Non-accessible points in extremally disconnected compact spaces * I

by

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Abstract. In this note we prove under assumption MA: 1. In each extremally disconnected compact space there exists a non-limit point of any countable discrete subset 2. In $G(I)$ (Gleason space over interval [0, 1]) there exists a non-limit point of any discrete subset of cardinality $<2^\omega$ 3. Kunen's Theorems on non-limit points from [11].

The paper deals with points in compact (Hausdorff) extremally disconnected spaces (and more generaly in F-spaces) which are not limit points of countable discrete subspaces. We show (Theorem 3) that in ZFC+(B) (where (B) is a known consequence of Martin's Axiom (MA) due to Booth [2]) in extremally disconnected compact spaces X of weight $X \leqslant 2^\omega$ such points exist. Under some more restrictive assumptions concerning the spaces we obtain much stronger conclusions. Namely, if the π-weight of X is countable (e.g. if X is the Gleason space over a compact metric space), then the space X contains a dense countable subset D such that no point of D is a limit point of any discrete subset of cardinality $<2^\omega$ (Theorem 5). This result can be applied, for example, to obtain the following fact (Kunen [11], ZFC+(MA)): $\beta N \setminus N$ contains a countable set, dense in itself, of points such that none of the points is a limit point of any discrete set of cardinality $<2^\omega$. We prove that if $cf(m) = m < 2^\omega$, then there exist points in βm which are not limit points of strongly discrete subsets (i.e. subsets A such that there exists a disjoint cover of A by open subsets of βm such that $U \cap A$ is a one-point set for each U of the cover) of βm having the cardinality $<m$. (Theorem 4). This result can be applied to obtain the existence in $\beta N \setminus N$ of points which are not P-points and which are not limit points of any countable discrete subset, but are limit points of discrete sets of cardinality ω_1.

Our result seems to be interesting in view of the following result by Kunen [11]: in $\beta N \setminus N$ there exist non-P-points which are not limit points of subsets having cardinality $<2^\omega$. This result by Kunen, and his other results from papers [11] and [10] concerning non-limit non-P-points of $\beta N \setminus N$, are shown to be consequences of our theorems concerning non-limit points in extremally disconnected spaces.

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All the results mentioned above were obtained under the assumption of (MA) or (B), which is weaker than (MA). However, as was recently shown by Kunen, results of this kind can be obtained in ZFC: namely, Kunen proved that in $\beta N\setminus N$ there exist non-P-points which are not limit point of countable discrete subsets; it would be interesting to obtain such a result for extremally disconnected compact spaces without isolated points (in particular, for those having weight $\leqslant 2^{\omega}$).

The problem of the existence of non-limit points of discrete countable subsets is related to other problems in topology. It is known (cf. Grizlov [7]) that if x is a limit point of a discrete countable subspace of a compact extremally disconnected space X without isolated points, then $X\setminus\{x\}$ is not normal.

The points of $\beta N\setminus N$ which are not limit points of discrete countable subsets are known to be minimal in the Rudin–Frolík ordering of $\beta N\setminus N$ (as regards these questions we refer the reader to M. E. Rudin [14] T. Frolík [6] and W. Comfort and S. Negrepontis [3]).

1. Preliminaries. Most of the theorems of this paper will be proved under the following assumption:

(B) (Booth [2]) Let $\mathfrak{F}$ be a family of infinite subsets of N such that $|\bigcap K| = \omega$ for each finite subfamily K of $\mathfrak{F}$. If $|\mathfrak{F}|<2^{\omega}$, then there exists an infinite set Z, $Z\subset\omega$, such that $|Z\setminus A|<\omega$ for each $A\in\mathfrak{F}$.

The statement (B) is known to be a consquence of Martin's Axiom (MA) (D. Booth [2] and D. Martin and R. Solovay [12]). Let $N^* = \beta N\setminus N$, where βN denotes the Čech–Stone compactification of the countable discrete space N.

Let X be a compact space, and let λ be a cardinal number. The closed set $F\subset X$ will be called a *$P(\lambda)$-set of X* if for each family $\mathfrak{F}$ of open neighbourhoods of F, if $|\mathfrak{F}|<\lambda$, then $F\subset\operatorname{int}\bigcap\mathfrak{F}$; if $\lambda = \omega_1$, then such a set will be called a *P-set*; if $F = \{x\}$ then x will be called a *$P(\lambda)$-point*.

Let $\chi(F, X)$ denote the character of a set F in a space X, i.e. the minimal cardinality of a base of neighbourhoods of F in X (sometimes we will write $\chi(F)$ for $\chi(F, X)$).

Under the assumption (B), each closed subset of N^* of the character $<2^{\omega}$ contains a $P(2^{\omega})$-point of N^*.

