Paracompactness in countable products via the Souslin operation

by

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Abstract. A proof is given that Čech-complete paracompact/metacompact spaces are countably productive using a Souslin-like operation. Generalization to p-spaces follows.

§ **1.** We offer a new proof and analysis of the following result and a new characterisation of paracompact Čech-complete spaces (also of paracompact p-spaces).

THEOREM. *If $(x_n\colon n \in \omega)$ is a sequence of Čech-complete spaces (or more generally, p-spaces) which are all Lindelöf paracompact/metacompact then their product is respectively Lindelöf paracompact/metacompact.*

We recall that a completely-regular space is Čech-complete if it is a $\mathscr{G}_\delta$ subset of some (equivalently, any) compactification.

The paracompact case was proved by Z. Frolík using perfect mappings [6] and was then generalised from Čech-complete spaces to p-spaces by A.V. Arhangel'skiĭ [0] and K. Morita [12]. The theorem was recently obtained for Čech-complete spaces by an application of "frames" in C. H. Dowker and D. Strauss [1]. The result was independently obtained by M. M. Čoban [2]. The present analysis was motivated by the observation that the Lindelöf case for Čech-complete spaces is an easy corollary of the following obvious

LEMMA. *A $\mathscr{G}_\delta$ subset of a compact Hausdorff space is Lindelöf if and only if it is a $\mathscr{K}_{\sigma\delta}$ subset.*

Since a countable product of spaces which are $\mathscr{K}_{\sigma\delta}$ in some compactification is readily seen to be $\mathscr{K}_{\sigma\delta}$ is some compactification, the Lindelöf case is immediate. For the other two cases we appeal to the preservation of Souslin representations and analyticity under countable products (Choquet [3], Frolík [5], C. A. Rogers [15]; note also J. E. Jayne [9]).

DEFINITION. Consider the following Souslin representation of a space X where A is an arbitrary set:

(1) $$X = \bigcup_{(a_1,a_2,\ldots)\in A^N} \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n),$$

where

(i) for each $n \in N$ and each $x \in X$

(2) $\{(a_1, \ldots, a_n) \in A^n\colon x \in V(a_1, \ldots, a_n)\}$ is finite,

and $\{V(a_1, \ldots, a_n)\colon a_1 \ldots a_n \in A\}$ is an open cover of X,

(ii) $K(\underline{a}) = \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n)$ is compact for each $\underline{a} = (a_1, a_2, \ldots) \in A^N$,

(iii) $V(a_1, \ldots, a_{n+1}) \subseteq V(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_{n+1} \in A$,

(iv) for each open $U \supseteq K(\underline{a})$ there is n so that $V(a_1, \ldots, a_n) \subseteq U$.

We call such a representation *meta-Souslin* or, respectively, *para-Souslin* if in place of (2) we require that for each $n \in N$ and $x \in X$ there is an open set W containing x so that

(3) $\{(a_1, \ldots, a_n) \in A^n\colon V(a_1, \ldots, a_n) \cap W \neq \emptyset\}$ is finite.

We note that the convergence condition (iii) appears and is studied in ([5], [14], [13]). When $|A| = k$ the representation is akin to A. H. Stone's k-analyticity [15]. Generalizations of Souslin representations were also studied in [10].

In Section 2 we shall prove Theorems 1–4 and in Section 3 we generalize all results to p-spaces.

THEOREM 1. *A space with a meta-Souslin/para-Souslin representation is respectively metacompact/paracompact.*

THEOREM 2. *A Čech-complete space is metacompact/paracompact if and only if it has a meta-Souslin/para-Souslin representation.*

THEOREM 3. *A countable product of meta-Souslin/para-Souslin spaces is again meta-Souslin/para-Souslin.*

I am grateful to Ken Kunen for pointing out that a para-Souslin space is necessarily Čech-complete. We thus have the following characterisation.

THEOREM 4. *A Hausdorff space is paracompact and Čech-complete if and only if it has a para-Souslin representation.*

It is *not* true, however, that meta-Souslin implies Čech-complete.

EXAMPLE. *The space Q of rationals is meta-Souslin in its usual topology.*

Proof. Enumerate Q as $\langle q_n\colon n \in \omega\rangle$. Taking $A = N$ choose $V(i_1, \ldots, i_n)$ to be, for $i_2 = \ldots = i_n = 1$, an open interval of rationals centered on q_{i_1} of diameter less than 2^{-n} and so small that it excludes the points q_m for $m < i_1$. Otherwise let $V(i_1, \ldots, i_n) = \emptyset$. Since $q_m \notin V(i_1, \ldots, i_n)$ for all $i_1 > m$ we see that:

$$|\{(i_1, \ldots, i_n)\colon x \in V(i_1, \ldots, i_n)\}| \leqslant m \quad \text{if} \quad x = q_m.$$

To see that (iv) holds in the definition of a meta-Souslin representation note that if $K(\underline{i}) = \bigcap_{n=1}^{\infty} V(i_1, \ldots, i_n) = \emptyset$, then for some $N > 1$, $i_N \neq 1$ (otherwise $K(\underline{i}) = K(i_1, 1, 1 \ldots) = \{q_{i_1}\}$) and so $V(i_1, \ldots, i_N) = \emptyset$. The rest is clear.

