By Proposition 1 from [7],

$$I(K, w(\mathscr{F}_1)\times w(\mathscr{F}_2))\leqslant I(v(\mathscr{F}_1)-X_1, w(\mathscr{F}_1))+I(v(\mathscr{F}_2), w(\mathscr{F}_2)).$$

It is easy to see that (Theorem 3.2 and Theorem 1.7 from [5]) the right part of this inequality is equal to $R\text{-}I(X_1, w(\mathscr{F}_1))+I(X_2, w(\mathscr{F}_2))$. Finally, it is easy to see that Theorem 3.2 completes the proof.

4.6. Corollary. *If X_1 is not realcompact and X_2 is realcompact, then*

$$R\text{-}I(X_1\times X_2, \beta X_1\times\beta X_2)\leqslant R\text{-}\mathrm{Ind}_0 X_1+\mathrm{Ind}_0 X_2.$$

Furthermore, if $X_1\times X_2$ is z-embedded in $\beta X_1\times\beta X_2$, then

$$R\text{-}\mathrm{Ind}_0(X_1\times X_2)\leqslant R\text{-}\mathrm{Ind}_0 X_1+\mathrm{Ind}_0 X_2.$$

It should be observed that the corresponding statements (Propositions 4.4, 4.5 and Corollary 4.6) hold also for covering realcompactness degree.

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The cohomological dimension of the ordered set of real numbers equals three

by

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Abstract. The purpose of the paper is to show that the cohomological dimension of the ordered set of real numbers equals three. An appropriate resolution is constructed.

We preserve the terminology and the notation of [1].

Let C be a small K-category where K denotes a commutative ring, then $C^e = C^*\otimes_K C$ is an enveloping category of C and $\mathrm{Hom}_C(= C$ for abbreviation) is a K-functor $C^e\to K\text{-Mod}$. The cohomological dimension $\dim_K C$ is defined as homological (projective) dimension of C in the category of K-functors $K\text{-Mod}^{C^e}$.

Any partially ordered set π may be viewed as a small category with a set of objects π and a unique map $x\to y$ for any $x\leqslant y$ in π. $\dim_K K\pi$ is denoted by $\dim_K\pi$, where $K\pi$ is a K-category generated by π. Let R denote the ordered set of real numbers.

The purpose of the present paper is to show that $\dim_K R = 3$ for any commutative ring K. We construct a particular projective resolution of R. In [1] Mitchell proved that $2\leqslant\dim_K R\leqslant 3$ assuming continuum hypothesis and expected this dimension to be 3; he proved even more, that $\dim_K R\leqslant n+2$ if $|R| = \aleph_n$.

I like to thank Dr. Andrzej Prószyński for correcting a detail of the proof.

1. We denote by $R(x, y)$ for $x, y\in R$ a K-free generator of $\mathrm{Hom}_{KR}(x, y)$ (i.e. a unique map $x\to y$ of R) if $x\leqslant y$ and zero in the opposite case. $\otimes$ means $\otimes_K$. We remind that $R(., a)\otimes R(b, .)$ denotes a K-functor $R^e\to K\text{-Mod}$ which is represented by the object (a, b) of R^e. It associates with an object (x, y) of R^e the free K-module on $R(x, a)\otimes R(b, y)$ if $x\leqslant a$, $b\leqslant y$ and zero in the opposite case. Functors $R(., a)\otimes R(b, .)$ are projective in the category $(K\text{-Mod})^{R^e}$ of K-functors.

We denote by Q the ordered set of 2-rational numbers, i.e., numbers of the form $m/2^n$ for some $n = 0, 1, \ldots$ and some integer m. We define a projective resolution $0\to P_2\xrightarrow{d_2} P_1\xrightarrow{d_1} P_0\xrightarrow{\varepsilon} Q\to 0$ of the functor Q in the category $(K\text{-Mod})^{Q^e}$ as follows

$$P_0 = \bigoplus_{a\in Q} Q(., a)\otimes Q(a, .),$$

$$P_1 = P_2 = \bigoplus_{n=0}^{\infty}\bigoplus_{m=-\infty}^{\infty} Q(., m/2^n)\otimes Q((m+1)/2^n, .)$$

and denote by t_a^0, $t_{m,n}^1$, $t_{m,n}^2$ the structural injections of direct summands. The maps ε, d_1, d_2 are defined by

$$(\varepsilon t_a^0)(x,y)\big(Q(x,a)\otimes Q(a,y)\big) = Q(x,y),$$
$$d_1 t_{m,n}^1 = t_{(m+1)/2^n}^0 - t_{m/2^n}^0,$$
$$d_2 t_{m,n}^2 = t_{m,n}^1 - t_{2m,n+1}^1 - t_{2m+1,n+1}^1.$$

