On a stability theorem for the fixed-point property

by

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Abstract. The author studies the relationship between the ordinary fixed point property (f.p.p.) and the proximate fixed point property (p.f.p.p.) for a class of spaces known as AANR's. He shows in particular that for a compact or a locally compact ANR, f.p.p. always implies p.f.p.p. This can be interpreted as to say that for such a space the fixed point property is stable in the sense that if the space possesses the fixed point property then a nearly continuous function always has a nearly invariant point. This generalizes a previous result of V. Klee. The author also shows that to each nearly continuous function on a compact ANR, a Lefschetz number can be assigned, which can be used to determine whether the function possesses any nearly invariant point. An application is then given for such functions on an n-sphere. Finally some related unsolved problems are discussed.

I. Introduction. In 1961, V. Klee published a stability theorem for the fixed-point property for compact absolute retracts [17]. As is well-known, every compact absolute retract A has the fixed-point property, i.e., if a function φ of A into itself is continuous, then some point of A is invariant under φ. Klee's theorem says roughly that if a function φ of A into itself is "nearly continuous", then some point of A is "nearly invariant" under φ. In this note we shall extend this result to a much wider class of spaces, a class which contains all the compact or locally compact absolute *neighborhood* retracts. (All manifolds and polyhedra are absolute neighborhood retracts, and an absolute retract is a *contractible* absolute neighborhood retract).

Specifically, let X be a topological space and (Y, ϱ) be a metric space. For a $\delta>0$, a function $f\colon X \to Y$ is said to be δ-*continuous* if each point x of X admits a neighborhood U_x such that the ϱ-diameter of the set $f(U_x)$ is at most δ. A metric space (M, ϱ) is said to have the *proximate fixed-point property* if for each $\varepsilon>0$, there exists a $\delta>0$ such that every δ-continuous function $f\colon M \to M$ possesses an ε-invariant point, i.e., a point $x \in M$ such that $\varrho(f(x), x) \leqslant \varepsilon$. In this note, we study the relationship between the ordinary fixed-point property (f.p.p.) and the proximate fixed-point property (p.f.p.p.) for a class of spaces known as approximative absolute neighborhood retracts (AANR). These are a generalization of the ordinary absolute neighborhood retracts. The definitions will be given in Section II. We shall show that for any locally compact and locally connected AANR, f.p.p. implies p.f.p.p., and for any compact AANR, f.p.p. and p.f.p.p. are equivalent to each other. (In particular, the fixed-point property is stable for these spaces). However, a noncompact AANR may possess p.f.p.p. without having f.p.p. These results will all be shown in Section II. Then in Section III, we shall shift our attention to individual mappings of a given space. We shall show in particular that for each compact ANR, there is a $\delta>0$ such that each δ-continuous function of the space can be assigned a Lefschetz number, which can be used to determine whether the function has any proximate fixed points. Finally, we shall list a few related questions in Section IV.

An interest in the concepts of ε-invariance and δ-continuity was motivated by mathematical modelling of physical problems where there may be strong empirical evidence for the δ-continuity of a function even though its full continuity cannot be tested experimentally. Indeed, in many applied situations, the conclusions as well as the hypotheses are naturally of a proximate nature. Moreover, there are spaces which have p.f.p.p. but not f.p.p. (see for instance Theorem 2.4 below). Thus, a proximate theory not only seems more natural in application, but may also be applied to a wider class of spaces. It is also of interest to note that L. E. J. Brouwer argues that only ε-fixed points have meaning for the intuitionist [5]. The proximate concepts of ε-invariance and δ-continuity studied here were introduced by V. Klee in 1961 [17]. An extensive study on these concepts was made by Klee and Yandl ([27], [18]), and by A. Finbow [7]. For the proximate fixed-point property for the multi-valued functions, see Muenzenberger [21], Muenzenberger and Smithson [22], Smithson [25], and Schirmer [24].

The author has been informed that the first two theorems and lemmas in Section II were also established by A. Finbow [7]. However, the approaches are different. The author is grateful to Professor Klee for introducing to him these proximate mathematical concepts, and he would also like to thank Dr. Finbow for reading over a preliminary version of this paper and making many useful comments.