REMARK AND PROBLEM. Let X be meta-Souslin with $A = N$. The compact valued mapping $\underline{a} \mapsto K(\underline{a})$ is upper-semicontinuous and inverse images of points under K are compact (by (2) and König's lemma). It follows from this that if X is a subspace of a Hausdorff space Y whose open sets are Souslin-$\mathscr{F}$ then X is Borelian-$\mathscr{G}(Y)$. Indeed

$$\Phi = \bigcup_{\underline{a} \in A^N} K(\underline{a}) \times \{\underline{a}\}$$

is closed in $Y \times A$ (by semi-continuity) and "vertical" sections are compact. Hence by Novikov's projection theorem, as generalized by D. G. Larman [11], $X = \mathrm{proj}_Y \Phi$ is the complement of a Souslin-$\mathscr{F}$ set and being analytic is Borelian-$\mathscr{G}(Y)$ by Lusin's Separation Theorem. (This argument was observed in [14].)

Thus X is in some sense absolutely Borel. Is every (metrizable) descriptive Borel set meta-Souslin? Or, perhaps, must a meta-Souslin set necessarily be $\mathscr{F}_{\sigma\delta}$ in any extension? Added in proof: Yes. See atend.

§ **2. Proofs of Theorems.** The proofs of the "meta" and "para" cases of Theorems 1–3 are almost identical. We therefore prove the "meta" cases first and then indicate what minor changes are needed for the "para" case.

Proof of Theorem 1. Assume X has a meta-Souslin representation as in the definition and let $\mathscr{U}$ be a given cover of X; we have to find an open point-finite refinement of $\mathscr{U}$ covering X. Observe that for any $\underline{a} \in A^N$ there is a finite number of sets, say $U_1, \ldots, U_k$ in $\mathscr{U}$ so that

$$K(\underline{a}) \subseteq U_1 \cup \ldots \cup U_k$$

(since $K(\underline{a})$ is compact), thus by clause (iv) there is N so that

$$V(a_1, \ldots, a_N) \subseteq U_1 \cup \ldots \cup U_k.$$

With this in mind we put

$$S = \{(a_1, \ldots, a_n) \in \bigcup_{m=1}^{\infty} A^m\colon \exists \mathscr{U}' \text{ finite subset of } \mathscr{U} \text{ with } V(a_1, \ldots, a_n) \subseteq \bigcup \mathscr{U}'\}$$

and let

$$T = \{(a_1, \ldots, a_n) \in S\colon (a_1, \ldots, a_{n-1}) \notin S\},$$

i.e., the set of sequences in S of minimal length. Note that T as a tree (ordered by extension of sequences) has no infinite branches. For each $(a_1, \ldots, a_n) \in T$ choose a finite system $\mathscr{U}(a_1, \ldots, a_n) \subseteq \mathscr{U}$ so that $V(a_1, \ldots, a_n) \subseteq \bigcup \mathscr{U}(a_1, \ldots, a_n)$ and let

$$\mathscr{V}(a_1, \ldots, a_n) = \{V(a_1, \ldots, a_n) \cap U\colon U \in \mathscr{U}(a_1, \ldots, a_n)\},$$

$$\mathscr{V} = \bigcup_{(a_1, \ldots, a_n) \in T} \mathscr{V}(a_1, \ldots, a_n).$$

We claim $\mathscr{V}$ is a point-finite refinement. To prove this consider for any $x \in X$ the tree

$$T_x = \{(a_1, \ldots, a_k)\colon \exists n \geqslant k (a_1, \ldots, a_k, \ldots, a_n) \in T \text{ and } x \in V(a_1, \ldots, a_k)\}.$$

Each level of T_x is finite by (2) and, by definition of T, T_x has no infinite branches. Hence by König's lemma T_x is finite. Thus by clause (i):

$$|\{V \in \mathscr{V}\colon x \in V\}| \leqslant |T_x| \max_{(a_1, \ldots, a_n) \in T_x \cap T} |\mathscr{U}(a_1, \ldots, a_n)|.$$

Evidently $\mathscr{V}$ is a cover: given $x \in X$, there is an $\underline{a}^* \in A^N$ with $x \in K(\underline{a}^*)$. By our initial step, there is a least n so that $V(a_1^*, \ldots, a_n^*)$ is finitely covered by $\mathscr{U}$. Thus $(a_1^*, \ldots, a_n^*) \in T$.