Then we have

$$d_1 d_2 t_{m,n}^2 = d_1(t_{m,n}^1 - t_{2m,n+1}^1 - t_{2m+1,n+1}^1)$$
$$= t_{(m+1)/2^n}^0 - t_{m/2^n}^0 - t_{(2m+1)/2^{n+1}}^0 + t_{2m/2^{n+1}}^0 - t_{(2m+2)/2^{n+1}}^0 + t_{(2m+1)/2^{n+1}}^0 = 0$$

and

$$\varepsilon(x,y)d_1(x,y)\,Q(x,m/2^n)\otimes Q((m+1)/2^n,y)$$
$$= \varepsilon(x,y)[Q(x,(m+1)/2^n)\otimes Q((m+1)/2^n,y) - Q(x,m/2^n)\otimes Q(m/2^n,y)]$$
$$= Q(x,y) - Q(x,y) = 0.$$

Since $\operatorname{Ker}\varepsilon(x,y)$ is generated by elements of the form $\gamma = Q(x,b)\otimes Q(b,y) - Q(x,a)\otimes Q(a,y)$ with $a<b$ then if $a = k/2^n$, $b = l/2^n$ we get

$$\gamma = \sum_{i=k}^{l-1} d_1\, Q(x, i/2^n)\otimes Q((i+1)/2^n, y) \in \operatorname{Im} d_1(x,y).$$

Let us assume that $\gamma_1 \in \operatorname{Ker} d_1(x,y)$ for $x<y$ (in the case $x \geqslant y$ we have $P_1(x,y) = 0$) then

$$\gamma_1 = \sum_{m,n} A_{m,n}\, Q(x,m/2^n)\otimes Q((m+1)/2^n,y)$$

with $A_{m,n}\in K$ and for $x\leqslant m/2^n<(m+1)/2^n\leqslant y$ we have

$$Q(x,m/2^n)\otimes Q((m+1)/2^n,y)$$
$$\equiv Q(x,2m/2^{n+1})\otimes Q((2m+1)/2^{n+1},y) + Q(x,(2m+1)/2^{n+1})\otimes Q((2m+2)/2^{n+1},y)$$

modulo $\operatorname{Im} d_2(x,y)$. Then for sufficiently large n we have

$$\gamma_1 \equiv \sum_{k=j}^{l} A'_k\, Q(x,k/2^n)\otimes Q((k+1)/2^n,y)$$

and $d_1\gamma_1 = 0$ implies

$$0 = A'_l\, Q(x,(l+1)/2^n)\otimes Q((l+1)/2^n,y) + \ldots$$

and the terms omitted do not contain the generator $Q(x,(l+1)/2^n)\otimes Q((l+1)/2^n,y)$. Thus $A'_l = 0$ and so all A'_k are zero. Consequently $\gamma_1\in\operatorname{Im} d_2(x,y)$ and the sequence

$$0\to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} Q \to 0$$

is a projective resolution.

It is easy to see that if we change in this resolution functors $Q(.,.)$ into $R(.,.)$ also in terms P_0, P_1, P_2 then we get a sequence $0\to\bar{P}_2\to\bar{P}_1\to\bar{P}_0\to R\to 0$ of

functors on R^e, which becomes exact on all objects (x,y) of R^e with $x\neq y$; it is not true for objects (x,x).

