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Ultraparacompactness in certain Pixley–Roy hyperspaces

by

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Abstract. A T_2-space Z is ultraparacompact if each open cover of Z has a disjoint open refinement. In this paper we present a sequence of results which guarantee that for certain spaces X, the Pixley–Roy hyperspace construction has the property that for each finite m and n, $(\mathscr{F}[X^m])^n$ is ultraparacompact. We also investigate ultrametrizability of certain PR-hyperspaces.

1. Introduction. This paper continues the study of the Pixley–Roy hyperspace initiated in [BFL]. Recall that for each space X, the space $\mathscr{F}[X]$, called the *Pixley–Roy hyperspace* of X, is the collection of all nonempty finite subsets of X topologized by using all sets of the form

$$[F, V] = \{F' \in \mathscr{F}[X]\colon F \subset F' \subset V\}$$

as a neighborhood base at $F \in \mathscr{F}[X]$, where V is allowed to be any open subset of X which contains F. In [BFL] we proved that if X is any first-countable subspace of any ordinal, then $\mathscr{F}[X]$ is metrizable. In [L_2] it was asserted that, for such an X, even $\mathscr{F}[X^2]$ is metrizable. In this paper we significantly sharpen (and simplify) both results by proving that if X is any subspace of any ordinal then for each m, $n \leqslant \omega_0$, $(\mathscr{F}[X^m])^n$ is *ultraparacompact*, i.e., each open cover admits an open refinement which partitions the space. (Indeed, we prove a stronger, but more technical, result — see Section 2.) It follows immediately that if X is a first-countable subspace of any ordinal then $(\mathscr{F}[X^m])^n$ is a Moore space (cf. [vD] or [L_2]) and is ultraparacompact and hence has a base of open sets which is the union of countably many subcollections, each of which is a disjoint open cover of the entire space. Spaces having such a base are called *ultrametrizable* and admit a compatible metric d which satisfies a very strong triangle inequality, viz., for any points x, y and z,

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}.$$

Another result in [BFL] characterized those generalized ordered spaces X built on a separable linearly ordered space (see Section 3 for definitions) for

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which $\mathcal{F}[X]$ is metrizable. In this paper we refine the techniques of [BFL] to show that for such spaces, if $\mathcal{F}[X]$ is metrizable then it is ultrametrizable.

Our terminology and notation for hyperspaces follows [vD] and [L_2]. Terminology and notation for ordered spaces generally follows [L_1].

After completing this paper, the authors learned that T. Przymusiński had also obtained our Theorem 2.4 in the special case where X is a subspace of an ordinal. Interested readers should consult his paper [P].

2. Pixley–Roy spaces and partial orderings. The main theorem in this section deals with T_1-spaces which can be partially ordered in a special way. Throughout this section, we will say that a partial ordering $\leqslant$ of a space Z is acceptable provided the set $L(z) = \{x \in Z: x \leqslant z\}$ is open for each $z \in Z$. Let us hasten to add that we view this notion as a technical convenience only, and not as a proper object of study in its own right. The property is named only to avoid repeating its definition over and over.

Before giving examples of spaces which can or cannot carry an acceptable partial ordering, let us observe

2.1. LEMMA. *If $\{Z_\alpha: \alpha \in A\}$ is a collection of spaces each admitting an acceptable partial ordering, then the box-product $\square\{Z_\alpha: \alpha \in A\}$ also has an acceptable partial ordering. In particular, if the index set A is finite then the usual product space $\Pi\{Z_\alpha: \alpha \in A\}$ has an acceptable partial ordering.*

Proof. If $\leqslant_\alpha$ is an acceptable partial ordering for Z_α define $\leqslant$ on $Z = \square\{Z_\alpha: \alpha \in A\}$ by the rule that $\langle z_\alpha: \alpha \in A\rangle \leqslant \langle z'_\alpha: \alpha \in A\rangle$ provided $z_\alpha \leqslant_\alpha z'_\alpha$ for each α. The order $\leqslant$ is acceptable because as sets

$$L(\langle z_\alpha: \alpha \in A\rangle) = \Pi\{L(z_\alpha): \alpha \in A\}. \ \blacksquare$$

2.2. EXAMPLES. The following types of spaces admit acceptable partial orderings:

a) any subspace of any ordinal;

b) any Sorgenfrey space, i.e., any linearly ordered set $(X, \leqslant)$ endowed with the topology having the collection $\{(x, y]: x < y, y \in X\}$ as a base of open sets.