II. The proximate fixed point property. To state our result, we need first describe a class of spaces known as approximate absolute neighborhood retracts (AANR). The concept of AANR was introduced by H. Noguchi in 1953 [23] (Noguchi used the term ε-ANR). It was then studied by A. Gmurczyk [9], A. Granas [12], and L. Górniewicz [10]. The concept was further generalized by M. H. Clapp in 1971 [6]. For further work on AANR and related concepts see the work of S. A. Bogatyĭ [1], V. Klee and Yandl [18]. In this note, we shall follow the more general definition of Clapp, except that we do not assume an AANR to be compact or even separable. The definition is given as follows.

DEFINITION 2.1. A subset X of a metric space (M,ϱ) is an *approximative neighborhood retract* of M if for each positive number ε, there is a neighborhood $U=U(\varepsilon)$ of X in M and a mapping $r_\varepsilon\colon U\to X$ such that for each x in X, $\varrho(r_\varepsilon(x),x)<\varepsilon$.

A metric space X is an *absolute approximative neighborhood retract* (AANR) if for every homeomorphism h mapping X onto a subset of a metric space M, the set $h(X)$ is an approximative neighborhood retract of M.

Remark 2.1. In Noguchi's original definition of approximative neighborhood retract, the neighborhood U is assumed to be independent of ε. The resulting AANR's turn out to be a proper subset of the AANR's defined above (see [6]). On the other hand, if we let ANR to denote the ordinary absolute neighborhood retract, it can be shown that ANR's form a proper subset of AANR's in the sense of Noguchi (see for instance [23, Example 3.6]). For more examples of AANR's see pp. 110 and 123 of [6].

DEFINITION 2.2. Let X be a topological space, (Y,ϱ) be a metric space and $f\colon X\to Y$ be a (not necessarily continuous) function. Let a positive number ε be given. By a *continuous ε-approximation* for f, we mean a continuous mapping $g\colon X\to Y$ such that $\varrho(f(x),g(x))\leqslant\varepsilon$ for every x of X.

LEMMA 2.1. *Let X be a paracompact topological space and V be a normed linear space. Then each δ-continuous function $f\colon X\to V$ has a continuous δ-approximation. In particular, if X is a convex subset of a normed linear space, then each δ-continuous function $f\colon X\to X$ has a continuous δ-approximation.*

Proof. Let a δ-continuous function $f\colon X\to V$ be given. We may choose a locally finite open covering $\{U_\alpha\}_{\alpha\in A}$ for X such that for each $\alpha\in A$, the set $f(U_\alpha)$ in V has a diameter $\leqslant\delta$. Then for each $\alpha\in A$, fix a point $p_\alpha\in U_\alpha$ and define $g_\alpha\colon U_\alpha\to V$ by $g_\alpha(x)=f(p_\alpha)$ for every $x\in U_\alpha$. Now, let $\{\lambda_\alpha\}_{\alpha\in A}$ be a partition of unity subordinating to the open covering $\{U_\alpha\}_{\alpha\in A}$ and define $g\colon X\to V$ by

$$g(x)=\sum_{\alpha\in A}\lambda_\alpha(x)g_\alpha(x).$$

g is continuous.

For any $y\in X$, y belongs to only finitely many U_α's, say $U_{\alpha_1},U_{\alpha_2},\ldots,U_{\alpha_n}$. Then

$$\begin{aligned}\|f(y)-g(y)\| &= \Big\|\sum_{\alpha\in A}\lambda_\alpha(y)f(y)-\sum_{\alpha\in A}\lambda_\alpha(y)g_\alpha(y)\Big\|\\ &\leqslant\sum_{i=1}^{n}\lambda_{\alpha_i}(y)\|f(y)-g_{\alpha_i}(y)\|\\ &=\sum_{i=1}^{n}\lambda_{\alpha_i}(y)\|f(y)-f(p_{\alpha_i})\|\\ &\leqslant\delta\sum_{i=1}^{n}\lambda_{\alpha_i}(y)\\ &\leqslant\delta\sum_{\alpha\in A}\lambda_\alpha(y)=\delta.\end{aligned}$$

Thus, g is a continuous δ-approximation for f.