2. Letters p, p', p_i, ... will denote real numbers which are not 2-rational and

$$p = c_0(p) + c_1(p)/2 + c_2(p)/2^2 + \ldots,$$

where $c_n(p) = 0, 1$ for $n = 1, 2, \ldots$ and $c_0(p) = [p]$. For $n = 0, 1, \ldots$ let $k_n(p)$ be such an integer that $k_n(p)/2^n < p < (k_n(p)+1)/2^n$. We put $r_n(p) = k_n(p)/2^n$, $s_n(p) = (k_n(p)+1)/2^n$ then $r_n(p)<p<s_n(p)$. We have

$$r_n(p) = r_{n+1}(p) \Leftrightarrow s_{n+1}(p) < s_n(p) \Leftrightarrow c_{n+1}(p) = 0$$

and similarly for $s_n(p) = s_{n+1}(p)$.

We define zero-dimensional component of a projective resolution

$$0\to T_3 \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} R \to 0$$

of the functor R in the category $(K\text{-Mod})^{R^e}$ as

$$T_0 = \bigoplus_{a\in R} R(.,a)\otimes R(a,.)$$

and let $u_a\colon R(.,a)\otimes R(a,.)\to T_0$ be the structural injection. T_0 is projective and an epimorphic map $\varepsilon\colon T_0\to R$ is defined by

$$(\varepsilon u_a)(x,y)R(x,a)\otimes R(a,y) = R(x,y).$$

It is easy to see that $\operatorname{Ker}\varepsilon(x,y)$ is zero for $x\geqslant y$ and is generated by elements of the form $R(x,b)\otimes R(b,y) - R(x,a)\otimes R(a,y)$ for $x\leqslant a<b\leqslant y$.

We define one-dimensional component of a resolution as

$$T_1 = \bigoplus_p [R(.,p)\otimes R(s_0(p),.)\oplus R(.,r_0(p))\otimes R(p,.)\oplus$$
$$\oplus \bigoplus_{\substack{n\geqslant 1\\ c_n(p)=0}} R(.,p)\otimes R(s_n(p),.)\oplus \bigoplus_{\substack{n\geqslant 1\\ c_n(p)=1}} R(.,r_n(p))\otimes R(p,.)]\oplus$$
$$\oplus \bigoplus_{n=0}^{\infty} \bigoplus_{m=-\infty}^{+\infty} R(.,m/2^n)\otimes R((m+1)/2^n,.)$$

and let $w_{p,0}$, $w_{0,p}$, $w_{p,n}$, $w_{n,p}$, $w_{m,n}$ be the structural injections. Clearly $T_1(x,y) = 0$ for $x\geqslant y$ and T_1 is projective.

The differential $d_1\colon T_1\to T_0$ is defined for $x\leqslant y$ by

$$d_1(x,y)R(x,a)\otimes R(b,y) = R(x,b)\otimes R(b,y) - R(x,a)\otimes R(a,y)$$

with $a, b\in R$ such that $x\leqslant a<b\leqslant y$.

It is clear that $\varepsilon d_1 = 0$. To see that $\operatorname{Im} d_1(x,y) = \operatorname{Ker}\varepsilon(x,y)$ let $x\leqslant a<b\leqslant y$ for some $a, b\in R$. Let us assume at first that both a, b are 2-irrational. Then there exists such $n>0$ that $a<s_n(a)\leqslant r_n(b)<b$ and consequently

$$\begin{aligned} R(x,b)\otimes R(b,y)-R(x,a)\otimes R(a,y) &= R(x,b)\otimes R(b,y)-R(x,r_n(b))\otimes R(r_n(b),y)+\\ &\quad +R(x,s_n(a))\otimes R(s_n(a),y)-R(x,a)\otimes R(a,y)+\\ &\quad +R(x,r_n(b))\otimes R(r_n(b),y)-R(x,s_n(a))\otimes R(s_n(a),y)\,. \end{aligned}$$

The first and the second terms are clearly in $\operatorname{Im} d_1(x,y)$. The third one is equal to $\sum_i d_1(x,y)R(x,i/2^n)\otimes R((i+1)/2^n,y)$ where $k_n(a)+1\leqslant i<k_n(b)$, then it belongs to $\operatorname{Im} d_1(x,y)$ too.

If one of a, b or both are 2-rational then the discussion is similar. We have proved that the sequence $T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} R \to 0$ is exact.