There is also a general construction which yields spaces with acceptable partial orderings.

c) Given any partially ordered set $(Z, \leqslant)$, the collection $\{L(z): z \in Z\}$ can be used as the basis for some topology on Z. In order to obtain a T_1 topology on Z, we must use the collection $\{L(z) - F: z \in Z, F$ is a finite subset of $Z - \{z\}\}$. Then $\leqslant$ will also be acceptable for any stronger topology on Z.

d) Any finite product, or any box product, of spaces of types a), b) or c). $\blacksquare$

2.3. EXAMPLES. Most "nice" spaces do not admit an acceptable partial order.

a) If X is any uncountable T_1-space which is second countable (having a countable network would suffice), then $\mathcal{F}[X]$ is not Lindelöf (since the collection $\{F \in \mathcal{F}[X]: \operatorname{card}(F) = 1\}$ is an uncountable closed discrete subspace of $\mathcal{F}[X]$) and satisfies the countable chain condition. If X could carry an acceptable partial ordering, it would follow from (2.4), below, that $\mathcal{F}[X]$ is ultraparacompact. But an ultraparacompact space satisfying the countable chain condition must be Lindelöf.

b) The Alexandroff double-arrow space A, i.e., the lexicographic product $[0, 1] \times \{0, 1\}$ endowed with the usual open interval topology, cannot carry an acceptable partial ordering in the light of (2.4), since $\mathcal{F}[A]$ is known to be non-paracompact (cf. [BFL], [L_2]). $\blacksquare$

We now turn to the main positive result of this section.

2.4. THEOREM. *Let Z be a T_1 space which has an acceptable partial order. Then $\mathcal{F}[Z]$ is ultraparacompact.*

Proof. Let Φ be any open covering of $\mathcal{F} = \mathcal{F}[Z]$. For each $F \in \mathcal{F}$ select a member $\mathcal{U}(F) \in \Phi$ containing F and an open subset $N(F)$ of Z having $[F, N(F)] \subset \mathcal{U}(F)$. We next define open sets $I(x, F)$ of Z for each $F \in \mathcal{F}$ and each $x \in F$ by induction on $\operatorname{card}(F)$. If $\operatorname{card}(F) = 1$ then $F = \{x\}$ and we let $I(x, F)$ be any open set in Z having $x \in I(x, F) \subset L(x) \cap N(F)$. Let $\mathcal{V}(F) = [F, I(x, F)]$. For induction hypothesis, suppose $k \geqslant z$ and that sets $I(x, F)$ have been defined for each $F \in \mathcal{F}$ having $\operatorname{card}(F) < k$ in such a way that:

(i) $I(x, F)$ is open and $x \in I(x, F) \subset L(x)$;

(ii) if F' is a nonempty proper subset of F and if $x' \in F'$ and $x \in F$ have $x \in I(x', F')$, then $I(x, F) \subset I(x', F')$;

(iii) the set $\mathcal{V}(F) = [F, \bigcup\{I(x, F): x \in F\}]$ is a subset of $\mathcal{U}(F)$.

