Let α be the first ordinal such that $p \in Y(\alpha)$. Then $\alpha \geqslant \lambda$. From minimality of α, it follows that $p \in D(\alpha) - \mathrm{cl}_S(\bigcup_{\beta<\alpha} Y(\beta))$ so that

$$p \notin \mathrm{cl}_S(\bigcup_{\beta<\alpha} Y(\beta)) \subset \mathrm{cl}_S(\bigcup_{\beta<\lambda} Y(\beta))$$

which is impossible. Hence $\bigcup_{\alpha<\lambda} Y(\alpha)$ is a relatively closed subset of X. According to (3.3), $\mathscr{F}[X]$ is metrizable. And yet the set E of condition (*) (where τ denotes the topology of X as a subspace of S) is all of X and so is uncountable. (Here we use the fact that X is dense in S and S has no "jumps", i.e., no points $a<b$ where $[a, b] = \{a, b\}$.)

3.7. Question (Przymusiński). For any space X, $\mathrm{ind}(\mathscr{F}[X]) = 0$ and (see [P]) if $\mathscr{F}[X]$ is normal, then $\dim(\mathscr{F}[X]) = 0$ (here dim denotes covering dimension.) Is there any space X for which $\dim(\mathscr{F}[X]) > 0$?

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On some test spaces in dimension theory

by

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Abstract. Let S and S_* denote the Sorgenfrey and Modified Sorgenfrey lines, respectively. Then the following result is proved in this paper: If X is any topological space, then $X \times S$ is strongly zero-dimensional if and only if $X \times S_*$ is strongly zero-dimensional.

1. Introduction. The question of whether $\dim(X \times Y) \leqslant \dim X + \dim Y$ for topological spaces X and Y has long been considered (see e.g., [G], p. 263 and 277). By $\dim X$, or the covering dimension of X, we mean the least integer, n, such that each finite cozero cover of X has a finite cozero refinement of order n. (A cover is of *order n* if and only if each point of the space is contained in at most $n+1$ elements of the cover. All spaces considered are completely regular.)

Researchers worked out the above problem but the recent discovery shows that Wage [W] and Przymusiński [Pr] construct a Lindelöf space X such that $\dim X = 0$ and X^2 is normal but $\dim(X^2) > 0$.

The aim of this paper is to give a full answer to one of the observations raised by Mrówka [Mr_2] in the conference of 1972 concerning the product problem which says: "Strong 0-dimensionality of various product spaces remains undecided. One group of such spaces are powers of certain generalizations of the Sorgenfrey space. Consider, for instance the product (reals) $\times [0, 1]$ ordered lexicographically and let S_* be this product with the Sorgenfrey topology (i.e., the base consists of half-open intervals).

S_* is N-compact and strong 0-dimensional, we do not know if S_*^2 is strongly 0-dimensional".

In this regard, Tan [Ta] showed that certain zero-sets in S_*^2 are countable intersection of clopen sets. However, he was unable to establish the strong zero-dimensionality of S_*^2.

The familiar Sorgenfrey space S is defined to be the space of real numbers with the class of all half open intervals $[a, b)$, $a<b$, as a base. It is a well-known fact that S is Lindelöf, first countable, N-compact and also has $\dim S = 0$.

A topological space X is called *zero-dimensional* if and only if X has a base consisting of clopen sets.

A Tychonoff space X is called *strongly zero-dimensional* provided that $\dim X = 0$.

The following theorem (see e.g., [G]) characterizes the class of all strongly zero-dimensional spaces.

1.1. THEOREM. *For a Tychonoff space X, the following conditions are equivalent:*

a) *X is strongly zero-dimensional.*

b) *βX is strongly zero-dimensional.*

c) *For every two disjoint zero-sets Z_1 and Z_2 of X, there exists a clopen set G of X such that $Z_1 \subset G$ and $G \cap Z_2 = \emptyset$.*

It can be easily seen now that a Lindelöf zero-dimensional space must be strongly zero-dimensional. Since S^2 fails to be Lindelöf, there is no easy way to determine $\dim S^2$. The fact that $\dim S^n = 0$ for all n was proved only in 1972 [Mr_1, Te]. Prior to that, several researchers have proved that $\dim S^2 = 0$ (see e.g., [N]), but their arguments could not be generalized, even to S^3. An interesting parallel is that Terasawa (private communication) has shown that S^2 is hereditarily strongly zero-dimensional; his proof cannot be generalized even to S^3.

Several theorems concerning $\dim S^n$ can be found in a more generalized way in Fora [F_1] and Fora [F_2]. In this paper, we are going to use the result "$\dim(S^n) = 0$ for all $n \in N$" to conclude "$\dim(S^n_*) = 0$ for all $n \in N$".

