Theorem 13.8 with the remarks on p. 188.) Now form F by adjoining to K any transcendental τ larger than all elements of K. Any point $p \in F$ is then the intersection of the sequence of intervals $(p - \tau^{-n}, p + \tau^{-n})$.

7. Concluding look at the reals and rationals. It seems appropriate to end with some observations on the real and rational structure spaces $\Sigma(R^d)$ and $\Sigma(Q^d)$ for d finite. We know that $\Sigma(Q^d)$ is compact, metric and totally disconnected. As $\Sigma(V)$ is obviously dense-in-itself when $\dim V \geqslant 2$, it follows from a well-known result [13] that $\Sigma(Q^d)$ is homeomorphic to the Cantor set for all finite $d \geqslant 2$. These spaces can, however, be distinguished on the basis of their convexity. Analogues of the classical theorems of Carathéodory and Helly for ordinary convex sets [2] can be proved for $\Sigma(V)$. As shown in [7] the Carathéodory and Helly numbers of $\Sigma(V)$ are d and $d+1$, respectively.

The spaces $\Sigma(R^d)$ are compact, first countable, and separable but not metric if $d \geqslant 2$. However, it is possible to distinguish $\Sigma(R^3)$ from $\Sigma(R^2)$ topologically. Since $\Sigma(R^2)$ is separable and carries an order topology, one can show that any discrete subspace of $\Sigma(R^2)$ is at most countable. (This can then be used to show that every subspace is separable.) But $\Sigma(R^3)$ contains an uncountable discrete subspace. Indeed, let T be a circle in R^3 contained in a plane away from the origin. Then for each $t \in T$ the convex set

$$\mathrm{conv}(T \sim \{t\} \cup \{-t\})$$

misses the origin and hence can be extended to a hypercone H_t. Let J be the (uncountable) collection of these hypercones. Then J is discrete since

$$J \cap (-t)^* = \{H_t\} \quad \text{for each } t.$$

It is unknown, whether or not one must resort to convexity to distinguish the higher structure spaces $\Sigma(R^d)$.

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