Para-Souslin case. This proceeds as before but now we show that $\mathscr{V}$ is locally-finite. Consider $x \in X$. Choose for each n an open set W_n with $x \in W_n$, so that

$$\{(a_1, \ldots, a_n) \in A^n\colon V(a_1, \ldots, a_n) \cap W_n \neq \emptyset\} \text{ is finite.}$$

We may assume $W_1 \supseteq W_2 \supseteq \ldots$ This time we take

$$T_x = \{(a_1, \ldots, a_k)\colon \exists (a_1, \ldots, a_k, \ldots, a_n) \in T \text{ and } V(a_1, \ldots, a_k) \cap W_k \neq \emptyset\}.$$

Here again T_x has finite levels and no infinite branches, hence is finite. Let

$$m = \max\{n\colon (a_1, \ldots, a_n) \in T_x\}.$$

Then with $W = W_{m+1}$, we have

$$|\{V \in \mathscr{V}\colon V \cap W \neq \emptyset\}| \leqslant |T_x| \max_{(a_1, \ldots, a_n) \in T_x} |\mathscr{U}(a_1, \ldots, a_n)|.$$

Indeed $V \in \mathscr{V}$ implies $V = U \cap V(a_1, \ldots, a_n)$ for some $(a_1, \ldots, a_n) \in T$ and $U \in \mathscr{U}(a_1, \ldots, a_n)$. Suppose $V \cap W \neq \emptyset$. Evidently if $n > m$, then since $(a_1, \ldots, a_{m+1}) \notin T_x$ we have

$$V(a_1, \ldots, a_{m+1}) \cap W_{m+1} = \emptyset$$

and so

$$V \cap W \subseteq V(a_1, \ldots, a_{m+1}) \cap W_{m+1} = \emptyset.$$

Thus $n \leqslant m$, but then $V(a_1, \ldots, a_n) \cap W_n \neq \emptyset$, hence $(a_1, \ldots, a_n) \in T_x$.

Note. The choice of T above is an implicit use of the paracompactness of the Baire space A^N (qua countable product of discrete spaces).

Proof of Theorem 2. We have to obtain a meta-Souslin representation of X given that X is metacompact and Čech-complete. Write $X = \bigcap_{n=1}^{\infty} G_n$ where each G_n is open in βX. Define inductively a sequence of *point-finite* open covers of X, $\langle \mathscr{V}_n\colon n \in N\rangle$, by demanding that

$\mathscr{V}_1$ refines $\{U \cap X\colon U \text{ open in } \beta X \text{ and } \operatorname{cl}_{\beta X} U \subseteq G_1\}$,

$\mathscr{V}_{n+1}$ refines $\{U \cap X\colon U \text{ open in } \beta X,\ \operatorname{cl}_{\beta X} U \subseteq G_{n+1} \text{ and, for some } V \in \mathscr{V}_n,\ \operatorname{cl}_X U \cap X \subseteq V\}$.

Note that the last collection is a cover of X because X is regular. If we allow $\emptyset \in \mathscr{V}_n$, then a set A (of cardinality $\sup_n |\mathscr{V}_n|$) may be found so that

(a) $\mathscr{V}_n = \{V(a_1, \ldots, a_n)\colon a_1, \ldots, a_n \in A\}$,

(b) $\operatorname{cl}_X V(a_1, \ldots, a_{n+1}) \subseteq V(a_1, \ldots, a_n)$

and the indexing in (a) satisfies (2) of clause (i) of the definition of meta-Souslin representation. We indicate a construction for (a) and (b): Write $\mathscr{V}_1 = \{V(\alpha)\colon \alpha < k_1\}$ where $k_1 = |\mathscr{V}_1|$. For each $\alpha < k_1$ define inductively,

$$\mathscr{V}_2(\alpha) = \{V \in \mathscr{V}_2\colon \operatorname{cl}_X V \subseteq V(\alpha)\} \setminus \bigcup_{\beta < \alpha} \mathscr{V}_2(\beta).$$

Now let

$$\mathscr{V}_2(\alpha) = \{V(\alpha_1, \alpha_2)\colon \alpha_2 < k_2\}$$

where $k_2 = |\mathscr{V}_2|$ and some or all members may be $\emptyset$ if k_2 is too large an index set. Continue inductively and then replace $k_1, k_2, \ldots$ by an upper bound. Since we do not repeat non-empty sets in this process, it is clear that (2) will hold.

We now observe that for $\underline{a} \in A^N$, we have

$$\bigcap_{n=1}^{\infty} \operatorname{cl}_{\beta X} V(a_1, \ldots, a_n) \subseteq \bigcap_{n=1}^{\infty} G_n = X,$$

hence

$$\begin{aligned} K(\underline{a}) &= \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n) \\ &= \bigcap_{n=1}^{\infty} \operatorname{cl}_X V(a_1, \ldots, a_n) = \bigcap_{n=1}^{\infty} X \cap \operatorname{cl}_{\beta X}(V(a_1, \ldots, a_n)) \\ &= \bigcap_{n=1}^{\infty} \operatorname{cl}_{\beta X} V(a_1, \ldots, a_n), \end{aligned}$$

since the latter is a subset of X. Thus $K(\underline{a})$ is compact.