2. The covering dimension of product of Modified Sorgenfrey lines. We write $X \underset{\text{top}}{=} Y$ in case X and Y are homeomorphic. The closure (boundary) of a set A in a space will be denoted by $\mathrm{Cl}\,A$ ($\mathrm{Bdry}\,A = \mathrm{Cl}\,A/\mathrm{Int}\,A$). N, R denote the set of all positive integers, the set of all real numbers, respectively.

We will start our results with the following:

2.1. LEMMA. *If X is a topological space which can be decomposed as a disjoint union of subspaces X_α, $\alpha < \beta$, where X_α's are clopen for $\alpha > 0$ and X_0 is C^*-embedded, then X is strongly zero-dimensional, provided all X_α's are.*

To prove Lemma 2.1, we need the following:

2.2. LEMMA. *If X satisfies the conditions stated in Lemma 2.1 and if D is a clopen set in X_0, then there exists a clopen set D_* in X such that $D_* \cap X_0 = D$.*

Proof of Lemma 2.2. Let $g\colon X_0 \to [0, 1]$ be defined by $g(D) = 1$ and $g(X_0/D) = 0$. Then g is a continuous map because D is a clopen set in X_0. Since X_0 is C^*-embedded, therefore we can find a continuous map $g_*\colon X \to [0, 1]$ such that $g_* | X_0 = g$. Since $g_*^{-1}[0, \frac{1}{4}] \cap X_\alpha$ and $g_*^{-1}[\frac{3}{4}, 1] \cap X_\alpha$ $(\alpha > 0)$ are two disjoint zero-sets in the strongly zero-dimensional space X_α, there exists a clopen set D_α in X_α (hence clopen in X) such that

$$g_*^{-1}[\tfrac{3}{4}, 1] \cap X_\alpha \subset D_\alpha \quad \text{and} \quad D_\alpha \cap g_*^{-1}[0, \tfrac{1}{4}] \cap X_\alpha = \emptyset .$$

Let $D_* = D \cup \bigcup_{\alpha>0} D_\alpha$. Then D_* is a closed set in X because D is a closed set in X and

$$\mathrm{Bdry}\big(\bigcup_{\alpha>0} D_\alpha\big) \subset g_*^{-1}[\tfrac{1}{4}, 1] \cap X_0 = D .$$

To prove that D_* is open, we let $x \in D_*$. Since each D_α $(\alpha > 0)$ is an open set, therefore we may assume $x \in D$, and consequently $g_*(x) = 1$. Hence

$$x \in g_*^{-1}(\tfrac{3}{4}, 1] \subset \bigcup_{\alpha>0} D_\alpha \cup D .$$

Hence D_* is an open set in X. It is clear that $D_* \cap X_0 = D$.

Proof of Lemma 2.1. Let Z_0 and Z_1 be any two disjoint zero-sets of X which are determined by a continuous map $f\colon X \to [0, 1]$ in such a way that $Z_i = f^{-1}(i)$ for $i = 0, 1$.

For each $\alpha > 0$, $f^{-1}[\frac{1}{2}, 1] \cap X_\alpha$ and $f^{-1}[0, \frac{1}{4}] \cap X_\alpha$ are two disjoint zero-sets of the strongly zero-dimensional space X_α. Therefore, there exist clopen sets K_α of X_α (hence clopen in X) such that

$$f^{-1}[\tfrac{1}{2}, 1] \cap X_\alpha \subset K_\alpha \quad \text{and} \quad K_\alpha \cap f^{-1}[0, \tfrac{1}{4}] \cap X_\alpha = \emptyset .$$

Observe that if $x \in \mathrm{Bdry}\big(\bigcup_{\alpha>0} K_\alpha\big)$, then $x \in X_0$ and $f(x) \geq \frac{1}{4}$. Since $f^{-1}[\frac{1}{4}, 1] \cap X_0$ and $f^{-1}[0, \frac{1}{6}] \cap X_0$ are two disjoint zero-sets of the strongly zero-dimensional space X_0, there exists a clopen set (in X_0) $D \subseteq X_0$ such that

$$f^{-1}[\tfrac{1}{4}, 1] \cap X_0 \subset D \quad \text{and} \quad D \cap f^{-1}[0, \tfrac{1}{6}] \cap X_0 = \emptyset .$$

By Lemma 2.2, we can find a clopen set D_* in X such that $D_* \cap X_0 = D$. Since $f^{-1}[\frac{1}{8}, 1] \cap X_\alpha$ and $f^{-1}(0) \cap X_\alpha$ $(\alpha > 0)$ are two disjoint zero-sets of the strongly zero-dimensional space X_α, so there exists a clopen set $C_\alpha \subset X_\alpha$ for which

$$f^{-1}(0) \cap X_\alpha \subset C_\alpha \quad \text{and} \quad C_\alpha \cap f^{-1}[\tfrac{1}{8}, 1] \cap X_\alpha = \emptyset .$$

Notice that $K_\alpha \cap C_\alpha = \emptyset$ because $K_\alpha \subset f^{-1}(\frac{1}{4}, 1]$ and $C_\alpha \subset f^{-1}[0, \frac{1}{8})$.