3. We define two dimensional components of a resolution as

$$T_2 = \bigoplus_p \Big[\bigoplus_{\substack{n\geqslant 1\\ c_n(p)=0}} R(.,p)\otimes R(s_n(p),.)\oplus \bigoplus_{\substack{n\geqslant 1\\ c_n(p)=1}} R(.,r_n(p))\otimes R(p,.)\oplus$$
$$\oplus \bigoplus_{n=0}^{\infty} R(.,r_n(p))\otimes R(s_n(p),.)\Big]\oplus \bigoplus_{n=0}^{\infty}\bigoplus_{m=-\infty}^{+\infty} R(.,m/2^n)\otimes R((m+1)/2^n,.)$$

and denote by $v_{p,n}$, $v_{n,p}$, $v_{n,p,n}$, $v_{m,n}$ the structural injections. For any n such that $c_n(p)=0$ we denote by $n^*>n$ the least index such that $c_{n^*}(p)=0$ and similarly for the case $c_n(p)=1$ and n^{**}.

Clearly $T_2(x,y)=0$ for $x\geqslant y$ and T_2 is projective.

The differential $d_2\colon T_2\to T_1$ is defined by

$$d_2 v_{p,n} = w_{p,n}-w_{p,n^*}-w_{s_n^*(p),s_n(p)}\,,$$
$$d_2 v_{n,p} = w_{n,p}-w_{n^{**},p}-w_{r_n(p),r_n^{**}(p)}\,,$$
$$d_2 v_{n,p,n} = w_{p,n}+w_{n,p}-w_{r_n(p),s_n(p)}\,,$$
$$d_2 v_{m,n} = w_{m,n}-w_{2m,n+1}-w_{2m+1,n+1}\,,$$

where we omit all "restriction" functors (f.i. w_{p,n^*} is in fact the composition $R(.,p)\otimes R(s_n(p),.)\to R(.,p)\otimes R(s_{n^*}(p),.)\xrightarrow{w_{p,n^*}} T_1$ where the first map is induced by the map $s_{n^*}(p)\to s_n(p)$) and for a fixed $n\geqslant 0$ and $m_1<m_2$ we write

$$w_{m_1/2^n,m_2/2^n}(x,y) = \sum_i w_{i,n}(x,y),\qquad m_1\leqslant i<m_2\,.$$

To simplify notation we use symbol $v_{p,n}$ (resp. $v_{n,p}$, $w_{p,n}$, $w_{n,p}$) also in the case $c_n(p)=1$ and we mean by it the structural injection $v_{p,k}\colon R(.,p)\otimes R(s_k(p),.)\to T_2$ where k is the least integer such that $s_k(p)=s_n(p)$ (we could write $v_{p,s_n(p)}$ instead of $v_{p,n}$).

It is quite easy to check that $d_1d_2=0$, f.i. if we denote for fixed x, y the generator $R(x,a)\otimes R(a,y)$ by $f(a)$, then

$$\begin{aligned} d_1 d_2 v_{p,n}(x,y)R(x,p)\otimes R(s_n(p),y) &= d_1(x,y)\big[R(x,p)\otimes R(s_n(p),y)-R(x,p)\otimes R(s_{n^*}(p),y)-\\ &\quad -\sum_i R(x,i/2^n)\otimes R((i+1)/2^n,y)\big]\\ &= f(s_n(p))-f(p)-f(s_{n^*}(p))+f(p)-f(s_n(p))+f(s_{n^*}(p)) = 0\,, \end{aligned}$$

where $k_{n^*}(p)+1\leqslant i<2^{n^*-n}[k_n(p)+1]$ and similarly for other terms.

