Case 2. If OC is a subset of $\pi_N h(K)$, there is a subcontinuum K_0 of K such that $\pi_N h(K_0) = OC$. Then $L = \{(x, y) | (x, y)$ is in $T \times T$ and there is a point z of K_0 such that $x = \pi_N(z)$ and $y = \pi_N h(z)\}$ is a subcontinuum of $T \times T$ such that $P_1(L)$ is a subset of OB and $P_2(L) = OC$. Thus, L contains a point of V_N. As before, this involves a contradiction.

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Convexity on a topological space

by

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Abstract. Although convexity is an attribute of subsets of linear spaces in general, we define convexity on topological spaces without linear structures paying attention to the concept of convex hull. Then some theorems which have been obtained in linear topological spaces are given in these spaces.

Takahashi [5] discussed a convexity on a metric space. In this paper, we discuss a convexity on a topological space without linear space structure. We introduce a convexity on a topological space and several concepts concerning the convexity, and obtain some theorems which generalize the theorems proved by Browder [1], Fan [2] and Sion [4]. All topological structures are implicitly assumed to satisfy Hausdorff separation axiom.

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1. Definitions and some elementary properties. Let X be a topological space, $\mathscr{A}(X)$ the family of all subsets of X and $\mathscr{F}(X)$ the family of all finite subsets of X. An *H-operator* on X is a mapping $\langle\cdot\rangle$ from $\mathscr{A}(X)$ into $\mathscr{A}(X)$ satisfying the following conditions:

(a) $\langle\emptyset\rangle = \emptyset$, where $\emptyset$ is the empty set;

(b) $\langle\{x\}\rangle = \{x\}$, $x \in X$;

(c) $\langle\langle A\rangle\rangle = \langle A\rangle$, $A \in \mathscr{A}(X)$;

(d) $\langle A\rangle = \bigcup\{\langle F\rangle : F \subset A, F \in \mathscr{F}(X)\}$.

The image $\langle A\rangle$ of A is said to be the *convex hull* of A. A *convex set* in X is a subset of X which is equal to its convex hull.

PROPOSITION 1. (i) *An H-operator is monotone, i.e. if $A \subset B$, then $\langle A\rangle \subset \langle B\rangle$.*

(ii) *The convex hull $\langle A\rangle$ of $A \in \mathscr{A}(X)$ is the smallest convex set containing A.*

(iii) *The entire space X and the empty set $\emptyset$ are convex sets.*

(iv) *If $\{C_\nu\}_{\nu\in I}$ is a family of convex sets, then $\bigcap_{\nu\in I} C_\nu$ is a convex set.*

(v) *If $\{C_\nu\}_{\nu\in I}$ is a family of convex sets such that for any two indices ν_1 and ν_2 there exists an index μ with $C_\mu \subset C_{\nu_1} \cap C_{\nu_2}$, then $\bigcup_{\nu\in I} C_\nu$ is a convex set.*

Proof. (i) Suppose $A\subset B$ and F is a finite subset of A. Since F is a finite subset of B, $\langle F\rangle\subset\langle B\rangle$ by (d). Hence $\langle A\rangle\subset\langle B\rangle$.

(ii) Suppose $A\in\mathscr{A}(X)$. If $A=\emptyset$, then the assertion is trivial by (a). Suppose $A\neq\emptyset$ and $x\in A$. Since $\{x\}=\langle\{x\}\rangle\subset\langle A\rangle$ by (i), $A\subset\langle A\rangle$. $\langle A\rangle$ is convex by (c). If B is a convex set containing A, then $\langle A\rangle\subset\langle B\rangle=B$ by (i).

(iii) The empty set $\emptyset$ is convex by (a). Since $X\subset\langle X\rangle$ by (ii), $X=\langle X\rangle$.

(iv) Put $C=\bigcap_{\nu\in I}C_\nu$. Since $\langle C\rangle\subset\langle C_\nu\rangle=C_\nu$ for $\nu\in I$, $\langle C\rangle\subset C$. Hence C is convex.

(v) Put $C=\bigcup_{\nu\in I}C$. To show C is convex, we need only to show $\langle F\rangle\subset C$ for finite subset F of C by (d). Since F is finite, there exists $\nu\in I$ such that $F\subset C_\nu$. Hence it follows $\langle F\rangle\subset C_\nu\subset C$.