Finally consider G open in X with $K(\underline{a}) \subseteq G$. Choose U open in βX with $G = U \cap X$, then

$$\bigcap_{n=1}^{\infty} \operatorname{cl}_{\beta X} V(a_1, \ldots, a_n) \subseteq U.$$

By compactness of βX there is N so that

$$\operatorname{cl}_{\beta X} V(a_1, \ldots, a_N) = \bigcap_{n=1}^{N} \operatorname{cl}_{\beta X} V(a_1, \ldots, a_n) \subseteq U.$$

Thus

$$V(a_1, \ldots, a_{N+1}) \subseteq X \cap \operatorname{cl}_{\beta X} V(a_1, \ldots, a_N) \subseteq X \cap U = G.$$

Proof of Theorem 3. This is entirely combinatorial. Let $\langle X_m\colon m \in N\rangle$ be a sequence of meta-Souslin (para-Souslin) spaces. We may assume that for one and the same set A

$$X_m = \bigcup_{\underline{a}\in A^N} \bigcup_{n=1}^{\infty} V_m(a_1, \dots, a_n)$$

is a meta- (para-) Souslin representation. Define

$$W(a_1, \dots, a_n) = \prod_{m=1}^{k} V_m(a_{2^{m-1}}, a_{3\cdot 2^{m-1}}, a_{5\cdot 2^{m-1}}, \dots, a_{(2s_m-1)2^{m-1}}) \prod_{m=k+1}^{\infty} X_m ,$$

where $n = 2^{k-1}(2s-1)$ and, for each $m \leqslant k$, s_m satisfies

$$2^{m-1}(2s_m-1) \leqslant n < 2^{m-1}(2s_m+1) .$$

It will be noticed that $k \leqslant n$ and that for each $m \leqslant k$ we have extracted from $a_1, \dots, a_n$ a subsequence corresponding to indices which are odd multiples of a fixed power (2^{m-1}) of 2. It is a routine exercise to check that

$$\bigcup_{\underline{a}\in A^N} \bigcap_{n=1}^{\infty} W(a_1, \dots, a_n)$$

is a meta- (para-) Souslin representation of $\prod_{n=1}^{\infty} X_n$. The combinatorial trick is standard in descriptive set theory.

Proof of Theorem 4. The proof is due to Kunen. Let X have a para-Souslin representation. By Theorem 1, X is paracompact and so is normal, hence completely regular. Thus βX exists. Recall that an open set U is said to be *regular* if $\operatorname{int}(\operatorname{cl} U) = U$ (of course if V is open then $\operatorname{int}(\operatorname{cl} V)$ is regular). Appealing to the condition (3) for a para-Souslin representation, construct a cover $\mathscr{U}_n$ of X consisting of regular open sets U, refining $\mathscr{V}_n$, so that for each U in $\mathscr{U}_n$ the set

$$\{(a_1, \dots, a_n):\ V(a_1, \dots, a_n) \cap U \neq \emptyset\} \tag{4}$$

is finite. Now define in βX the open set

$$G_n = \bigcup \{\operatorname{int}_{\beta X} \operatorname{cl}_{\beta X} U:\ U \in \mathscr{U}_n\} .$$

Clearly $X \subseteq \bigcap_{n=1}^{\infty} G_n$. We prove that this is in fact an equality.

Consider $p \in \bigcap_{n=1}^{\infty} G_n \setminus X$. For each n choose $U_n \in \mathscr{U}_n$ so that $p \in \operatorname{int}\operatorname{cl}_{\beta X} U_n$ and choose a zero set Z_n of X with

$$p \in \operatorname{cl}_{\beta X} Z_n \subseteq \operatorname{int}\operatorname{cl}_{\beta X} U_n ,$$

(because $\{p\} = \bigcap_{Z\in p} \operatorname{cl}_{\beta X} Z$, cf. [7]). Thus (on intersection with X) $Z_n \subseteq U_n$. Observe that since $Z_i \in p$ all i, we always have $Z_1 \cap \dots \cap Z_n \neq \emptyset$. Now let

$$T_p = \{(a_1, \dots, a_n):\ (\exists Z \in p) Z \subseteq V(a_1, \dots, a_n)\} .$$