We show that $\operatorname{Ker} d_1(x,y)=\operatorname{Im} d_2(x,y)$. This is obvious for $x\geqslant y$, then let us assume that $x<y$ and $\gamma\in\operatorname{Ker} d_1(x,y)$. Thus γ is a linear form in generators

$$\begin{aligned} &R(x,p)\otimes R(s_n(p),y) &&\text{for}\quad x\leqslant p<s_n(p)\leqslant y\,,\\ &R(x,r_n(p))\otimes R(p,y) &&\text{for}\quad x\leqslant r_n(p)<p\leqslant y\,,\\ &R(x,m/2^n)\otimes R((m+1)/2^n,y) &&\text{for}\quad x\leqslant m/2^n<(m+1)/2^n\leqslant y \end{aligned}$$

for some finite number of p's, n's and m's. If $p<y$ and $c_n(p)=1$, then for some $j>n$ such that $c_j(p)=0$ we have $p<s_j(p)<y$. Thus

$$R(x,r_n(p))\otimes R(p,y)\equiv R(x,r_j(p))\otimes R(p,y)\equiv -R(x,p)\otimes R(s_j(p),y)\,,$$

where $\equiv$ denotes the congruence modulo $\operatorname{Im} d_2(x,y)+\overline{P}_1(x,y)$, $\overline{P}_1(x,y)$ being the last group of terms in $T_1(x,y)$. In γ several terms with fixed p and different n may appear and we can choose one sufficiently large j for such n.

If some of p's is equal to y then for some sufficiently large $l>n$ we have

$$R(x,r_n(y))\otimes R(y,y)\equiv R(x,r_l(y))\otimes R(y,y)$$

for all needed n. Also for terms of the first type we can choose sufficiently large $k>n$ such that $c_k(p)=1$ and

$$R(x,p)\otimes R(s_n(p),y)\equiv R(x,p)\otimes R(s_k(p),y)$$

for all needed n. Thus we have a congruence

$$\gamma\equiv\sum_{i=1}^{t} A_i R(x,p_i)\otimes R(s_{j_i}(p_i),y)+AR(x,r_l(y))\otimes R(y,y)$$

where $x\leqslant p_1<\ldots<p_t<y$ and $A=0$ if y is 2-rational. Since $\equiv$ is mapped by d_1 into a congruence modulo $\overline{P}_0(x,y)$, then

$$0=d_1(x,y)\gamma\equiv\sum_i A_i\big(f(s_j(p_i))-f(p_i)\big)+A\big(f(y)-f(r_l(y))\big)$$

which implies $A_1=\ldots=A_t=A=0$ and $\gamma\in\operatorname{Im} d_2(x,y)+\overline{P}_1(x,y)$. We know that

$$\operatorname{Ker}\big(d_1(x,y)\,|\,\overline{P}_1(x,y)\big)=d_2(x,y)\overline{P}_2(x,y)$$

by the remark at the end of Part 1. Thus we have proved that the sequence $T_2\xrightarrow{d_2}T_1\xrightarrow{d_1}T_0$ is exact.

4. We define three-dimensional component of a resolution as

$$T_3 = \bigoplus_p \bigoplus_{n=0}^{\infty} R(., r_n(p)) \otimes R(s_n(p), .)$$

and denote by $t_{p,n}$ the structural injections. Clearly $T_3(x, y) = 0$ for $x \geqslant y$ and T_3 is projective.

The differential $d_3\colon T_3 \to T_2$ is defined by

$$d_3 t_{p,n} = v_{n,p,n} - v_{n+1,p,n+1} - v + v_{k_n(p),n}$$

where $v = v_{p,n}$ if $c_{n+1}(p) = 0$ and $v = v_{n,p}$ if $c_{n+1}(p) = 1$ and we omit all "restriction" functors.

If $c_{n+1}(p) = 0$, then

$$\begin{aligned} d_2 d_3 t_{p,n} &= d_2(v_{n,p,n} - v_{n+1,p,n+1} - v_{p,n} + v_{k_n(p),n}) \\ &= w_{n,p} + w_{p,n} - w_{r_n(p),s_n(p)} - w_{n+1,p} - w_{p,n+1} + w_{r_{n+1}(p),s_{n+1}(p)} - \\ &\quad - w_{p,n} + w_{p,n^*} + w_{s_n^*(p),s_n(p)} + w_{k_n(p),n} - w_{2k_n(p),n+1} - w_{2k_n(p)+1,n+1} = 0 \end{aligned}$$

because $n^* = n+1$, $w_{n+1,p} = w_{n,p}$. If $c_{n+1}(p) = 1$ the computation is similar. We have proved that $d_2 d_3 = 0$.