Now, let X be a convex subset of a normed linear space V, and $f\colon X\to X(\subset V)$ be a δ-continuous function. Define a continuous δ-approximation $g\colon X\to V$ as above. By the fact that $\{\lambda_\alpha\}_{\alpha\in A}$ is a partition of unity subordinating to a locally finite open covering $\{U_\alpha\}_{\alpha\in A}$, it is easy to see that for each $x\in X$, the point $g(x)$ lies in the convex hull of the set $\{g_{\alpha_i}(x)|\ i=1,2,\ldots,n\}=\{f(p_{\alpha_i})|\ i=1,2,\ldots,n\}$, where $U_{\alpha_1},U_{\alpha_2},\ldots,U_{\alpha_n}$ are the only sets in the open covering $\{U_\alpha\}_{\alpha\in A}$ which contain x.

But X is convex and each $f(p_{\alpha_i})$ belongs to X. Hence, $g(x) \in X$ for each $x \in X$. Thus, $g\colon X \to X$ is a continuous δ-approximation to f.

We remark in passing that V. Klee proved a similar statement for compact convex polyhedra in a finite-dimensional normed linear space [17, Proposition 2]. Our lemma is a generalization of Klee's proposition.

THEOREM 2.1. *Let X be a compact,* AANR *with a metric ϱ. For each $\varepsilon>0$, there exists a $\delta>0$ such that every δ-continuous function $f\colon X\to X$ has a continuous ε-approximation.*

Proof. It is well known that X can be embedded as a subset of some normed linear space V (in fact, since X is compact, X can be embedded as a subset of the Hilbert cube). Let X_0 be the image of such an embedding. By an argument similar to that used in the proof of [17, Proposition 5], it can be shown that to prove the theorem for the space X, it is sufficient to prove the assertion for the set X_0.

Let an $\varepsilon>0$ be given, we shall find a $\delta>0$ such that every δ-continuous function $f\colon X_0\to X_0$ has a continuous ε-approximation. Since X_0 is the image of an embedding of an AANR, there exists an open neighborhood U of X_0 in V and a continuous mapping $r\colon U\to X_0$ such that for each $x\in X_0$, $\|r(x)-x\|<\frac{1}{6}\varepsilon$. We claim that there exists an open set W in V, $X_0\subset W\subset U$, such that for each $x\in W$, $\|r(x)-x\|<\frac{1}{2}\varepsilon$. This can be seen as follows. For each point x in V and each positive number s, we let $B(x;s)$ be the open ball in V with center x and radius s. Now, for each point $p\in X_0$, let

$$W(p) = \{x\in U\cap B(p;\tfrac{1}{6}\varepsilon)\mid\ r(x)\in B(r(p);\tfrac{1}{6}\varepsilon)\}\,.$$

Then let $W = \bigcup\{W(p)\mid\ p\in X_0\}$. Clearly, W is open in V and $X_0\subset W\subset U$. Furthermore, for each $x\in W$, $x\in W(p)$ for some $p\in X_0$. Then

$$\begin{aligned}\|r(x)-x\| &\leqslant \|r(x)-r(p)\|+\|r(p)-p\|+\|p-x\|\\ &< \tfrac{1}{6}\varepsilon+\tfrac{1}{6}\varepsilon+\tfrac{1}{6}\varepsilon = \tfrac{1}{2}\varepsilon\,.\end{aligned}$$

We may now choose a $\delta>0$ such that

1. $\delta<\frac{1}{2}\varepsilon$,
2. $\delta<\operatorname{dist}(X_0, V-W)$.