Now consider a set $F_0 \in \mathcal{F}$ with $\operatorname{card}(F_0) = k$, and let $x_0 \in F_0$. Define a set $P(x_0, F_0)$ by $P(x_0, F_0) = \{(x', F'): x' \in F'$ and F' is a proper subset of F_0 and $x_0 \in I(x', F')\}$. Clearly $P(x_0, F_0)$ is finite, so that there is an open set $I(x_0, F_0)$ satisfying

$$x_0 \in I(x_0, F_0) \subset N(F_0) \cap L(x_0) \cap \left(\bigcap\{I(x', F'): (x', F') \in P(x_0, F_0)\}\right).$$

Then the three induction hypotheses are satisfied for the pair (x_0, F_0), and the induction continues, defining $I(x, F)$ for every $F \in \mathcal{F}$ and every $x \in F$. Observe that the second induction hypothesis yields

(iv) if $\emptyset \neq F'' \subset F' \subset F$ satisfy $F \in \mathcal{V}(F')$ and $F' \in \mathcal{V}(F'')$, then $F \in \mathcal{V}(F'')$ from which we can deduce

(v) if $F \in \mathcal{F}$, then there is an $F' \in \mathcal{F}$ satisfying $F' \subset F$, $F \in \mathcal{V}(F')$ and F' is minimal, where we define that an element S of $\mathcal{F}$ is *minimal* provided $S \notin \mathcal{V}(T)$ for every proper nonempty subset T of S.

Therefore the collection

$$\Psi = \{\mathcal{V}(F): F \in \mathcal{F} \text{ and } F \text{ is minimal}\}$$

is an open cover of $\mathcal{F}$ which refines Φ. We assert that Ψ is a disjoint open cover, as required to establish ultraparacompactness. To prove that assertion it will be enough to prove that if S and T are minimal and if $\mathscr{V}(S)\cap\mathscr{V}(T)\neq\emptyset$, then $S=T$. We begin by decomposing S and T in a special way. Let S_1 be the set of maximal elements of S. Let S_2 be the set of maximal elements of $S-\bigcup\{I(x,S_1)\colon x\in S_1\}$ and, in general, let S_{k+1} be the set of maximal elements of

$$S-\bigcup\{I(x,S_1\cup\ldots\cup S_k)\colon x\in S_1\cup\ldots\cup S_k\}.$$

Since S is finite, this induction terminates at some stage m. Then

$$S\subset\bigcup\{I(x,S_1\cup\ldots\cup S_m)\colon x\in S_1\cup\ldots\cup S_m\}$$

so that if we write $S'=S_1\cup\ldots\cup S_m$, $S\in\mathscr{V}(S')$. But S is minimal and $S'\subset S$ so that $S=S'=S_1\cup\ldots\cup S_m$. In an analogous way we write $T=T_1\cup\ldots\cup T_n$ where T_{k+1} is the set of maximal elements of

$$T-\bigcup\{I(y,T_1\cup\ldots\cup T_k)\colon y\in T_1\cup\ldots\cup T_k\}\quad\text{whenever}\quad k<n.$$

Because $\mathscr{V}(S)\cap\mathscr{V}(T)\neq\emptyset$ we have

$$(*)\qquad S\cup T\subset\big(\bigcup\{I(x,S)\colon x\in S\}\cap\big(\bigcup\{I(y,T)\colon y\in T\}\big).$$

Fix $s\in S_1$. According to $(*)$, $s\in I(t,T)$ for some $t\in T$ and so $s\leqslant t$. Then there is a $t_1\in T_1$ having $t\leqslant t_1$. Again according to $(*)$, there is an $s'\in S$ with $t_1\in I(s',S)\subset L(s')$ so that $s\leqslant t\leqslant t_1\leqslant s'$. Then maximality of s yields $s=t_1=s'$ so that $s\in T_1$. Hence $S_1\subset T_1$. An analogous argument shows that $T_1\subset S_1$ so that $T_1=S_1$.