We show that $\operatorname{Ker} d_2(x, y) = \operatorname{Im} d_3(x, y)$. This is obvious for $x \geqslant y$ then let us assume that $x < y$ and $\gamma \in \operatorname{Ker} d_2(x, y)$. Then γ is a linear form in generators

$$\begin{aligned} &R(x, p) \otimes R(s_n(p), y) && \text{for} \quad x \leqslant p < s_n(p) \leqslant y, \\ &R(x, r_n(p)) \otimes R(p, y) && \text{for} \quad x \leqslant r_n(p) < p \leqslant y, \\ &R(x, r_n(p)) \otimes R(s_n(p), y) && \text{for} \quad x \leqslant r_n(p) < s_n(p) \leqslant y, \\ &R(x, m/2^n) \otimes R((m+1)/2^n, y) && \text{for} \quad x \leqslant m/2^n < (m+1)/2^n \leqslant y \end{aligned}$$

for some finite number of p's, n's and m's. Let $\equiv$ be the congruence modulo $\operatorname{Im} d_3(x, y)$. By the definition of $d_3 t_{p,n}$ it follows that $\gamma \equiv \gamma_1$ where γ_1 is of the form

$$\begin{aligned} \gamma_1 = &\sum_{i=1}^{r} \sum_{j=1}^{t_i} A_{ij} R(x, p_i) \otimes R(s_{n_{ij}}(p_i), y) + \\ &+ \sum_{i=1}^{r'} \sum_{j=1}^{t'_i} A'_{ij} R(x, r_{n'_{ij}}(p'_i)) \otimes R(p'_i, y) + \\ &+ \sum_{i=1}^{r''} A''_i R(x, r_{n''_i}(p''_i)) \otimes R(s_{n''_i}(p''_i), y) + z \end{aligned}$$

where $x \leqslant p_1 < \ldots < p_r < y$, $x < p'_1 < \ldots < p'_{r'} \leqslant y$, $x < p''_1 < \ldots < p''_{r''} < y$, $n_{i1} < \ldots < n_{it_i}$, $n'_{i1} < \ldots < n'_{it'_i}$ and $c_{n_{ij}}(p_i) = 0$, $c_{n'_{ij}}(p'_i) = 1$, $z \in \overline{P}_2(x, y)$ and $d_2 \gamma_1 = 0$. We denote by G_1, G_2, G_3 the first, second and the third summand of γ_1, so $\gamma_1 = G_1 + G_2 + G_3 + z$. Let us assume that $A_{rt_r} \neq 0$ and put $p = p_r$, $n = n_{rt_r}$ then

$$d_2(x, y) R(x, p) \otimes R(s_n(p), y) = R(x, p) \otimes R(s_n(p), y) - R(x, p) \otimes R(s_{n^*}(p), y) + z_1,$$

where $z_1 \in \overline{P}_1(x, y)$. We have moreover

(1) $$\begin{aligned} d_2(x, y) R(x, r_l(p)) \otimes R(s_l(p), y) \\ = R(x, r_l(p)) \otimes R(p, y) + R(x, p) \otimes R(s_l(p), y) + z' \end{aligned}$$

for some $z' \in \overline{P}_1(x, y)$. We consider two cases: (a) all $p'_1, \ldots, p'_{r'}$ are different from p, (b) $p'_k = p$ for some k.

Let us assume that (a) holds. The generator $R(x, p) \otimes R(s_{n^*}(p), y)$ does not appear in $d_2 G_2$ and appears only once in $d_2 G_1$. The generator $R(x, r_{n^*}(p)) \otimes R(p, y)$ does not appear in $d_2 G_1$ and $d_2 G_2$. From (1) for $l = n^*$ and $d_2 \gamma_1 = 0$ it follows that $A_{rt_r} = 0$ contrary to our assumption.