Now, consider any δ-continuous function $f\colon X_0\to X_0$. By Lemma 2.1, there exists a continuous δ-approximation $g\colon X_0\to V$. From our choice of δ, it is clear that $r\circ g\colon X_0\to X_0$ is a well-defined continuous function, and for each $x\in X_0$,

$$\begin{aligned}\|f(x)-rg(x)\| &\leqslant \|f(x)-g(x)\|+\|g(x)-rg(x)\|\\ &< \delta+\tfrac{1}{2}\varepsilon<\tfrac{1}{2}\varepsilon+\tfrac{1}{2}\varepsilon = \varepsilon\,.\end{aligned}$$

This finishes the proof.

THEOREM 2.2. *Let X be a compact* AANR. *X possesses f.p.p. if and only if X possesses p.f.p.p.*

Proof. It is easy to see that for a compact metric space, p.f.p.p. always implies f.p.p. On the other hand, if X is a compact AANR which possesses f.p.p., then for any $\varepsilon>0$, there is a $\delta>0$ such that each δ-continuous $f\colon X\to X$ has a continuous ε-approximation g. Then any fixed point for g is an ε-invariant point for f. Hence, X also possesses p.f.p.p.

THEOREM 2.3. *Let X be a locally compact and locally connected* AANR. *If X has f.p.p., then f must also have p.f.p.p.*

Proof. By a theorem of V. Klee [16, Theorem 2.7], if X is a locally compact and locally connected metric space which has f.p.p., then X must also be compact. Hence the assertion follows immediately from Theorem 2.2.

Remark 2.2. It is well known that any topological manifold or any locally compact CW complex is an ANR (see [13] and [26]). On the other hand, an ANR is always a locally connected AANR. Thus, for a compact manifold or CW complex, f.p.p. and p.f.p.p. are equivalent, and for a general locally finite CW complex, f.p.p. implies p.f.p.p.

The converse of Theorem 2.3 is false, even for such a nice space as the n-dimensional unit open ball. In the following, we shall show that each n-dimensional open ball has p.f.p.p. On the other hand, they clearly lack f.p.p. To describe our next result, we shall adopt the following convention: if $\{(X_i,\varrho_i)\mid\ i=1,2,3,\ldots\}$ is a family of metric spaces with $\varrho_i(X_i)\leqslant 1$ for each i, then the product space $\prod_{i=1}^{\infty}X_i$ is also considered as a metric space with the metric $\varrho=\sum_{i=1}^{\infty}2^{-i}\varrho_i$.

THEOREM 2.4. *Let X be any of the following spaces:*

1. *X = an open ball in the euclidean space E^n, or*

2. *$X=\prod_{i\in A}I_i$, the product space of a family of intervals, where A is either finite or countably infinite and for each $i\in A$, I_i is the open, the closed, or the half-open unit interval.*

Let an $\varepsilon>0$ be given. Then for each positive number $\delta<\varepsilon$, every δ-continuous function of X possesses an ε-invariant point.

Outline of a proof. The idea is basically that used in Proposition 3.4 and Corollary 3.5 of [18]. However, by means of our Lemma 2.1, we are able to give a sharper estimate for the δ in terms of the ε for the spaces considered here. Let $\varepsilon>0$ be given and let $\delta<\varepsilon$ be an arbitrary positive number. Suppose $f\colon X\to X$ is δ-continuous. By Lemma 2.1, f has a continuous δ-approximation $g\colon X\to X$. Now, let $\tau=\frac{1}{2}(\varepsilon-\delta)$. Following the idea of Proposition 3.4 and Corollary 3.5 of [18] (see also [8]), it is not difficult to find a compact subset $Y\subset X$ and a continuous mapping $r\colon X\to Y$ such that

1. Y has the f.p.p.
2. For each $x\in X$, $\varrho(x, r(x))<\tau$ where ϱ is the metric in X. If we let $j\colon Y\to X$ be the inclusion function, then any fixed point of $r\circ g\circ j\colon Y\to Y$ is an ε-invariant point for f. The details of the proof are omitted.