The proof of this well-known fact is the same as the proof of the existence of P-points, which was carried out by W. Rudin in [15] under CH and by A. Blass [1] under (MA) and therefore is omitted.

For the notion of an F-space we refer the reader to [3] and [17]. We shall use only the following equivalent property of being a totally disconnected compact F-space (see e.g. Walker [17]): if X is a compact totally disconnected space, then X is an F-space iff for each countable family $\mathscr{A}$ of pairwise disjoint closed-and-open subsets of X, if $\mathscr{B}\subset\mathscr{A}$, then there exists a closed-and-open set $V\subset X$ such that $V\cap\bigcup(\mathscr{A}\setminus\mathscr{B}) = \emptyset$ and $V\supset\bigcup\mathscr{B}$.

From this reformulation it immediately follows that any compact extremally disconnected space is an F-space.

Let $\mathscr{A}$ be a countable infinite set of pairwise disjoint closed-and-open subsets of X; if $\operatorname{cl}\bigcup\mathscr{A} = X$, then $\mathscr{A}$ will be called a *quasi-partition of X*.

If $\mathscr{A} = \{A^n|\ n\in\omega\}$ is a quasi-partition of a compact F-space X, then the map $\pi_{\mathscr{A}}\colon X\overset{\text{onto}}{\longrightarrow}\beta N$ given by the formula

$$\pi_{\mathscr{A}}(x) = \bigcap\{\operatorname{cl}K|\ K\subset N \text{ and } x\in\operatorname{cl}\cup\{A^n|\ n\in K\}\}$$

for $x\in X$, is continuous and open. Moreover, if $x\in A^n$, then $\pi_{\mathscr{A}}(x) = n$.

Conversely, if an open continuous map $\pi\colon X\overset{\text{onto}}{\longrightarrow}\beta N$ from a compact F-space is given, then $\mathscr{A} = \{\pi^{-1}(n)|\ n\in N\}$ forms a quasi-partition of X such that $\pi = \pi_{\mathscr{A}}$.

LEMMA 1. *Let X be a compact totally disconnected F-space. If for each point $x\in X$ and each open neighbourhood V of x there exists a quasi-partition $\mathscr{A}$ such that $x\in U\subset V$ for some U from $\mathscr{A}$, then X is homeomorphic to the inverse limit*

$$Y = \varprojlim\{\beta N, p^{\alpha}_{\beta}|\ \alpha, \beta\in\Sigma\}$$

such that each projection $\pi_{\alpha}\colon Y\overset{\text{onto}}{\longrightarrow}\beta N$ is open, where $|\Sigma|\leqslant w(X)^{\omega}$ and $p^{\alpha}_{\beta}\colon N\overset{\text{onto}}{\longrightarrow}N$ are onto.

Proof. Let $\mathfrak{A}$ be the collection of all quasi-partitions of X. Clearly, $|\mathfrak{A}|\leqslant w(X)^{\omega}$.

Consider a partial order on $\mathscr{A}$, assuming that $\mathscr{A}\prec\mathscr{B}$ if $\mathscr{B}$ is a refinement of $\mathscr{A}$, i.e. if for each $V\in\mathscr{B}$ there is a $U\in\mathscr{A}$ such that $V\subset U$.

The set $\mathfrak{A}$ is directed under $\prec$. In fact $\sup\{\mathscr{A},\mathscr{B}\} = \{A\cap B|\ A\in\mathscr{A}, B\in\mathscr{B}\}\setminus\{\emptyset\}$.

Let $\mathscr{A} = \{A^n|\ n\in\omega\}$; we shall assume that each quasi-partition has a fixed enumeration.

Let $\mathscr{B} = \{B^m|\ m\in\omega\}$ be a quasi-partition such that $\mathscr{A}\prec\mathscr{B}$. We define a function $p^{\mathscr{B}}_{\mathscr{A}}\colon N\overset{\text{onto}}{\longrightarrow}N$ such that $p^{\mathscr{B}}_{\mathscr{A}}(m) = n$ if $B^m\subset A^n$.

Let $\pi^{\mathscr{B}}_{\mathscr{A}} = \beta p^{\mathscr{B}}_{\mathscr{A}}\colon \beta N\overset{\text{onto}}{\longrightarrow}\beta N$.

Let us consider the space $Y = \varprojlim\{\beta N, \pi^{\mathscr{B}}_{\mathscr{A}}|\ \mathscr{A},\mathscr{B}\in\mathfrak{A}\}$. Since $\pi^{\mathscr{B}}_{\mathscr{A}}\pi_{\mathscr{B}} = \pi_{\mathscr{A}}$, where $\pi_{\mathscr{A}}$ are projections, there exists a continuous map $\pi^*\colon X\to Y$ induced by $\{\pi_{\mathscr{A}}|\ \mathscr{A}\in\mathfrak{A}\}$.