For induction hypothesis, suppose $k\leqslant m$ and that we have established $S_i=T_i$ for $1\leqslant i<k$. Let $s\in S_k$. From $(*)$, there is some $t\in T$ having $s\in I(t,T)$. We assert $t\notin\bigcup\{I(y,T_1\cup\ldots\cup T_{k-1})\colon y\in T_1\cup\ldots\cup T_{k-1}\}$. For suppose there is a

$$t'\in T_1\cup\ldots\cup T_{k-1}=T'$$

having $t\in I(t',T')$. We could then apply (ii), above, to conclude

$$s\in I(t,T)\subset I(t',T').$$

But by induction hypothesis, $T'=S_1\cup\ldots\cup S_{k-1}$ so that

$$s\in\bigcup\{I(x,S_1\cup\ldots\cup S_{k-1})\colon x\in S_1\cup\ldots\cup S_{k-1}\},$$

contrary to our definition of the set S_k. Therefore

$$t\in T-\bigcup\{I(y,T_1\cup\ldots\cup T_{k-1})\colon y\in T_1\cup\ldots\cup T_{k-1}\}$$

and so there must be a member $t_k\in T_k$ having $t\leqslant t_k$. But then since $I(t,T)\subset L(t)$, we have $s\leqslant t_k$. Applying $(*)$ to t_k we find a point $\hat{s}\in S$ having $t_k\in I(\hat{s},S)$. We assert

$$\hat{s}\notin\bigcup\{I(x,S_1\cup\ldots\cup S_{k-1})\colon x\in S_1\cup\ldots\cup S_{k-1}\}.$$

For suppose there is an $s'\in S'=S_1\cup\ldots\cup S_{k-1}$ with $\hat{s}\in I(s',S')$. Because S' is a proper subset of S it follows from (ii) that $t_k\in I(\hat{s},S)\subset I(s',S')$. But by induction hypothesis, $S'=T_1\cup\ldots\cup T_{k-1}$ so that we are forced to conclude

$$t_k\in\bigcup\{I(y,T_1\cup\ldots\cup T_{k-1})\colon y\in T_1\cup\ldots\cup T_{k-1}\}$$

contradicting the fact that $t_k\in T_k$. Therefore

$$\hat{s}\in S-\bigcup\{I(x,S_1\cup\ldots\cup S_{k-1})\colon x\in S_1\cup\ldots\cup S_{k-1}\}$$

so that there is an element $s^*\in S_k$ having $\hat{s}\leqslant s^*$. But then, combining $t_k\in I(\hat{s},S)\subset L(\hat{s})$ with $s\leqslant t_k$ and $\hat{s}\leqslant s^*$ we obtain $s\leqslant t_k\leqslant\hat{s}\leqslant s^*$. Because both s and s^* are maximal elements of the set

$$S-\bigcup\{I(x,S_1\cup\ldots\cup S_{k-1})\colon x\in S_1\cup\ldots\cup S_{k-1}\}$$

we conclude that $s=t_k$, thereby showing that $S_k\subset T_k$. An analogous argument shows that $T_k\subset S_k$ so that $T_k=S_k$ and the induction continues and proves

$$S=\bigcup\{S_i\colon 1\leqslant i\leqslant m\}=\bigcup\{T_i\colon 1\leqslant i\leqslant m\}\subset T.$$

An analogous argument shows that $T\subset S$, so $T=S$. ■

The study of products of Pixley–Roy spaces is greatly facilitated by the next theorem.

2.5. Theorem. *Let $X_1,X_2,\ldots,X_n$ be any T_1-spaces. Then the space $\mathscr{F}[X_1]\times\mathscr{F}[X_2]\times\ldots\times\mathscr{F}[X_n]$ is homeomorphic to a closed subspace of $\mathscr{F}[X_1\times X_2\times\times\ldots\times X_n]$.*

Proof. For notational simplicity we consider the case where $n=2$. The general case can be treated similarly, or can be deduced by an obvious inductive use of the special case.