Let $\boldsymbol{R}$ be the set of all functions from a countably infinite set N into the real number system R which are zero except at a finite number of points of N, i.e. $\boldsymbol{R}$ is the direct sum $\sum_{i\in N}R_i$ where $R_i=R$ for all $i\in N$. The topology and the linear space structure on $\boldsymbol{R}$ are the usual ones. Suppose that a topological space X and an H-operator $\langle\cdot\rangle$ on X are given. Let $\mathscr{H}(X)$ be a subfamily $\{\langle F\rangle\colon F\in\mathscr{F}(X)\}$ of $\mathscr{A}(X)$. For $H\in\mathscr{H}(X)$, a mapping φ from H into $\boldsymbol{R}$ is called a *structure mapping* on H, if it has the following properties:

(a) The mapping φ is an into-homeomorphism;

(b) If $A\subset H$, then $\varphi(\langle A\rangle)=\langle\varphi(A)\rangle$, where $\langle\varphi(A)\rangle$ is the usual convex hull of $\varphi(A)$ in $\boldsymbol{R}$.

PROPOSITION 2. (i) *If a subset A of H is convex, then $\varphi(A)$ is convex.*

(ii) *If $\tilde{A}\subset\varphi(H)$, then $\varphi^{-1}(\langle\tilde{A}\rangle)=\langle\varphi^{-1}(\tilde{A})\rangle$. Hence if $\tilde{A}$ is convex, then $\varphi^{-1}(\tilde{A})$ is convex.*

Proof. (i) Since $\langle\varphi(A)\rangle=\varphi(\langle A\rangle)=\varphi(A)$, $\varphi(A)$ is convex.

(ii) Since $\varphi(\langle\varphi^{-1}(\tilde{A})\rangle)=\langle\varphi(\varphi^{-1}(\tilde{A}))\rangle=\langle\tilde{A}\rangle$, $\langle\varphi^{-1}(A)\rangle=\varphi^{-1}(\langle\tilde{A}\rangle)$.

Let S_H be the set of all structure mappings on H. When S_H is nonempty for each $H\in\mathscr{H}(X)$, an element Φ of the product $\prod_{H\in\mathscr{H}(X)}S_H$ is said to be a *structure* on X with respect to the H-operator $\langle\cdot\rangle$. A *convex space* $(X,\langle\cdot\rangle,\Psi)$ is a triple consisting of a topological space X, an H-operator $\langle\cdot\rangle$ on X and a structure Φ on X with respect to $\langle\cdot\rangle$.

A nonempty convex set Y in a convex space $(X,\langle\cdot\rangle,\Phi)$ is also a convex space. The topology on Y is the relative topology induced from X. The H-operator $\langle\cdot\rangle_Y$ on Y is defined as follows. For $A\in\mathscr{A}(Y)$, $\langle A\rangle_Y=\langle A\rangle$. The structure Φ_Y is the restriction of Φ to $\mathscr{H}(Y)$. The convex space $(Y,\langle\cdot\rangle_Y,\Phi_Y)$ is said to be a *subspace* of the convex space $(X,\langle\cdot\rangle,\Phi)$.

2. Examples of convex spaces. (I) A convex subset X of a real linear topological space E is an example of a convex space. The topology of X is the relative topology induced from E. The H-operator is the usual convex hull determined by the linear space structure of E. Suppose $H\in\mathscr{H}(X)$. Since the linear subspace V spanned by H is finite dimensional, there exists a topological isomorphism I from V into $\boldsymbol{R}$. The restriction $\Phi(H)$ of I to H is a structure mapping on H. We always take the structure Φ of this type whenever we regard a convex subset of a real linear topological space as a convex space.

(II) The n-dimensional real projective space $P^n(R)$ is another example of a convex space. $P^n(R)$ is the quotient space of $R^{n+1}\setminus\{0\}$ by the following equivalence relation $\sim$: $x\sim y$ if and only if there exists a nonzero real number t such that $x=ty$. We use the following notations:

$\pi\colon R^{n+1}\setminus\{0\}\to P^n(R)$ is the quotient mapping,

$S^n=\{x\in R^{n+1}\colon \|x\|=1$, where

$$\|x\|=((x^1)^2+(x^2)^2+\ldots+(x^{n+1})^2)^{1/2}\ \text{for}\ x=(x^1,x^2,\ldots,x^{n+1})\},$$

$$S^n_+=\{x\in S^n\colon x^{n+1}>0\}\cup\{x\in S^n\colon x^n>0,\ x^{n+1}=0\}\cup$$
$$\cup\{x\in S^n\colon x^{n-1}>0,\ x^n=x^{n+1}=0\}\cup\ldots\cup\{x\in S^n\colon x^1>0,\ x^2=\ldots=x^{n+1}\}.$$

For each $\tilde{x}\in P^n(R)$ there exists a unique element $x\in S^n_+$ such that $\pi(x)=\tilde{x}$, so we denote by λ the mapping which corresponds each $\tilde{x}\in P^n(R)$ to the element $x\in S^n_+$. The mapping λ is a bijection from $P^n(R)$ onto S^n_+, and λ^{-1} is continuous as λ^{-1} is the restriction of π to S^n_+. We define a mapping $\theta\colon S^n_+\to R^n$ by

$$\theta(x^1,\ldots,x^n,x^{n+1})=(x^1,\ldots,x^n)\quad\text{for}\quad(x^1,\ldots,x^n,x^{n+1})\in S^n_+.$$

Then θ is an into-homeomorphism and the image $\theta(S^n_+)$ is a convex subset of R^n. We denote by η the composition $\theta\circ\lambda\colon P^n(R)\to R^n$. Then η is injective and the inverse η^{-1} is continuous. We define an H-operator on $P^n(R)$ by

$$\langle A\rangle=\eta^{-1}(\langle\eta(A)\rangle)\quad\text{for}\quad A\subset P^n(R).$$

For $H\in\mathscr{H}(P^n(R))$ the structure mapping $\Phi(H)$ on H is the restriction of η to H.

3. Some theorems. The following theorem is obtained by Browder [1] when X is a compact convex subset of a real linear topological space. The method of the proof of the theorem is same as [1].

THEOREM 1. *Let $(X,\langle\cdot\rangle,\Phi)$ be a compact convex space and T be a mapping from X into $\mathscr{A}(X)$, where for $x\in X$, Tx is a nonempty convex set in X. Suppose further that for $y\in X$, $T^{-1}y=\{x\in X\colon y\in Tx\}$ is open in X. Then there exists $x_0\in X$ such that $x_0\in Tx_0$.*

Proof. Since Tx is nonempty for $x\in X$, $\{T^{-1}y\}_{y\in X}$ is an open covering of X. Since X is compact, there exists a finite subset $F=\{y_1,y_2,\ldots,y_n\}$ of X such that $\{T^{-1}y_i\}_{i=1}^n$ is an open covering of X. Setting $H=\langle F\rangle$ and $A_i=H\cap T^{-1}y_i$ for $i=1,2,\ldots,n$, $\{A_i\}_{i=1}^n$ is an open covering of H. Let $\varphi=\Phi(H)$ be the structure mapping on H. Putting $\tilde{H} = \varphi(H) = \langle\varphi(F)\rangle$ and $\tilde{A}_i = \varphi(A_i)$, $\{\tilde{A}_i\}_{i=1}^n$ is an open covering of $\tilde{H}$. Since $\tilde{H}$ is compact, there exists a continuous partition of unity $\{g_1, g_2, \ldots, g_n\}$ subordinate to $\{\tilde{A}_i\}$. A mapping f from $\tilde{H}$ into itself is defined by

$$f(\tilde{x}) = \sum_{i=1}^{n} g_i(\tilde{x})\tilde{y}_i \quad \text{for} \quad \tilde{x} \in \tilde{H},$$

where $\tilde{y}_i = \varphi(y_i)$ for $i = 1, 2, \ldots, n$. Since f is continuous, f has a fixed point $\tilde{x}_0 \in \tilde{H}$ by the Brouwer fixed point theorem. Put $x_0 = \varphi^{-1}(\tilde{x}_0)$. If the set of indices i's such that $g_i(\tilde{x}_0) \neq 0$ is $\{i_1, i_2, \ldots, i_m\}$, then