Since for each point $x\in X$ and each open neighbourhood V of x there exists a map $\pi_{\mathscr{A}}$ such that $x\in\pi^{-1}_{\mathscr{A}}(n)\subset V$ for some $n\in N$, thus for any points $x\neq y\in X$ there is a map $\pi_{\mathscr{A}'}\colon X\overset{\text{onto}}{\longrightarrow}\beta N$ such that $\pi_{\mathscr{A}'}(x)\neq\pi_{\mathscr{A}'}(y)$. Consequently, π^* is a homeomorphism. ∎

COROLLARY. *Each infinite compact extremally disconnected space is homeomorphic to the inverse limit $\varprojlim\{\beta N, \beta p^{\alpha}_{\beta}|\ \alpha, \beta\in\Sigma\}$, where $|\Sigma| = w(X)$.*

Proof. By Efimov's theorem from [5], $w(X) = w(X)^{\omega}$ for extremally disconnected compact spaces. ∎

If B is a Boolean algebra, then by $\operatorname{st}(B)$ we denote the Stone space of B.

Let $U(m) = \operatorname{st}(\mathscr{P}(m)/\mathscr{P}^{<m}(m))$, where m is a cardinal and $\mathscr{P}^{<m}(m)$ is the ideal of all subsets of cardinality less than m. The space $U(m)$ is an F-space (see Comfort

and Negrepontis [3]). If m is regular, then the set $U(m)$ is embedded in $\beta m = \mathrm{st}(\mathscr{P}(m))$ as a $P(m)$-set.

By Lemma 1, if $\mathrm{cf}(m)>\omega$ then $U(m)$ is homeomorphic to

$$Y = \varprojlim\{\beta N, \beta p^\alpha_\beta|\ \alpha, \beta \in \Sigma\},$$

where $|\Sigma|\leqslant m^\omega$ and the projections π_α: $Y \xrightarrow{\text{onto}} \beta N$ are induced by the set of all quasi-partitions of Y. If $\mathrm{cf}(m) = \omega$, then there are no open continuous maps from $U(m)$ onto βN.

The metric separable space X is called *measurable* if there exists a σ-additive measure ν: $B \to [0, 1]$, where B is the algebra of all Borel subsets of X such that $\nu(U)>0$ if U is an open nonempty set, and $\nu(\{x\}) = 0$ for $x \in X$, and ν is a regular Borel measure (see Halmos [8]).

Let $M(X) = B/I$, where $I = \{A \in B|\ \nu(A) = 0\}$. It is known that $M(X)$ is a complete Boolean algebra (see Halmos [8]).

Let $G(X)$ be the Gleason space over a regular space X, i.e. $G(X) = \mathrm{st}(\mathrm{RO}(X))$, where $\mathrm{RO}(X)$ is the complete Boolean algebra of regularly open sets in X.

A subsets D of X is said to be *strongly discrete* in the space X if there exists a disjoint family $\{V_d|\ d \in D\}$ of open subsets of X such that $V_d \cap D = \{d\}$.

Countably discrete subsets of any regular space X are obviously strongly discrete.

The reader is referred to books by Comfort and Negrepontis [3] (ultrafilters, βN, F-spaces, Gleason spaces, ...) Engelking [4] (character, weight, ...) Sikorski [15] (Boolean algebras) and to Halmos [8] (measure, measure algebras, ...).

2. Main results. Let $<_*$ denote the canonical well-ordering on $2^\omega\times 2^\omega$ (the so called max-lex, see T. Jech [9], p. 11).

Let $\mathfrak{F}$ be a centred family of closed-and-open subsets of X. We shall say that the open continuous map π: $X \xrightarrow{\text{onto}} \beta N$ *refines* $\mathfrak{F}$ if for each $V \in \mathfrak{F}$ there exists a finite subset F of N such that $\pi(X\setminus V)\subset F$.

Let X be a space and let $\{\pi_\alpha|\ \alpha<2^\omega\}$ be a family of open maps π_α: $X \xrightarrow{\text{onto}} \beta N$.