Define a function $p\colon\mathscr{F}[X_1]\times\mathscr{F}[X_2]\to\mathscr{F}[X_1\times X_2]$ by the rule that $p(A,B)=A\times B$. Let $\mathscr{R}=\{A\times B\colon A\in\mathscr{F}[X_1],\ B\in\mathscr{F}[X_2]\}$. Then $\mathscr{R}$ is the image of p and p is a homeomorphism because $p([A,U]\times[B,V])=\mathscr{R}\cap[A\times B,U\times V]$. To see that $\mathscr{R}$ is closed in $\mathscr{F}[X_1\times X_2]$, let $F\in\mathscr{F}[X_1\times X_2]-\mathscr{R}$. Let F_i be the projection of F onto X_i; then $F\subset F_1\times F_2$ and the containment is proper. Choose $(p,q)\in(F_1\times F_2)-F$. Then $[F,(X_1\times X_2)-\{(p,q)\}]$ is a neighborhood of F in $\mathscr{F}[X_1\times X_2]$ which is disjoint from $\mathscr{R}$. ■

2.6. Corollary. *Let Z be any T_1-space which admits an acceptable partial order. Let $m,n\leqslant\omega_0$. Then $(\mathscr{F}[Z^m])^n$ is ultraparacompact.*

Proof. According to (2.6), $(\mathscr{F}[Z^m])^n$ embeds as a closed subspace of $\mathscr{F}[Z^{mn}]$, and the latter space is ultraparacompact in the light of (2.1) and (2.4). The usual argument shows that ultraparacompactness is inherited by closed subspaces, so the proof is complete. ■

It is easy to see that (2.6) cannot be sharpened to assert that $\mathscr{F}[Z^{\omega_0}]$ is ultraparacompact, as our next example shows.

2.7. Example. Let X be any nondegenerate T_1 space. Then $\mathscr{F}[X^{\omega_0}]$ is not even normal.

Proof. Choose distinct points a, b in X. As a subspace of X. $D=\{a,b\}$ is discrete and D^{ω_0}, which is the Cantor set, is a subspace of X^{ω_0}. But then $\mathscr{F}[D^{\omega_0}]$ is a *closed* subspace of $\mathscr{F}[X^{\omega_0}]$ since X^{ω_0} is T_1 (cf. [vD]) and it is well-known that $\mathscr{F}[D^{\omega_0}]$ is not normal. ■

2.8. QUESTION. Suppose that Z is a T_1-space which admits an acceptable partial ordering. Must $(\mathscr{F}[Z])^{\omega_0}$ be ultraparacompact? We do not know the answer even in the case where Z is a subspace of an ordinal. ■

2.9. Remark. In Section 3 we will prove that for any space X, $\mathscr{F}[X]$ is ultraparacompact whenever $\mathscr{F}[X]$ is paracompact — see (3.2).

3. Ultrametrizability of PR-spaces. The authors wish to thank T. Przymusiński for helpful observations which greatly simplified and improved results in this section of an earlier version of our paper.

There are three general situations in which metrizability of $\mathscr{F}[X]$ is well-understood:

1) where X is a first-countable space which admits an acceptable partial order in the sense of Section 2 (because then $\mathscr{F}[X]$ is paracompact in the light of (2.4) and is a Moore space [vD], whence $\mathscr{F}[X]$ is metrizable);

2) where X is a generalized ordered space constructed from a separable linearly ordered space — see [BFL] for details and (3.5) below for a related example;

3) where X is a subspace of a Souslin line — see [R] for details, and (3.4), below.

The purpose of this section is to show that in each of those situations, and indeed for *any* space X, the hyperspace $\mathscr{F}[X]$ is ultrametrizable whenever it is metrizable.

We begin with a lemma, pointed out to us by Przymusiński. The proof is straightforward.