We have just shown that T_p is infinite, because $\mathscr{U}_n$ refines $\mathscr{V}_n$. We next claim that T_p is finitely branching. For, consider any $(a_1, \dots, a_n)$ with $V(a_1, \dots, a_n) \cap U_n = \emptyset$. Thus if $Z \subseteq V(a_1, \dots, a_n)$ then $Z \cap Z_n \subseteq Z \cap U_n = \emptyset$, hence $Z \notin p$ by our earlier observation. By finiteness of (4), T_p is finitely branching. Thus by König's lemma there is a branch $\underline{a}^* \in A^N$ passing through T_p. Choose $Z_n^* \in p$ with $Z_n^* \subseteq V(a_1^*, \dots, a_n^*)$. Now $p \notin K(\underline{a}^*)$ so there is a zero-set $Z_0 \in p$ with $Z_0 \cap K(\underline{a}^*) = \emptyset$ (since by definition $K(\underline{a})$ is compact). By clause (iv) of the definition of a para-Souslin representation, there exists N so that $Z_0 \cap V(a_1^*, \dots, a_N^*) = \emptyset$. But

$$\emptyset \neq Z_0 \cap Z_1^* \cap \dots \cap Z_N^* \subseteq Z_0 \cap V(a_1) \cap \dots \cap V(a_1, \dots, a_N) .$$

This is a contradiction. Hence $X = \bigcap_{n=1}^{\infty} G_n$. This completes our proof.

§ 3. Generalization to p-spaces. We recall that a space X is called a *p-space* (Arhangel'skiĭ [1]), if there is a sequence $\langle \mathscr{U}_n:\ n \in N\rangle$ of families of open subsets of βX such that for $x \in X$

$$x \in \bigcap_{n=1}^{\infty} \operatorname{St}(x, \mathscr{U}_n) \subseteq X ,$$

where $\operatorname{St}(x, \mathscr{U}_n) = \bigcup \{U \in \mathscr{U}_n:\ x \in U\}$.

Theorems 1–3 continue to hold, provided we *weaken* clause (iv) to read:

(iv)′ for each $\underline{a} \in A^N$ with $K(\underline{a}) \neq \emptyset$ and for every open set U of x with $U \supseteq K(\underline{a})$, there is n so that $V(a_1, \dots, a_n) \subseteq U$.

We shall call such representations *weakly meta-Souslin* or *weakly para-Souslin* according as (2) or (3) is invoked in clause (i). We now have:

THEOREMS.

1′. *Every weakly meta-Souslin/weakly para-Souslin space is metacompact/paracompact.*

2′. *A p-space is metacompact/paracompact if and only if it is weakly meta-Souslin/weakly para-Souslin.*

3′. *A countable product of weakly meta-Souslin/weakly para-Souslin spaces is again weakly meta-Souslin/weakly para-Souslin.*

4′. *A Hausdorff space is a paracompact p-space if and only if it is weakly para-Souslin.*

Proof. It is easily checked that weakening clause (iv) does not affect the proofs of Theorems 1 and 3. With regard to Theorem 2, one defines $\mathscr{V}_n$ inductively so that

$\mathscr{V}_1$ refines $\{U \cap X:\ U$ open in βX and $(\exists U' \in \mathscr{U}_1) \operatorname{cl}_{\beta X} \subseteq U'\}$,

$\mathscr{V}_{n+1}$ refines $\{U \cap X:\ U$ open in βX and $(\exists U' \in \mathscr{U}_{n+1}) \operatorname{cl}_{\beta X} U \subseteq U'$ and $(\exists V \in \mathscr{V}_n) \operatorname{cl}_X U \cap X \subseteq V\}$.

Writing as before $V(a_1, \dots, a_n) = X \cap U(a_1, \dots, a_n)$ with $U(a_1, \dots, a_n)$ open in βX, we need to show that

$$K(\underline{a}) = \bigcap_{n=1}^{\infty} V(a_1, \dots, a_n)$$

is compact. So let $x \in K(\underline{a})$ (if $K(\underline{a}) = \emptyset$, this is already compact) and consider that

$$x \in K(\underline{a}) \subseteq \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X} U(a_1, \ldots, a_n).$$

Thus for each n, $x \in \mathrm{cl}_{\beta X} U(a_1, \ldots, a_n)$; but by definition of $\mathscr{V}_n$ there is $U' \in \mathscr{U}_n$ so that

$$x \in \mathrm{cl}_{\beta X} U(a_1, \ldots, a_n) \subseteq U'.$$

Hence

$$\mathrm{cl}_{\beta X} U(a_1, \ldots, a_n) \subseteq \mathrm{St}(x, \mathscr{U}_n).$$

Thus

$$\bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X} U(a_1, \ldots, a_n) \subseteq \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{U}_n) \subseteq X;$$

that is, for $K(\underline{a}) \neq \emptyset$ we have

$$\begin{aligned} K(\underline{a}) &= \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n) \\ &= \bigcap \mathrm{cl}_X V(a_1, \ldots, a_{n+1}) \\ &= \bigcap X \cap \mathrm{cl}_{\beta X} U(a_1, \ldots, a_{n+1}) \\ &= \bigcap_{n=1}^{\infty} \mathrm{cl}_{\beta X} U(a_1, \ldots, a_n). \end{aligned}$$

which implies clause (ii) and (iv)′.