If $\{B^\alpha_\beta|\ (\alpha, \beta)<_* (\alpha_0, \beta_0)\}$ is a family of closed non-void subsets of X, and $\{\mathfrak{F}_\alpha|\ \alpha<\lambda\}$ is a collection of centred families of closed-and-open subsets of X, where $\lambda\leqslant 2^\omega$, then let

(1) $$K^{\alpha_0}_{\beta_0} = \bigcap\{B^\alpha_\beta|\ (\alpha, \beta)<_* (\alpha_0, \beta_0)\},$$

(2) $$L^{\alpha_0}_{\beta_0} = \begin{cases} K^{\alpha_0}_{\beta_0}, \text{ if } K^{\alpha_0}_{\beta_0} \cap \bigcap \mathfrak{F}_{\alpha_0} = \emptyset \text{ or if } \alpha_0\geqslant\lambda, \text{ or if } i(\alpha_0) \text{ does not exist,} \\ K^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{i(\alpha_0)}(N^*), \text{ otherwise,}\end{cases}$$

where $i(\alpha_0)$ is the least of the ordinals i such that $K^{\alpha_0}_{\beta_0} \cap \pi_i^{-1}(N^*) \neq \emptyset$, and π_i refines the family $\mathfrak{F}_{\alpha_0}$.

Let $\{U_\alpha|\ \alpha<2^\omega\}$ be a standard base for N^*.

LEMMA 2. (B) *Let X be a compact space and let $\{\pi_\alpha|\ \alpha<2^\omega\}$ be a family of open continuous maps π_α: $X \xrightarrow{\text{onto}} \beta N$. Let $\{\mathfrak{F}_\alpha|\ \alpha<\lambda\}$, where $\lambda\leqslant 2^\omega$, be a collection of centred families of closed-and-open sets such that $|\mathfrak{F}_\alpha|<2^\omega$. Then there exists a family of non-empty closed sets $\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$ having the following properties (the sets L^α_β are defined in accordance with* (1) *and* (2)):

(3) $B^\alpha_\beta\subset B^\alpha_{\beta'}$, *for* $(\alpha', \beta')<_*(\alpha, \beta)<_*(\alpha_0, \beta_0)$, *for all* $\alpha_0, \beta_0<2^\omega$.

(4) *if* $L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(N^*) = \emptyset$, *then* $B^{\alpha_0}_{\beta_0} = L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(\{n\}) \neq \emptyset$, *for some* $n \in N$,

(5) *if* $L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(N^*) \neq \emptyset$, *then* $B^{\alpha_0}_{\beta_0} = \pi^{-1}_{\alpha_0}(U)\subset L^{\alpha_0}_{\beta_0}$, *where U is a closed-and-open set in N^*, such that* $U\subset U_{\beta_0}$, *if* $L^{\beta_0}_{\alpha_0}\cap\pi^{-1}_{\alpha_0}(U_{\beta_0})\neq\emptyset$, *and* $U\subset N^*\setminus U_{\beta_0}$, *otherwise*,

(6) $\chi(B^{\alpha_0}_{\beta_0})<2^\omega$, *for each* (α_0, β_0).

Proof. The family $\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$ will be constructed by induction. Assume that $2^\omega\times 2^\omega$ is well-ordered by $<_*$, and assume that we have formed our construction for all $(\alpha, \beta)<_*(\alpha_0, \beta_0)$. We have $K^{\alpha_0}_{\beta_0} \neq \emptyset$ and $\chi(K^{\alpha_0}_{\beta_0})<2^\omega$.

By the definition of $L^{\alpha_0}_{\beta_0}$, $L^{\alpha_0}_{\beta_0}$ is a non-empty closed set and $\chi(L^{\alpha_0}_{\beta_0})\leqslant\chi(K^{\alpha_0}_{\beta_0})\cdot\omega$ and, consequently, $\chi(L^{\alpha_0}_{\beta_0})<2^\omega$.

If $L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(N^*) = \emptyset$, then $B^{\alpha_0}_{\beta_0} = L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(\{n\})$ and $\chi(B^{\alpha_0}_{\beta_0})\leqslant\chi(L^{\alpha_0}_{\beta_0})\cdot\omega<2^\omega$.

If $L^{\alpha_0}_{\beta_0} \cap \pi^{-1}_{\alpha_0}(N^*) \neq \emptyset$, then $\chi(\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}) \cap N^*, N^*)\leqslant\chi(\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}), \beta N)\cdot\omega<2^\omega$.

If $\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}) \cap N^* \cap U_{\beta_0} \neq \emptyset$, then by (B) there exists a closed-and-open set U in N^*, $U\subset U_{\beta_0}$, such that $U\subset\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}) \cap N^* \cap U_{\beta_0}$.

In this case let $B^{\alpha_0}_{\beta_0} = \pi^{-1}_{\alpha_0}(U)$.

If $\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}) \cap N^* \cap U_{\beta_0} = \emptyset$, then by (B) there exists a closed-and-open set U in N^*, $U\subset N^*\setminus U_{\beta_0}$ such that $U\subset\pi_{\alpha_0}(L^{\alpha_0}_{\beta_0}) \cap N^*\setminus U_{\beta_0}$. Let $B^{\alpha_0}_{\beta_0} = \pi^{-1}_{\alpha_0}(U)$.