$$\tilde{x}_0 = \sum_{k=1}^{m} g_{i_k}(\tilde{x}_0)\tilde{y}_{i_k}$$

and $\tilde{x}_0 \in \tilde{A}_{i_k}$ for $k = 1, \ldots, m$. Hence $x_0 \in A_{i_k} \subset T^{-1}y_{i_k}$, i.e. $y_{i_k} \in Tx_0$ for $k = 1, \ldots, m$. Since Tx_0 is convex, $\langle y_{i_1}, \ldots, y_{i_m}\rangle \subset Tx_0$, where we write $\langle y_{i_1}, \ldots, y_{i_m}\rangle$ instead of $\langle\{y_{i_1}, \ldots, y_{i_m}\}\rangle$. Hence we have

$$\begin{aligned} x_0 = \varphi^{-1}(\tilde{x}_0) &= \varphi^{-1}\Big(\sum_{k=1}^{m} g_{i_k}(\tilde{x}_0)\tilde{y}_{i_k}\Big) \\ &\in \varphi^{-1}(\langle\tilde{y}_{i_1}, \ldots, \tilde{y}_{i_m}\rangle) \\ &= \langle y_{i_1}, \ldots, y_{i_m}\rangle \subset Tx_0 . \end{aligned}$$

Before stating the next theorem, we give some definitions. Let $(X, \langle\cdot\rangle, \Phi)$ be a convex space and f a real-valued function on X. The function f is said to be *convex* if $f \circ \Phi(H)^{-1}$ is convex in the usual sense for $H \in \mathscr{H}(X)$, *quasi-convex* if the set $\{x \in X: f(x) \leqslant c\}$ is convex for $c \in R$ and *quasi-concave* if $-f$ is quasi-convex. When X is a convex subset of a real linear topological space, these definitions coincide with the usual ones.

Proposition 3. (i) *If f and g are convex and r is a nonnegative number, then $f+g$ and rf are convex.*

(ii) *If f is convex, then f is quasi-convex,*

(iii) *f is quasi-convex if and only if the set $\{x \in X: f(x) < c\}$ is convex for $c \in R$.*

Proof. (i) Let $H \in \mathscr{H}(X)$ and $\varphi = \Phi(H)$ be the structure mapping on H. Since $(f+g) \circ \varphi^{-1} = f \circ \varphi^{-1} + g \circ \varphi^{-1}$ and $(rf) \circ \varphi^{-1} = r(f \circ \varphi^{-1})$, $(f+g) \circ \varphi^{-1}$ and $(rf) \circ \varphi^{-1}$ are convex. Hence $f+g$ and rf are convex.

(ii) Let $A_c = \{x \in X: f(x) \leqslant c\}$ for $c \in R$. To show A_c is convex, it is sufficient to show that $\langle F\rangle \subset A_c$ for finite subset F of A_c. Let $\varphi = \Phi(\langle F\rangle)$ be the structure mapping on $\langle F\rangle$ and $\tilde{A}_c = \{\tilde{x} \in \langle\varphi(F)\rangle: f \circ \varphi(\tilde{x}) \leqslant c\}$. Since $\tilde{A}_c$ is convex by hypothesis and $\varphi(F) \subset \tilde{A}_c$, $\varphi(\langle F\rangle) = \langle\varphi(F)\rangle = \tilde{A}_c$. Therefore $\langle F\rangle = \varphi^{-1}(\tilde{A}_c) \subset A_c$.

(iii) Suppose f is quasi-convex. The equality

$$\{x \in X: f(x) < c\} = \bigcup_{d<c} \{x \in X: f(x) \leqslant d\}$$

holds. Hence, by (v) of Proposition 1, $\{x \in X: f(x) < c\}$ is convex. Conversely suppose $\{x \in X: f(x) < d\}$ is convex for $d \in R$. The equality

$$\{x \in X: f(x) \leqslant c\} = \bigcap_{d<c} \{x \in X: f(x) < d\}$$

holds. Hence, by (iv) of Proposition 1, $\{x \in X: f(x) \leqslant c\}$ is convex.

The next theorem is obtained by Fan [2] when X is a compact convex subset of a real linear topological space.