3.1. LEMMA. *Suppose a space Z can be written as $Z = \bigcup\{Z(n)\colon n<\omega_0\}$ where each $Z(n)$ is a discrete (but not necessarily closed) subspace of Z and where each set $\bigcup\{Z(n)\colon 0\leqslant n\leqslant k\}$ is closed in Z. Suppose also that $\operatorname{ind}(X) = 0$ (i.e., each point of Z has a clopen neighborhood base). Then the following are equivalent:*

a) *Z is ultraparacompact;*

b) *Z is paracompact;*

c) *Z is collectionwise normal;*

d) *Z is strongly collectionwise Hausdorff, i.e., if D is a closed discrete subspace of Z then there is a discrete collection $\{U(d)\colon d\in D\}$ of open subsets of Z having $d\in U(d)$ for each $d\in D$.* ■

3.2. COROLLARY. *For any space X, the following are equivalent:*

a) *$\mathscr{F}[X]$ is ultraparacompact;*

b) *$\mathscr{F}[X]$ is paracompact;*

c) *$\mathscr{F}[X]$ is collectionwise normal;*

d) *$\mathscr{F}[X]$ is strongly collectionwise Hausdorff.* ■

3.3. THEOREM. *For any space X, $\mathscr{F}[X]$ is ultrametrizable if and only if $\mathscr{F}[X]$ is metrizable.*

Proof. To prove the non-trivial half, suppose $\mathscr{F}[X]$ is metrizable. Then it is paracompact, whence ultraparacompact, and has a development $\{\Psi(n)\colon n<\omega_0\}$. For each $n<\omega_0$ let $\Psi'(n)$ be a disjoint open cover of $\mathscr{F}[X]$ which refines $\Psi(n)$. Then $\bigcup\{\Psi'(n)\colon n<\omega_0\}$ is the required base for $\mathscr{F}[X]$. ■

In [R, Theorem 2] a characterization is given of those subspaces X of a connected Souslin line for which $\mathscr{F}[X]$ is metrizable. It is possible to reformulate part of Rudin's result, obtaining a condition which is internal to X (i.e., does not refer to the connected Souslin line which contains X) and is slightly easier to use as a sufficient condition for metrizability of $\mathscr{F}[X]$. The proof requires only minor modifications of Rudin's proof, and the reader is referred to [R] for details.

3.4. PROPOSITION. *Let X be a first-countable generalized ordered space. Suppose $X = \bigcup\{Y(\alpha)\colon \alpha<\omega_1\}$ where:*

a) *each $Y(\alpha)$ is countable;*

b) *if $\alpha<\beta<\omega_1$, then $Y(\alpha)\subset Y(\beta)$;*

c) *the set $\{\alpha<\omega_1\colon \bigcup_{\beta<\alpha} Y(\beta)$ is not closed$\}$ is not stationary.*

Then $\mathscr{F}[X]$ is ultrametrizable. ■

3.5. Remark. In [BFL] we characterized metrizability of $\mathscr{F}[X]$ in terms of the structure of X, where X was a generalized ordered space constructed on a separable linearly ordered space. It is natural to ask whether separability of the underlying linearly ordered space could be relaxed, perhaps to countable cellularity. The answer is "consistently, no."

3.6. EXAMPLE. If there is a Souslin space, then there is a linearly ordered space X which is dense-in-itself, uncountable, and satisfies the countable chain condition and for which $\mathscr{F}[X]$ is metrizable. However, X does not satisfy the conditions of [BFL, Theorem I] since, in particular, the set

$$E = \{x\in X\colon \text{each neighborhood of } x \text{ contains points on both sides of } x\}$$

is not countable.

Let S be a connected Souslin line. Write $S = \bigcup\{K(\alpha)\colon \alpha<\omega_1\}$ where each $K(\alpha)$ is a compact metrizable subspace of S. Each $K(\alpha)$ contains a countable dense set $D(\alpha)$. Define sets $Y(\alpha)$ recursively on $\alpha<\omega_1$ by the rule that

$$Y(\alpha) = [D(\alpha)-\operatorname{cl}_S(\bigcup_{\beta<\alpha} Y(\beta))]\cup[\bigcup_{\beta<\alpha} Y(\beta)].$$