The above inductive construction of $B^{\alpha_0}_{\beta_0}$ can be used to construct B^0_0 if we put $K^0_0 = X$ and L^0_0 equal to K^0_0 or $\pi^{-1}_{i(0)}(N^*)$, where $i(0)$ is the minimal i such that $\pi_i(N^*) \cap \bigcap \mathfrak{F}_0 \neq \emptyset$ and π_i refines $\mathfrak{F}_0$.

Thus the sets B^α_β are defined. From the construction it follows that B^α_β's satisfy the desired conditions. ∎

LEMMA 3. (B) *Let X be a compact space, let $\{\pi_\alpha|\ \alpha<2^\omega\}$ be a family of open maps π_α: $X \xrightarrow{\text{onto}} \beta N$, and let $\{\mathfrak{F}_\alpha|\ \alpha<\lambda\}$, be a collection of families of closed-and-open sets, such that $\lambda<2^\omega$, $\mathfrak{F}_\alpha$ is centred and $|\mathfrak{F}_\alpha|<2^\omega$ for each $\alpha<\lambda$. Let the family $\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$ satisfy conditions* (3)–(6).

Then, for every $\alpha<2^\omega$, either $\pi_\alpha(x) \in N$ or $\pi_\alpha(x)$ is a $P(2^\omega)$-point in N^*, for each point $x \in \bigcap\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$.

Proof. Let $x \in \bigcap\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$. By (4), if $\pi_\alpha(\{x\}) \cap N^* = \emptyset$, then $\pi_\alpha(x) = n$ for an $n \in N$.

If $\pi_\alpha(x) \in N^*$, then by (3) $\pi_\alpha(x)$ has a linearly ordered base. Hence, $\pi_\alpha(x)$ is in this case a $P(2^\omega)$-point of N^*. ∎

THEOREM 1. (B) *Let X be a compact space and let $\{\pi_\alpha|\ \alpha<2^\omega\}$ be a family of open maps π_α: $X \xrightarrow{\text{onto}} \beta N$. There exists a point $x \in X$ such that for each $\alpha<2^\omega$ either $\pi_\alpha(x)$ belongs to N or $\pi_\alpha(x)$ is a $P(2^\omega)$-point of N^*.*

Proof. Take $\lambda = 0$ in Lemmas 2 and 3. Then, by Lemma 2, there exists a family $\{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$ satisfying conditions (3)–(6). Since that family is centred, by Lemma 3 there exists a point x such that either $\pi_\alpha(x) \in N$ or $\pi_\alpha(x)$ is a $P(2^\omega)$-point in N^*. ∎

THEOREM 2. (B) *Let X be a compact extremally disconnected space of weight 2^ω and let $\{\pi_\alpha|\ \alpha<2^\omega\}$ be a family of open maps π_α: $X \xrightarrow{\text{onto}} \beta N$ such that for each centred family $\mathfrak{F}$, $|\mathfrak{F}|<2^\omega$, of closed-and-open subsets of X there is an α such that π_α refines $\mathfrak{F}$. Then there exist a point $x \in X$, such that for each α either $\pi_\alpha(x) \in N$ or $\pi_\alpha(x)$ is a $P(2^\omega)$-point of N^*, and for each family $\mathfrak{F}$, $|\mathfrak{F}|<2^\omega$, of closed-and-open neighbourhoods of x, if $\operatorname{int} \bigcap \mathfrak{F} = \emptyset$, then there is an α such that π_α refines $\mathfrak{F}$ and $\pi_\alpha(x) \in N^*$.*

Proof. Let $\{\mathfrak{F}_\alpha|\ \alpha<2^\omega\}$ be the collection of all centred families of closed-and-open subsets of X such that:

(i) $|\mathfrak{F}_\alpha|<2^\omega$, for all $\alpha<2^\omega$,

(ii) $\operatorname{int} \bigcap \mathfrak{F}_\alpha = \emptyset$.

Apply Lemmas 2 and 3 to the family $\{\mathfrak{F}_\alpha|\ \alpha<2^\omega\}$. Then the point

$$x \in \bigcap \{B^\alpha_\beta|\ \alpha, \beta<2^\omega\}$$

is the desired one.