To see that the Souslin-representation "covers" X, note that for any $x \in X$ the infinite tree

$$T_x = \{(a_1, \ldots, a_n) \colon x \in V(a_1, \ldots, a_n)\}$$

if finitely branching both in the case of point-finiteness and, *a fortiori*, local-finiteness assumptions. By König's lemma there is a branch $\underline{a}^* \in A^N$ through T_x and we have $x \in \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n^*) = K(\underline{a}^*)$.

Minor alterations to the proof of Theorem 4 yield Theorem 4′ as follows. In place of G_n consider

$$\mathscr{G}_n = \{\mathrm{int}_{\beta X} \mathrm{cl}_{\beta X} U \colon U \in \mathscr{U}_n\}.$$

Then for fixed $x \in X$ assume $p \in \bigcap_{n=1}^{\infty} \mathrm{St}(x, \mathscr{G}_n) \setminus X$ and consider, instead of T_p, the tree

$$T_{p,x} = \{(a_1, \ldots, a_n) \colon \exists Z \in p Z \subseteq V(a_1, \ldots, a_n) \text{ and } x \in V(a_1, \ldots, a_n\}.$$

As before this is infinite and finitely branching. If $\underline{a}^* \in A^N$ is a branch through $T_{p,x}$, then $x \in K(\underline{a}^*) \neq \emptyset$.

Remarks. 1. The Lindelöf case of the main theorem holds good for p-spaces, because we may choose $\mathscr{V}_n$ (as above) to be countable subcovers. Thus $A = N$ and we obtain a "weak Souslin representation" (clauses (ii)–(iv)′ obeyed). Now the refinement $\mathscr{V}$ of Theorem 1 is countable, from which it is a quick step to obtain a countable subcover from $\mathscr{U}$. We have:

A p-space is Lindelöf if and only if it has a weak Souslin representation.

2. Any metric space is metacompact and so is weakly meta-Souslin. (Let $V(a_1, \ldots, a_n)$ refine the balls of radius less than 2^{-n}.)

§ 4. **Some further generalizations.** It is possible to modify clause (i) to obtain other classes of spaces preserved under countable products. We note two examples relating to perfect normality and paracompactness.

4.1. A class of perfectly normal spaces. Consider a space representable in the form

$$X = \bigcup_{\underline{a} \in A^N} \bigcap_{n=1}^{\infty} V(a_1, \ldots, a_n)$$

where in addition to clauses (i) and (iv) of Section 1 we also require that

(i)′ $\mathscr{V}_n = \{V(a_1, \ldots, a_n) \colon a_1, \ldots, a_n \in A\}$ is, for each n, σ-closure preserving, i.e., $\mathscr{V}_n = \bigcup_{m=1}^{\infty} \mathscr{V}_n^m$ and, for each m, $\mathrm{cl} \bigcup \mathscr{V}'_m = \bigcup \{\mathrm{cl}\, V \colon V \in \mathscr{V}'_m\}$ for any subset $\mathscr{V}'_m$ of $\mathscr{V}_n^m$ (meaning that $\mathscr{V}_n^m$ is closure preserving).

(iii)′ $\mathrm{cl}_X V(a_1, \ldots, a_{n+1}) \subseteq V(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_{n+1} \in A$, and, moreover, each $K(\underline{a})$ is *empty or a singleton.*

Such a space X is necessarily perfectly normal. For, given G open in X, let

$$\mathscr{V} = \{V(a_1, \ldots, a_n) \colon V(a_1, \ldots, a_n) \subseteq G\}.$$

By (i)′ write $\mathscr{V} = \bigcup \mathscr{V}^{(n)}$. Then by (iii)′ and (iv)

$$G = \bigcup_n \mathrm{cl}(\bigcup \mathscr{V}^{(n+1)})$$

so that G is a countable union of closed sets. To deduce normality consider two disjoint closed subsets A, B of X. Take G above to be $X \setminus B$. Write $V_n = \bigcup \mathscr{V}^{(n)}$, then $A \subseteq \bigcup_n V_{(n+1)}$ and

$$B \cap \mathrm{cl}_X V_{(n+1)} = \emptyset.$$

The sets $V_{(n+1)}$ may now be used to separate A and B (compare Wilansky [17] p. 77).

This class of spaces is obviously countably productive. It is equally clear that a space thus representable (perhaps "analytically normal" might be a name for this) has the further property that any open cover has an open σ-closure preserving refinement and so is paracompact.