Let $U = \big(D_*/\bigcup_{\alpha>0} C_\alpha\big) \cup \bigcup_{\alpha>0} K_\alpha$. Then U is a clopen set in X for which $Z_1 \subset U$ and $U \cap Z_0 = \emptyset$ (see the observation below).

OBSERVATION. (i) *U is an open set in X.*

(ii) *U is a closed set in X.*

(iii) *$Z_1 \subset U$ and $U \cap Z_0 = \emptyset$.*

(i) Let $x \in U$. Since each K_α $(\alpha > 0)$ is an open set in X, so we may assume $x \in D_*/\bigcup_{\alpha>0} C_\alpha$. Now, either $x \in X_{\alpha_0}$ for some $\alpha_0 > 0$ or $x \in X_0$. In the first case, we get

$$x \in D_* \cap (X_{\alpha_0}/C_{\alpha_0}) \subset D_*/\bigcup_{\alpha>0} C_\alpha$$

(notice that both D_* and $X_{\alpha_0}/C_{\alpha_0}$ are clopen sets in X). In the last case, we get $x \in D$ and consequently $f(x) > \frac{1}{6}$.

Since f is a continuous map and D_* is open, so there exists an open set E in X such that

$$x \in E \subset f^{-1}(\tfrac{1}{6}, 1] \cap D_* .$$

Since for each $\alpha>0$, $C_\alpha \subset f^{-1}[0, \frac{1}{8})$, therefore $C_\alpha \cap E = \emptyset$ and consequently $E \cap \bigcup_{\alpha>0} C_\alpha = \emptyset$. Hence

$$x \in E \subset D_* / \bigcup_{\alpha>0} C_\alpha \subset U.$$

(ii) Since each C_α ($\alpha>0$) is an open set in X, therefore $D_* / \bigcup_{\alpha>0} C_\alpha$ is a closed set. Also, notice that

$$\mathrm{Bdry}\big(\bigcup_{\alpha>0} K_\alpha\big) \subset f^{-1}[\tfrac{1}{4}, 1] \cap X_0 \subset D \subset U.$$

Hence U is a closed set in X.

(iii) Let $x \in Z_1$. Then $f(x) = 1$. If $x \in X_0$, then $x \in D \subset U$. If $x \in X_\alpha$ for some $\alpha>0$, then $x \in K_\alpha \subset U$.

It is clear that $Z_0 \cap U = \emptyset$, and this completes the proof of Lemma 2.1.

We shall now state the main result concerning the Modified Sorgenfrey line S_*.

2.3. THEOREM. *Let X be any Tychonoff space. Then $X \times S_*$ is strongly zero-dimensional if and only if $X \times S$ is.*

Proof. It is clear that $X \times S$ is strongly zero-dimensional whenever $X \times S_*$ is strongly zero-dimensional. Now, suppose that $X \times S$ is strongly zero-dimensional. Let $X_0 = X \times (R \times \{1\}) \subset X \times S_*$. Then X_0 is C^*-embedded and $X_0 \underset{\text{top}}{=} X \times S$.

Consequently X_0 is a strongly zero-dimensional space. For each real number r, define

$$S(r) = \{(x, (r, t)) \colon x \in X,\ t \in [0, 1)\}.$$

Then $S(r)$ is a clopen subset of $X \times S_*$ and moreover

$$S(r) \underset{\text{top}}{=} X \times S \quad \text{for all } r \in R.$$

Now apply Lemma 2.1 with $X_\alpha = S(r)$ ($\alpha \neq 0$) on the topological space $X \times S_*$ to complete the proof of the theorem.

2.4. COROLLARY. *The space S_*^n ($n \in N$) is strongly zero-dimensional.*

Since $\dim(S^2) = 0$ (see e.g., [F_2]), so $\dim(S \times S_*) = 0$ (by Theorem 2.3) and henceforth $\dim(S_* \times S) = 0$. Apply Theorem 2.3 again, get $\dim(S_* \times S_*) = 0$. Now, use the induction principle together with $\dim(S^n) = 0$ to get $\dim(S_*^n) = 0$ for all $n \geq 2$.

2.5. COROLLARY. *If Y is any strongly zero-dimensional metrizable space, then $S_* \times Y$ is strongly zero-dimensional.*

The result follows immediately from the fact that a product of a perfectly normal Hausdorff strongly zero-dimensional space and a metrizable strongly zero-dimensional space is strongly zero-dimensional (see [Pe], p. 354).

We can still conclude several corollaries to Theorem 2.3, but the best thing to observe is that S_* does not give us any additional trouble in the product Problem since we can always replace S_* by S according to Theorem 2.3.

Actually, S has much nicer properties than S_*. So, dealing with S is much easier than dealing with S_*.

At the end of this paper, I would like to point out that the proof and the result which are given here are less complicated and, in some sense, more general than those given in Fora [F_1] and Tan [Ta].

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