Indeed, by Lemma 3, either $\pi_\alpha(x) \in N$ or $\pi_\alpha(x)$ is a $P(2^\omega)$-point in N^*. Let $\mathfrak{K}$ be a family of closed-and-open neighbourhoods of x, such that $|\mathfrak{K}|<2^\omega$. We can assume that $\operatorname{int} \bigcap \mathfrak{K} = \emptyset$. There is an $\alpha_0<2^\omega$ such that $\mathfrak{F}_{\alpha_0} = \mathfrak{K}$. Since $x \in \bigcap \{B^\alpha_\beta|\ \alpha, \beta<2\} \cap \bigcap \mathfrak{F}_{\alpha_0}$ and since $\chi(K^{\alpha_0}_0 \cap \bigcap \mathfrak{F}_{\alpha_0})<2^\omega$, by the assumption of the theorem there exists an α' such that $\pi_{\alpha'}$ refines the closed-and-open base for $K^{\alpha_0}_0 \cap \bigcap \mathfrak{F}_{\alpha_0}$ of minimal cardinality. Hence, $\pi^{-1}_{i(\alpha_0)}(N^*) \cap K^{\alpha_0}_0 \cap \bigcap \mathfrak{F}_{\alpha_0} \neq \emptyset$. By the definition of $B^{\alpha_0}_0$, $B^{\alpha_0}_0 \subset \pi^{-1}_{i(\alpha_0)}(N^*)$, $\pi_{i(\alpha_0)}(x) \in N^*$, and $\pi_{i(\alpha_0)}$ refines $\mathfrak{F}_{\alpha_0}$. ■

3. Inaccessible points of a compact extremally disconnected space.

THEOREM 3. (B) *Let X be a compact extremally disconnected space of weight 2^ω. There exists an $x \in X$ such that x is not a limit point of any countable discrete subset of X.*

Proof. Let $\{\pi_{\mathscr{A}}|\ \mathscr{A} \in \mathfrak{A}\}$ be the family of all open maps $\pi_{\mathscr{A}}$: $X \xrightarrow{\text{onto}} \beta N$. By Theorem 1, there is a point $x \in X$ such that either $\pi_{\mathscr{A}}(x) \in N$ or $\pi_{\mathscr{A}}(x)$ is a $P(2^\omega)$-point of N^* for all $\mathscr{A} \in \mathfrak{A}$.

Suppose on the contrary that $x \in \operatorname{cl} D \setminus D$, where $|D| = \omega$ and D is a discrete subset of X.

There exists a quasi-partition $\mathscr{B} \in \mathfrak{A}$ of X such that if $y, z \in D$ and $y \neq z$, then $\pi_{\mathscr{B}}(y) \neq \pi_{\mathscr{B}}(z)$ and $\pi_{\mathscr{B}}(D) \subset N^*$.

Hence $\pi_{\mathscr{B}}(x) \in N^*$ and $\pi_{\mathscr{B}}(x)$ is a $P(2^\omega)$-point of N^*. We have $\pi_{\mathscr{B}}(x) \notin \operatorname{cl}(\pi_{\mathscr{B}}(D))$, and hence $x \notin \operatorname{cl} D$, a contradiction. ■

LEMMA 4. *Let D, $|D| \leqslant m<2^\omega$, be a strongly discrete subset of $U(m)$, where $\operatorname{cf}(m)>\omega$. Then there exists a quasi-partition $\mathscr{A}$ of $U(m)$ such that $\pi_{\mathscr{A}}(y) \neq \pi_{\mathscr{A}}(z)$ for $z \neq y$ and $\pi_{\mathscr{A}}(D) \subset N^*$. Moreover, if $x \in \operatorname{cl} D \setminus D$, then $\pi_{\mathscr{A}}(x) \notin \pi_{\mathscr{A}}(D)$.*

Proof. Let $\{K_d|\ d \in D\}$ be a family of almost disjoint subsets of N and let $\{U_d|\ d \in D\}$ be a partition of m such that $d \in \operatorname{cl}_{\beta m} U_d$. By the non-measurability of m (Ulam [16]) there exists a partition $\{U^n_d|\ n \in K_d\}$ of U_d, such that, for each $n, d \notin \operatorname{cl}_{\beta m} U^n_d$. The quasi-partition $\mathscr{A} = \{\operatorname{cl}_{\beta m} \cup \{U^n_d|\ d \in D\} \cap U(m)|\ n \in N\}$ is the desired one. ■

THEOREM 4. (B) *In the space $U(m)$, where $\omega<\operatorname{cf}(m) \leqslant m<2^\omega$, there exists a point x which is not a limit point of any strongly discrete subset of $U(m)$ of cardinality not greater than m.*

Proof. By Theorem 1 there exists a point $x \in U(m)$ such that for each $\mathscr{B} \in \mathfrak{A}$ either $\pi_{\mathscr{A}}(x) \in N$ or $\pi_{\mathscr{B}}(x)$ is a $P(2^\omega)$-point, where $\mathfrak{A}$ is the set of all quasi-partitions of $U(m)$.

This point is not a limit point of any strongly discrete set of cardinality $\leqslant m$.