III. ε-Invariant points for a given function. Having considered spaces with the proximate fixed point property, we now consider individual functions of a given space. Let X be a metric space which does not necessarily have the proximate fixed-point property, and ε be a positive number. For a given δ-continuous function $f\colon X\to X$, how can we determine whether f has an ε-invariant point? For the analogous question on the fixed-points for a continuous function of a compact ANR, one of the most important tools is the Lefschetz fixed-point theorem (first proved by S. Lefschetz for manifolds [19], and by H. Hopf for polyhedra [14], and generalized by S. Lefschetz for all compact ANR spaces [20]). In the following, we shall show that a Lefschetz number can also be assigned to a suitable δ-continuous function of certain compact AANR's, and this number can be used in a sufficient condition for the function to possess an ε-invariant point. To describe these AANR's, recall that a topological space X is said to be of a *finite homology type* if (i) the homology groups $H_q(X, Q)$ of X with rational coefficients are finite dimensional for all q, and (ii) $H_q(X, Q) = 0$ for all but finitely many q's. We shall consider compact AANR spaces of a finite homology type. These spaces include, in particular, all the compact ANR's.

THEOREM 3.1. *For each compact* AANR *space X of a finite homology type and for each positive number ε, there exists a $\delta>0$ such that each δ-continuous function $f\colon X\to X$ can be assigned a unique Lefschetz number $\Lambda(f)$ in such a manner that the following conditions are satisfied*:

1. *If $f\colon X\to X$ is continuous, $\Lambda(f)$ = the usual Lefschetz number of f.*
2. *If $f\colon X\to X$ is δ-continuous and $g\colon X\to X$ is a continuous δ-approximation for f, then $\Lambda(f)=\Lambda(g)$.*
3. *For any δ-continuous function $f\colon X\to X$, if $\Lambda(f)\neq 0$, then f must have an ε-invariant point.*

Proof. We first note that for each compact AANR space X of a finite homology type with a metric ϱ, there exists a $\delta_1>0$ such that for any two continuous mappings $f, g\colon X\to X$, if $\varrho(f(x), g(x))<\delta_1$ for all $x\in X$, then f and g are homotopic (see [11, Theorem 2.3, p. 11]).

Now let an $\varepsilon>0$ be given. First fix a positive number $\delta_1\leqslant 2\varepsilon$ as above. Then use Theorem 2.1 to fix a $\delta<\delta_1/2$ such that each δ-continuous function $f\colon X\to X$ has a continuous $(\delta_1/2)$-approximation g. Since any two such continuous $(\delta_1/2)$-approximations for f are homotopic, we may define the Lefschetz number of f to be the Lefschetz number of any such continuous $(\delta_1/2)$-approximation g of f. This definition of Lefschetz number for the δ-continuous functions clearly satisfies the conditions 1 and 2 of the theorem. Furthermore, if $\Lambda(f)\neq 0$ for some δ-continuous function f. Then $\Lambda(g)\neq 0$ for any continuous $(\delta_1/2)$-approximation g of f (by Theorem 2.1, there exists at least one such g). Since $\delta_1/2\leqslant\varepsilon$, any fixed point for g is an ε-invariant point for f.

Remark 3.1. Even though the Lefschetz number of a δ-continuous function f is defined by means of that of a continuous approximation of f, it is possible in some cases to compute $\Lambda(f)$ directly from f by looking at the image under f of a finite ε-dense subset of the space. This can be seen as follows. For a given δ-continuous function f of a compact AANR of finite homology type, from the proofs of our Lemma 2.1 and Theorem 2.1, a continuous approximation to f can always be made to agree with f on *any* finite ε-dense subset of the space for a sufficient small ε. Since the Lefschetz number for a continuous function can be found by looking at the image of a suitable finite ε-dense subset of the space (for instance, when the space is a simplicial complex, the ε-dense subset is the set of vertices of a subdivision), it is also possible to do the same for a δ-continuous function.

Remark 3.2. For a general compact AANR, it is well-known that two continuous function of the space can be arbitrarily close to each other without being homotopic. Therefore, the proof of our Theorem 3.1 is not valid for a general compact AANR. We do not know whether the theorem is still true for such a space.

Remark 3.3. Our Theorems 2.1 and 3.1 do not give an explicit dependence of the δ's on the given ε's. However, in certain cases, when the space is already embedded in a normed linear space, it is possible to estimate such a δ. For instance, if X is a euclidean n-sphere in E^{n+1} with a radius R, it can be shown that for any $\varepsilon>0$, a $\delta<\operatorname{Inf}(\varepsilon/2, R/2)$ will satisfy both Theorems 2.1 and 3.1.