Let us assume that (b) holds. The generator $R(x, p) \otimes R(s_{n^*}(p), y)$ does not appear in $d_2 G_2$ and appears only once in $d_2 G_1$. So it must appear in $d_2 G_3$. By (1) for $l = n^*$ and $d_2 \gamma_1 = 0$ it follows that the generator $R(x, r_{n^*}(p)) \otimes R(p, y)$ appears in $d_2 G_2$, then it appears in $d_2 G_3$ too. Let α be the least integer such that $r_\alpha(p) = r_{n^*}(p)$ thus

$$R(x, r_{n^*}(p)) \otimes R(p, y) = R(x, r_\alpha(p)) \otimes R(p, y)$$

and by

$$d_2(x, y) R(x, r_l(p)) \otimes R(p, y) = R(x, r_l(p)) \otimes R(p, y) - R(x, r_{l^{**}}(p)) \otimes R(p, y) + z_1$$

(where $z_1 \in \overline{P}_1(x, y)$) it follows that in $d_2 G_2$ appears either $R(x, r_{\alpha^{**}}(p)) \otimes R(p, y)$ or $R(x, r_\beta(p)) \otimes R(p, y)$ where $\beta^{**} = \alpha$. These generators do not appear in $d_2 G_1$ consequently in $d_2 G_3$ appear $R(x, r_\alpha(p)) \otimes R(p, y)$ and either $R(x, r_{\alpha^{**}}(p)) \otimes R(p, y)$ or $R(x, r_\beta(p)) \otimes R(p, y)$. This is impossible, because $r_\alpha(p) < r_{\alpha^{**}}(p)$, $r_\beta(p) < r_\alpha(p)$ and in G_3 we have only one term for each $p''_1, \ldots, p''_{r''}$. We have proved that $A_{ij} = 0$ for all i, j.

The proof that all A'_{ij} are zero is analogous to the proof in case (a).

By (1) it follows easily that all A''_i are zero too.

Consequently $\gamma_1 = z \in \overline{P}_2(x, y)$ and $d_2 z = 0$ implies $z = 0$ and finally $\gamma \in \operatorname{Im} d_3(x, y)$. We have proved that the sequence $T_3 \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1$ is exact.

5. We show that the sequence $0 \to T_3 \xrightarrow{d_3} T_2$ is exact, i.e., that $\operatorname{Ker} d_3(x, y) = 0$. This is obvious for $x \geqslant y$ then let us assume that $x < y$ and $\gamma \in \operatorname{Ker} d_3(x, y)$. Let

$$\gamma = \sum_{i=1}^{l} \sum_{j=1}^{t_i} A_{ij} R(x, r_{n_{ij}}(p_i)) \otimes R(s_{n_{ij}}(p_i), y)$$

where $x < p_1 < \ldots < p_l < y$, $n_{i1} < \ldots < n_{it_i}$ and let us assume $A_{ln_{lt_l}} \neq 0$. We put $n = n_{lt_l}$, $p = p_l$ then

$$0 = d_3(x, y) = A_{ln}[R(x, r_n(p)) \otimes R(s_n(p), y) - R(x, r_{n+1}(p)) \otimes R(s_{n+1}(p), y)] + \ldots$$

and the second term in the brackets appears only once, so $A_{ln} = 0$ — contradiction. Consequently $\gamma = 0$ and we have proved that the sequence

$$0 \to T_3 \xrightarrow{d_3} T_2 \xrightarrow{d_2} T_1 \xrightarrow{d_1} T_0 \xrightarrow{\varepsilon} R \to 0$$

is a projective resolution of R.

6. Now we can prove the main theorem of the paper.

THEOREM. *The cohomological dimension* $\dim_K R$ *of the ordered set of real numbers* R *is equal to three for any commutative ring* K.

Proof. It is sufficient to prove that the sequence $0 \to T_3 \xrightarrow{d_3} T_2$ does not split. Suppose the converse, thus there exists a map $g\colon T_2 \to T_3$ such that $gd_3 = 1_{T_3}$.

Let us remark that a non-zero map $R(.,a)\otimes R(b,.) \to R(.,a')\otimes R(b',.)$ exists if and only if $a \leqslant a'$ and $b' \leqslant b$, moreover each map $R(.,a)\otimes R(b,.) \to \bigoplus_{\lambda} R(.,a_\lambda)\otimes R(b_\lambda,.)$ factorizes through a finite direct sum.