Indeed, assume that $x \in \operatorname{cl} D \setminus D$ for some strongly discrete set and $|D| \leqslant m$. By Lemma 4, there exists an $\mathscr{A} \in \mathfrak{A}$ such that $\pi_{\mathscr{A}}(D) \subset N^*$ and $\pi_{\mathscr{A}}(y) \neq \pi_{\mathscr{A}}(z)$ for $y, z \in D \cup \{x\}$ and $y \neq z$. Since $\pi_{\mathscr{A}}(x)$ is a $P(2^\omega)$-point of N^*, we have $\pi_{\mathscr{A}}(x) \notin (\pi_{\mathscr{A}}(D)) \supset \pi_{\mathscr{A}}(\operatorname{cl} D)$, which is a contradiction. ■

COROLLARY. (B) *If $m<2^\omega$ and m is regular, then in βm there exists a point x such that x is not a limit point of any strongly discrete set of cardinality $<m$.*

This follows from Theorem 4 in view of the fact that $U(m)$ is a $P(m)$-subset of βm. ■

Denote by $\pi w(X)$ the π-weight of X, i.e. the minimal cardinality of π-bases of X; a family $\mathscr{B}$ of non-empty open sets is called a *π-base* of X if each non-empty open set contains a member of $\mathscr{B}$; a π-base $\mathscr{B}$ is non-archimedean if $V \subset U$ or $U \subset V$ or $U \cap V = \emptyset$, for each two members of $\mathscr{B}$.

LEMMA 5. (B) *Let X be a dense-in-itself extremally disconnected compact space such that $\pi w(X) = \omega$ and let $\mathfrak{S}$ be a centred family of closed-and-open subsets of X. If $|\mathfrak{S}|<2^\omega$, then there exists an open map π: $X \xrightarrow{\text{onto}} \beta N$ such that π refines $\mathfrak{S}$.*

Proof. Let $\{U_n|\ n \in \omega\}$ be a closed-and-open non-archimedean π-base of X. For each $S \in \mathfrak{S}$ let $A_S = \{n \in \omega|\ U_n \subset S\}$. Since $\mathfrak{S}$ is centred, the family $\{A_S|\ S \in \mathfrak{S}\}$ has the uniform intersection property (i.e. intersections of its finite subfamilies are infinite).

By (B), there exists an infinite $A \subset \omega$ such that $A \setminus A_S$ is finite for each S.

Let $C \subset A$ be an infinite set such that $\{U_n|\ n \in C\}$ is a pairwise disjoint family. The quasi-partition $\mathscr{A} = \{U_n|\ n \in C\} \cup \{\{X \setminus \operatorname{cl} \bigcup \{U_n|\ n \in C\}$ induces a map $\pi_{\mathscr{A}}$ refining $\mathfrak{S}$. ■

Remark. If X is a dense-in-itself extremally disconnected compact space with a countable Suslin number, then for each nowhere dense set D there exists an open map π: $X \xrightarrow{\text{onto}} \beta N$ such that $\pi(D) \subset N^*$.

THEOREM 5. (B) *Let X be a extremally disconnected compact space with $\pi w(X) = \omega$. Then there exists a point $x \in X$ such that x is not a limit point of any discrete set D, $|D|<2^\omega$.*

Proof. Let $\mathfrak{A}$ be the set of all quasi-partitions of X. Since $|\mathfrak{A}| = 2^\omega$, there exists a point x satisfying the conditions of Theorem 2.

Let D, $|D|<2^\omega$, be a discrete set such that $x \in \operatorname{cl} D \setminus D$. For each point $d \in D$ there exists a closed-and-open V_d such that $x \in V_d$ and $d \notin V_d$.

By Lemma 5 there exists an $\mathscr{A} \in \mathfrak{A}$ such that $\pi_{\mathscr{A}}$ refines $V = \{V_d|\ d \in D\}$ and $\pi_{\mathscr{A}}(x) \in N^*$.

There exists a $\mathscr{B} \in \mathfrak{A}$ such that $\pi_{\mathscr{B}}(D) \subset N^*$ (by the remark before Theorem 5).

Let $\mathscr{C}$ be a quasi-partition refining $\mathscr{A}$ and $\mathscr{B}$. It is obvious that $\pi_{\mathscr{C}}(x) \notin \pi_{\mathscr{C}}(D)$. Indeed, $\pi_{\mathscr{A}}(d) \in N$ for each $d \in D$ but $\pi_{\mathscr{A}}(x) \in N^*$. Because $\pi_{\mathscr{C}}(x) \notin \pi_{\mathscr{C}}(D)$ and $\pi_{\mathscr{C}}(x)$ is a $P(2^{\omega})$-point of N^*, $\pi_{\mathscr{C}}(x) \notin \pi_{\mathscr{C}}(\mathrm{cl}\, D \setminus D)$; a contradiction. ■

COROLLARY. (B) *In a dense-in-itself extremally disconnected compact space with* $\pi w(X) = \omega$ *there exists a countable dense-in-itself set of points such that none of them is a limit point of any discrete set of cardinality* $<2^{\omega}$.