Each $Y(\alpha)$ is countable and $Y(\alpha)\subset Y(\alpha')$ whenever $\alpha<\alpha'<\omega_1$. Let

$$X = \bigcup\{Y(\alpha)\colon \alpha<\omega_1\}$$

and topologize X as a subspace of S. It is easy to see that X is dense in S so that X, in its relative topology, is a linearly ordered space which satisfies the countable chain condition (but is not separable). We assert that for each limit ordinal λ, the set $\bigcup\{Y(\alpha)\colon \alpha<\lambda\}$ is closed in X. For suppose some $p\in X$ belongs to

$$\operatorname{cl}_S(\bigcup_{\alpha<\lambda} Y(\alpha))-\bigcup_{\alpha<\lambda} Y(\alpha).$$

Let α be the first ordinal such that $p \in Y(\alpha)$. Then $\alpha \geqslant \lambda$. From minimality of α, it follows that $p \in D(\alpha) - \mathrm{cl}_S(\bigcup_{\beta<\alpha} Y(\beta))$ so that

$$p \notin \mathrm{cl}_S(\bigcup_{\beta<\alpha} Y(\beta)) \subset \mathrm{cl}_S(\bigcup_{\beta<\lambda} Y(\beta))$$

which is impossible. Hence $\bigcup_{\alpha<\lambda} Y(\alpha)$ is a relatively closed subset of X. According to (3.3), $\mathscr{F}[X]$ is metrizable. And yet the set E of condition (*) (where τ denotes the topology of X as a subspace of S) is all of X and so is uncountable. (Here we use the fact that X is dense in S and S has no "jumps", i.e., no points $a<b$ where $[a, b] = \{a, b\}$.)

3.7. QUESTION (Przymusiński). For any space X, $\mathrm{ind}(\mathscr{F}[X]) = 0$ and (see [P]) if $\mathscr{F}[X]$ is normal, then $\dim(\mathscr{F}[X]) = 0$ (here dim denotes covering dimension.) Is there any space X for which $\dim(\mathscr{F}[X])>0$?

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On some test spaces in dimension theory

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Abstract. Let S and S_* denote the Sorgenfrey and Modified Sorgenfrey lines, respectively. Then the following result is proved in this paper: If X is any topological space, then $X \times S$ is strongly zero-dimensional if and only if $X \times S_*$ is strongly zero-dimensional.

1. Introduction. The question of whether $\dim(X \times Y) \leqslant \dim X + \dim Y$ for topological spaces X and Y has long been considered (see e.g., [G], p. 263 and 277). By $\dim X$, or the covering dimension of X, we mean the least integer, n, such that each finite cozero cover of X has a finite cozero refinement of order n. (A cover is of *order n* if and only if each point of the space is contained in at most $n+1$ elements of the cover. All spaces considered are completely regular.)

Researchers worked out the above problem but the recent discovery shows that Wage [W] and Przymusiński [Pr] construct a Lindelöf space X such that $\dim X = 0$ and X^2 is normal but $\dim(X^2)>0$.

The aim of this paper is to give a full answer to one of the observations raised by Mrówka [Mr_2] in the conference of 1972 concerning the product problem which says: "Strong 0-dimensionality of various product spaces remains undecided. One group of such spaces are powers of certain generalizations of the Sorgenfrey space. Consider, for instance the product (reals)$\times[0, 1]$ ordered lexicographically and let S_* be this product with the Sorgenfrey topology (i.e., the base consists of half-open intervals).

S_* is N-compact and strong 0-dimensional, we do not know if S_*^2 is strongly 0-dimensional".

In this regard, Tan [Ta] showed that certain zero-sets in S_*^2 are countable intersection of clopen sets. However, he was unable to establish the strong zero-dimensionality of S_*^2.

The familiar Sorgenfrey space S is defined to be the space of real numbers with the class of all half open intervals $[a, b)$, $a<b$, as a base. It is a well-known fact that S is Lindelöf, first countable, N-compact and also has $\dim S = 0$.

A topological space X is called *zero-dimensional* if and only if X has a base consisting of clopen sets.

A Tychonoff space X is called *strongly zero-dimensional* provided that $\dim X = 0$.