We denote by $\pi_p\colon T_3 \to \bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$ a projection map thus

$$\pi_p g v_{p,n}\colon R(.,p)\otimes R(s_n(p),.) \to \bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$$

and $\pi_p g v_{p,n} = 0$ because $p > r_n(p)$ for all n. Similarly we show that $\pi_p g v_{n,p} = 0$. There exists a countable set $A \subset R$ such that all maps $gv_{m,n}$ factorizes through $\bigoplus_{p\in A}\bigoplus_n$. We fix a 2-irrational element $p \in R\setminus A$, then $\pi_p g v_{m,n} = 0$ for all m, n and by the above equalities we get

$$\begin{aligned}\pi_p t_{p,n} = \pi_p g d_3 t_{p,n} &= \pi_p g(v_{n,p,n} - v_{n+1,p,n+1} - v + v_{k_n(p),n})\\ &= \pi_p g v_{n,p,n} - \pi_p g v_{n+1,p,n+1}\\ &= g_n - g_{n+1}\end{aligned}$$

where $g_n = \pi_p g v_{n,p,n}$, $n = 0, 1, \ldots$ and $v = v_{p,n}$ or $v_{n,p}$. Maps $t_n = \pi_p t_{p,n}$ are structural injections of a direct sum $\bigoplus_n R(.,r_n(p))\otimes R(s_n(p),.)$ and let π_n be a corresponding projection onto nth summand. Then we have $g_n = t_n + g_{n+1}$ and since g_0 factorizes through a finite sum then there exists such $k \geqslant 0$ that $\pi_n g_0 = 0$ for $n > k$. Since

$$g_0 = t_0 + g_1 = t_0 + t_1 + g_2 = \ldots = t_0 + \ldots + t_{k+1} + g_{k+2}$$

then $\pi_{k+1} g_0 = \pi_{k+1} t_{k+1} = 1$ because

$$\pi_{k+1} g_{k+2}\colon R(.,r_{k+2}(p))\otimes R(s_{k+2}(p),.) \to R(.,r_{k+1}(p))\otimes R(s_{k+1}(p),.)$$

is a zero map. We get a contradiction thus d_3 does not split.

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The space of maximal convex sets

by

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Abstract. This paper is devoted to the study of *hypercones* (also called "semispaces") — maximal convex sets missing a point in a vector space of arbitrary dimension over a totally ordered field. It is shown how the hypercones can be organized into a compact Hausdorff space with an intrinsic system of "convex" sets. Relationships of "convexity" to topology on the hypercone space are studied. Also metrizability, separatility, and other topological properties of the hypercone space are characterized in terms of the underlying ordered field and the dimension of the given vector space.

1. Introduction. Maximal convex sets missing a given point have arisen naturally in connection with separation properties of convex sets. In a series of lectures in 1951, T. S. Motzkin [15] described such sets in detail for 3-dimensional Euclidean space and used them in some separation theorems. Köthe [12] also makes use of the maximal convex sets missing a point — which he calls *hypercones* — in his proof of the geometric form of the Hahn–Banach Theorem. Related ideas are attributed by Kelley, Namioka, et al. [10] to Stone and Kakutani. In 1955 Preston Hammer [4] independently noted some of the elementary properties of hypercones in real vector spaces. (Although Hammer's term "semispace" is often used, Köthe's term "hypercone" seems better. There is a larger class of convex sets deserving to be called "semispaces", and the term "semispace" is inappropriate in situations as in [7] where spaces more general than vector spaces are treated.)

The first deeper results on hypercones were given in [11] by V. L. Klee, who gave (among other results) a complete characterization of hypercones in real vector spaces. In the setting of topological vector spaces, a certain class of hypercones compatible with the topological structure was studied by C. E. Moore [14]. Using his separation results, Moore achieved a new characterization of reflexive Banach spaces.

In this paper the structure of hypercones will again be investigated — not individually, but rather as a space. It will be shown that the collection of hypercones can be made into a totally disconnected compact Hausdorff space which can be endowed in a natural way with a notion of "convexity". The questions of metrizability, separability and first countability of the hypercone space will be investigated. In addition, we shall study the basic similarities and dissimilarities of its convexity structure with ordinary convexity.