Proof. By Theorem 5, in each set of a countable π-base we can find a point which is not a limit point of any discrete set of cardinality $<2^{\omega}$. ■

The following two theorems seem to be known (cf. the proofs of the corresponding theorems in Jech [9] p. 108–109).

THEOREM A. (MA) *Let* $Y = \mathrm{st}(M(X))$, *where* X *is a measurable separable metric space. Then each subset of* Y *of cardinality* $<2^{\omega}$ *is nowhere dense.*

THEOREM B. (MA) *Let* $Y = \mathrm{st}(M(X))$, *where* X *is a measurable separable metric space. Then, for each centred family* $\mathfrak{S}$, $|\mathfrak{S}|<2^{\omega}$ *of closed-and-open subsets of* Y, *there exists a quasi-partition* $\mathscr{A}$ *of* Y *such that* $\pi_{\mathscr{A}}$ *refines* $\mathfrak{S}$.

Now we can prove

THEOREM 6 (Kunen [11]). *In the space* $\mathrm{st}(M(X))$, *where* X *is a measurable separable metric space, there exists a point* x *such that* x *is not a limit point of any set of cardinality* $<2^{\omega}$.

Proof. By Theorem 2, there exists a point $x \in \mathrm{st}(M(X))$ such that either $\pi_{\mathscr{A}}(x) \in N$ or $\pi_{\mathscr{A}}(x)$ is a $P(2^{\omega})$-point of N^*, where $\mathscr{A}$ is an arbitrary quasi-partition of Y.

Suppose that $x \in \mathrm{cl}\, D \setminus D$, where $|D|<2^{\omega}$. The set D is nowhere dense and there exists a quasi-partition $\mathscr{B}$ of Y such that $\pi_{\mathscr{B}}(D) \subset N^*$.

For each point $d \in D$ there exists a closed-and-open set V_d, such that $d \notin V_d$ and $x \in V_d$. By Theorem B, there exists a $\pi_{\mathscr{A}_0}$, such that $\pi_{\mathscr{A}_0}$ refines $\mathfrak{B} = \{V_d|\ d \in D\}$ and $\pi_{\mathscr{A}_0}(x) \in N^*$. Let $\mathscr{C}$ be a quasi-partition refining $\mathscr{A}_0$ and $\mathscr{B}$. We have $\pi_{\mathscr{C}}(x) \notin \pi_{\mathscr{C}}(D)$. Indeed, for each $d \in D$, $\pi_{\mathscr{A}_0}(d) \in N$ but $\pi_{\mathscr{A}_0}(x) \in N^*$. Since $|D|<2^{\omega}$, we have $|\pi_{\mathscr{C}}(D)|<2^{\omega}$. Since $\pi_{\mathscr{C}}(x)$ is a $P(2^{\omega})$-point of N^*, $\pi_{\mathscr{C}}(x) \notin \mathrm{cl}\, \pi_{\mathscr{C}}(D)$, which is a contradiction. ■

4. Corollaries for βN. Kunen [11] has proved, assuming (B), that: each compact extremally disconnected space of weight $\leqslant 2^{\omega}$ can be embedded in $\beta N \setminus N$ as a $P(2^{\omega})$-set.

From this fact we shall derive, using the results from our paper, the following corollaries:

COROLLARY 1. (MA) (Kunen [11]). *In* $\beta N \setminus N$ *there exists a non-P-point that is not a limit point of any set of cardinality* $<2^{\omega}$.

Proof. Clearly, each non-isolated point of an extremally disconnected compact space of weight 2^{ω} is a non-P-point; by Theorem 6 and Kunen's Theorem cited above we have our theorem. ■

COROLLARY 2. (B) (Kunen [11]). *In* $\beta N \setminus N$ *there exists a countable dense-in-itself subset of points which are not limit points of any discrete subset of cardinality* $<2^{\omega}$.

Proof. By Kunen's theorem cited above and the corollary to Theorem 5.

COROLLARY 3. (B) *For each regular cardinal* m, $m<2^{\omega}$, *in* $\beta N \setminus N$ *there exists a point* x *such that* x *is not a* P-*point*, x *is a limit point of a strongly discrete set of cardinality* m, *and* x *is not a limit point of any strongly discrete set of cardinality* $<m$.

Proof. Immediately follows from the corollary to Theorem 4 and Kunen's Theorem since βm is an extremally disconnected space. ■

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