In fact, for euclidean spheres, many classical fixed-point theorems for continuous functions can be carried over for the δ-continuous functions. As a sample, we shall state the following theorem, whose proof is straightforward once we observe that for each $\delta<R/2$, any δ-continuous function of the sphere can be approximated within $\sqrt{2\delta}$ by a continuous function (cf. Theorem 2.1). The detail for the proof will be omitted.

THEOREM 3.2. *Let S^n be a euclidean n-sphere with a radius R, and δ be any positive number $<R/2$. If n is an even integer, then each δ-continuous function $f\colon S^n\to S^n$ either has a $\sqrt{2}\,\delta$-invariant point, or sends some point, within a distance $\sqrt{2}\,\delta$, to its antipodal point.*

IV. Some questions. Recently, Klee and Yandl introduced the concepts of *strong proximate absolute neighborhood retract* (SPANR), and *proximate absolute neighborhood retract* (PANR) [27], [18]. It is known that (see [7])

$$\mathrm{ANR}\Rightarrow \mathrm{AANR}_N\Leftrightarrow \mathrm{SPANR}\Rightarrow \mathrm{FANR}\Rightarrow \mathrm{PANR}$$

where AANR_N stands for the absolute approximative neighborhood retract in the sense of Noguchi [23], FANR stands for the fundamental absolute neighborhood retract introduced by K. Borsuk [4], and all the spaces are assumed to be compact. Since each AANR_N is an AANR in the sense of Clapp (AANR_C), and we have proved that for a compact AANR_C, f.p.p. and p.f.p.p. are equivalent (Theorem 2.2), for each AANR_N or each SPANR, f.p.p. and p.f.p.p. must also be equivalent. On the other hand, for a compact PANR, f.p.p. does not necessarily imply p.f.p.p. This can be seen as follows. In [17, p. 45], Klee described a plane continuum K which has f.p.p. but lacks p.f.p.p. With the help of [18, Theorem 2.3], it is easy to see that the plane continuum K is in fact a PANR. In view of these, it is natural to ask the following.

QUESTION 4.1. For a compact FANR, does f.p.p. always imply p.f.p.p.?

A positive answer would also settle a question of Finbow that whether compact FANR's are the same as compact PANR's [7]. Finbow also suggested the following two problems in a communication. First note that by our Remark 3.2, our method of assigning Lefschetz numbers may not determine a unique number for a δ-continuous function on a compact AANR. Suppose such a function f is given. We may then consider a set $L_\varepsilon(f)$, the set of Lefschetz numbers obtained from the continuous ε-approximations of f. If f is continuous, we may let $L(f) = \bigcap_{\varepsilon>0} L_\varepsilon(f)$.

QUESTION 4.2. What do $L_\varepsilon(f)$ and $L(f)$ say about f in general? What does $L(\mathrm{id}_X)$ say about the space X?

In our Theorem 3.1, we have used two properties of certain compact AANR space X. (1) There is a $\delta>0$ such that any two δ-close continuous maps of X are homotopic. (2) For each $\varepsilon>0$, there exists a $\delta>0$ such that each δ-continuous function of X has an ε-approximation.

The second of these characterizes compact AANR_C's. The first does not quite characterize compact ANR's, though a slightly stronger property does (see [15, p. 114, Theorem 1.3]).

QUESTION 4.3. What kind of spaces satisfies the conjunction of properties (1) and (2)?

Finally, we shall list a question raised by Professor Klee in a private communication.

QUESTION 4.4. Suppose X is a compact metric space which is connected and locally connected. If X has f.p.p., must it also have p.f.p.p.?

Note that a finite-dimensional compact metric space X is an ANR if and only if it is locally contractible (see [2, p. 240] or [3, p. 122]), if there exists a counter-example to Question 4.4, then by our Theorem 2.2, the space must either be infinite dimensional or not locally contractible.

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