4.2. A class of paracompact spaces.

THEOREM. *Let X have a Souslin representation with clause* (i) *modified to require that for each n $\{V(a_1, \ldots, a_n) \colon a_1, \ldots, a_n \in A\}$ is σ-discrete* (cf. Hansell [8]). *Then X is paracompact.*

Proof. We argue as in Theorem 1. Let $\mathscr{U}$ be an open cover of X and for each n let

$$\{V(a_1, \ldots, a_n): a_1, \ldots, a_n \in A\} = \bigcup_{m=1}^{\infty} V_m^n$$

with each V_m^n discrete. Now put

$$S_m^n = \{(a_1, \ldots, a_n): V(a_1, \ldots, a_n) \in V_m^n \text{ and } \exists \mathscr{U}' \text{ finite subset of } \mathscr{U} \text{ with } V(a_1, \ldots, a_n) \subseteq \bigcup \mathscr{U}'\}.$$

We have

$$A^N = \bigcup_{n,m} \bigcup_{(a, \ldots a_n) \in S_m^n} \{\underline{b} \in A^N: (b_1, \ldots, b_n) = (a_1, \ldots, a_n)\}.$$

Let

$$T_m^n = \{(a_1, \ldots, a_n) \in S_m^n: \text{non } \exists k<n \text{ non } \exists h (a_1, \ldots, a_k) \in S_h^k\}.$$

Then, since T_m^n picks out sequences of minimal length,

$$A^N = \bigcup_{n,m} \bigcup_{(a_1, \ldots a_n) \in T_m^n} \{\underline{b} \in A^N: (b_1, \ldots, b_n) = (a_1, \ldots, a_n)\}$$

and the summands are thus mutually disjoint. Now for $(a_1, \ldots, a_n) \in T_m^n$ choose a finite system $\mathscr{U}(a_1, \ldots, a_n) \subseteq \mathscr{U}$ covering $V(a_1, \ldots, a_n)$. Then

$$\mathscr{V}(a_1, \ldots, a_n) = \{U \cap V(a_1, \ldots, a_n): U \in \mathscr{U}(a_1, \ldots, a_n)\}$$

is finite, so that

$$\mathscr{V}_{n,m} = \bigcup_{(a_1, \ldots, a_n) \in T_m^n} \mathscr{V}(a_1, \ldots, a_n)$$

is σ-discrete (since V_m^n is discrete) and finer than $\mathscr{U}$. Hence

$$\mathscr{U} = \bigcup_{n,m} \mathscr{U}_{n,m}$$

is a σ-discrete open refinement of $\mathscr{U}$ covering X. Thus X is paracompact (cf. Engelking [4] p. 376).

Again the property envisaged by this kind of Souslin representation is preserved under countable products.

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Added in proof. We have two observations to make.

1. Any regular meta-Souslin space X with representation (1) subject to A being countable is an $\mathscr{F}_{\sigma\delta}$ subset of any regular space Y containing X. To see this choose for each $a_1, \ldots, a_n$ in A a set $W(a_1, \ldots, a_n)$ open in Y so that

$$V(a_1, \ldots, a_n) = W(a_1, \ldots, a_n) \cap X.$$

Now put

$$T(a_1, \ldots, a_n) = \{(a_1, \ldots, a_n, \ldots, a_m);\ \mathrm{cl}_Y W(a_1, \ldots, a_m) \subseteq W(a_1, \ldots, a_n)\},$$

$$Y_k^n = \{y \in Y: (\exists \text{ distinct } \sigma_1, \ldots, \sigma_k \text{ in } A^n)\ y \in W(\sigma_1) \cap \ldots \cap W(\sigma_k)\}.$$

Then Y_k^n is open and so

$$Z = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} Y \setminus Y_k^n$$

is $\mathscr{F}_{\sigma\delta}$ and by hypothesis on the representation of X we have $X \subseteq Z$. Assuming, without loss of generality, that X is dense in Y, we obtain $X \subseteq W$, where

$$W = \bigcap_{n=1}^{\infty} \bigcup_{(a_1, \ldots, a_n)} \bigcup_{(b_1, \ldots, b_m) \in T(a_1, \ldots, a_n)} \mathrm{cl}_Y W(b_1, \ldots, b_m).$$

But W is an $\mathscr{F}_{\sigma\delta}$ set in Y and finally, by König's Lemma, we have

$$X = Z \cap W.$$

I am grateful to Leo Harrington for the idea of using the set Z.

2. Józef Chaber has pointed out to us that a completely regular space with a weakly meta-Souslin representation is necessarily a p-space. We thus have a characterization of meta-compact p-spaces. To prove this observe that the open covers $\mathfrak{V}_n = \{V(a_1, \ldots, a_n): a_1, \ldots, a_n \in A\}$ satisfy the condition (p) that if $V_n \in \mathfrak{V}_n$ and $\bigcap_{n=1}^{\infty} V_n \neq \emptyset$ then for any centered family $\mathscr{A}$ such that for each n there is A in $\mathscr{A}$ with $A \subseteq V_n$ we have $\bigcap \{\overline{A}: A \in \mathscr{A}\} \neq \emptyset$. We omit the details.

References

[0] A. V. Arhangel'skiĭ, *On a class of spaces containing all metric and all bicompact spaces.* Dokl. Akad. Nauk. SSSR 151 (1963), pp. 751–754.

[1] C. H. Dowker and D. Strauss, *Sums in the category of frames*, Houston J. Math. 3 (1976), pp. 17–32.

[2] M. M. Čoban, *On the theory of p-spaces*, Soviet Math. Dokl. 11 (1970), pp. 1257–1260.

[3] G. Choquet, *Ensembles K-analytiques et K-souslinien*, Annl. Inst. Fourier, Grenoble 9 (1959), pp. 75–89.

[4] R. Engelking, *General Topology*, Warszawa 1977.

[5] Z. Frolík, *On the descriptive theory of sets*, Czech. J. Math. 13 (88) (1963), pp. 561–573.

[6] — *On the topological product of paracompact spaces*, Bull. Acad. Polon. Sci. 8 (1960), pp. 747–750.

[7] L. Gillman and M. Jerison, *Rings of Continuous Functions*, New York 1960.

[8] R. W. Hansell, *On the non-separable theory of K-Borel and K-Souslin sets*, Gen. Top. and Appl. 3 (1973), pp. 161–195.

[9] J. Jayne, *Structure of analytic Hausdorff spaces*, Mathematika 23 (1976), pp. 208–211.

[10] A. H. Kruse, *Souslinoid and analytic sets in a general setting*, Memoirs AMS 86.

[11] D. G. Larman, *Projecting and uniformising Borel sets with $\mathcal{K}_\sigma$-sections*, Mathematika 19 (1972), pp. 231–244. Also 20 (1973), pp. 233–246.

[12] K. Morita, *Products of normal spaces with metric spaces*, Math. Ann. 154 (1964), pp. 365–382.

[13] A. J. Ostaszewski, *On Lusin's separation principle in Hausdorff spaces*, Proc. LMS 27 (3), (1973), pp. 649–666.

[14] A. J. Ostaszewski, *Descriptive set theory in Hausdorff spaces*, Ph. D. thesis, University College, London (1973).
[15] C. A. Rogers, *Analytic sets in Hausdorff spaces*, Mathematika 11 (1964), pp. 1–8.
[16] A. H. Stone, *Non-separable Borel sets*, Dissertationes Math. 28 (1962).
[17] A. Wilansky, *Topology for Analysis*, London 1970.

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Sur les fonctions de deux variables équicontinues par rapport à une variable

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Résumé. Dans cet article on considère la mesurabilité (la propriété de Baire) d'une fonction $f\colon X\times Y\to R$ définie sur le produit cartésien d'un espace topologique X et d'un espace mesurable Y (et d'un espace topologique Y) dont toutes les sections f^y sont équicontinues et toutes les sections f_x sont mesurables (ont la propriété de Baire).

Soient (X, T_1) et (Y, T_2) des espaces topologiques et R l'espace des nombres réels. Si $X = Y = R$ et $T_1 = T_2$ est la topologie euclidienne et si $f\colon R\times R\to R$ est une fonction, la mesurabilité (au sens de Lebesgue) de toutes les sections $f_x(y) = f(x, y)$ $(x, y\in R)$ et la continuité de toutes les sections $f^y(x) = f(x, y)$ impliquent la mesurabilité (au sens de Lebesgue sur le plan $R\times R$) de la fonction f. Ce théorème ne reste plus vrai dans le cas des fonctions réelles définies sur le produit de deux espaces topologiques. En effet, l'hypothèse du continu implique l'existence d'une fonction $f\colon R\times R\to R$ non mesurable au sens de Lebesgue et telle que toutes ses sections f_x sont mesurables au sens de Lebesgue et toutes ses sections f^y sont approximativement continues (donc continues relativement à la topologie de densité [2]). Dans l'article [3] les auteurs examinent les cas particuliers dans lesquels le théorème considéré reste vrai.

Dans la première partie de cet article je démontre que l'équicontinuité de toutes les sections f_x d'une fonction $f\colon X\times Y\to R$ (X et Y étant des espaces topologiques) et la μ_2-mesurabilité (propriété de Baire) de toutes les sections f^y impliquent la $\overline{\mu_1\times\mu_2}$-mesurabilité (propriété de Baire) de la fonction f, en admettant des hypothèses supplémentaires sur les mesures μ_1 et μ_2.

Dans la deuxième partie je démontre que l'équicontinuité approximative de toutes les sections f_x d'une fonction $f\colon I\times I\to R$, où $I = [0, 1]$, et que la propriété de Baire par rapport à la topologie euclidienne de toutes les sections f^y impliquent la propriété de Baire par rapport à la topologie euclidienne sur $I\times I$ de la fonction f.