Theorem 2. *Let $(X, \langle\cdot\rangle, \Phi)$ be a compact convex space. Let $\{f_\nu\}_{\nu \in I}$ be a family of real-valued lower semicontinuous convex functions defined on X. Then there exists an $x \in X$ satisfying*

$$f_\nu(x) \leqslant c \quad \textit{for} \quad \nu \in I,$$

if and only if, for any finite set of indices $\nu_1, \nu_2, \ldots, \nu_n$ of I and for any n nonnegative numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 1$, there exists a $y \in X$ satisfying

$$\sum_{i=1}^{n} \lambda_i f_{\nu_i}(y) \leqslant c .$$

Proof. The "only if" part is easy. We prove the "if" part. Suppose that for each $x \in X$ there exists $\nu \in I$ such that $f_\nu(x) > c$. Setting $G_\nu = \{x \in X: f_\nu(x) > c\}$, $\{G_\nu\}_{\nu \in I}$ is an open covering of X. Since X is compact, there exists a finite subcovering $\{G_{\nu_1}, \ldots, G_{\nu_n}\}$ of $\{G_\nu\}_{\nu \in I}$. Let $\{g_1, \ldots, g_n\}$ be a continuous partition of unity subordinate to $\{G_{\nu_1}, \ldots, G_{\nu_n}\}$ and put

$$D(x, y) = \sum_{i=1}^{n} g_i(x) f_{\nu_i}(y) \quad \text{for} \quad (x, y) \in X \times X$$

and

$$d(x) = D(x, x) \quad \text{for} \quad x \in X .$$

Since d is lower semicontinuous on X by Lemma 3 of [6], d takes its minimum m. Hence we have

$$d(x) \geqslant m > c \quad \text{for} \quad x \in X .$$

We define a mapping T from X into $\mathscr{A}(X)$ by

$$Tx = \{y \in X: D(x, y) < m\} \quad \text{for} \quad x \in X .$$

Then Tx is nonempty and convex by hypothesis and $T^{-1}y = \{x \in X: D(x, y) < m\}$ is open as g_i's are continuous. Hence by Theorem 1 there exists $x_0 \in X$ such that $x_0 \in Tx_0$, i.e. $d(x_0) < m$. This is a contradiction.

The following theorem is obtained by Sion [4] when X and Y are compact convex subsets of real linear topological spaces. The method of the proof of the theorem is same as [4].

Theorem 3. *Let $(X, \langle\cdot\rangle, \Phi)$ and $(Y, [\cdot], \Psi)$ be compact convex spaces and f a real-valued function on $X \times Y$ such that $f(\cdot, y)$ is quasi-concave and upper semicontinuous* *on X for $y \in X$ and $f(x, \cdot)$ is quasi-convex and lower semicontinuous on Y for $x \in X$. Then we have*

$$\max_{x\in X}\min_{y\in Y} f(x, y) = \min_{y\in Y}\max_{x\in X} f(x, y) .$$

We need some lemmas. The following two lemmas and their proofs are in [4].

LEMMA 1. *Let S be an n-dimensional simplex with vertices $a_0, \ldots, a_n$. If $\{A_0, \ldots, A_n\}$ is an open covering of S and $S\setminus A_i$ is convex for $i = 0, \ldots, n$ and $a_i \notin A_j$ for $i \neq j$ $(i, j = 0, \ldots, n)$, then $\bigcap_{i=0}^{n} A_i \neq \emptyset$.*

LEMMA 2. *Let $a_0, \ldots, a_n$ be elements of R^k where $k<n$. Then*

$$\bigcap_{i=0}^{n} \langle a_0, \ldots, \hat{a}_i, \ldots, a_n\rangle \neq \emptyset ,$$

where we indicate by $\hat{a}_i$ that this element is to be omitted.

LEMMA 3. *Let $(X, \langle\cdot\rangle, \Phi)$ be a convex space, Y a finite set and f a real-valued function on $X\times Y$ such that $f(\cdot, y)$ is quasi-concave and upper semicontinuous on X for $y \in Y$. Suppose, in addition, that Y is minimal with respect to the property: for each $x \in X$ there exists a $y \in Y$ such that $f(x, y)<c$. Then there exists $x_0 \in X$ such that $f(x_0, y)<c$ for all $y \in Y$.*

Proof. Setting $Y = \{y_0, \ldots, y_n\}$ and $A_i = \{x \in X\colon f(x, y_i)<c\}$ for $i = 0, \ldots, n$ $\{A_0, \ldots, A_n\}$ is an open covering of X and $X\setminus A_i$ is convex for $i = 1, \ldots, n$. By the minimality of Y, for each i there exists $a_i \in X$ such that $a_i \notin A_j$ for $j \neq i$. Since $\{a_0, \ldots, \hat{a}_i, \ldots, a_n\}\subset X\setminus A_i$ and $X\setminus A_i$ is convex for $i = 0, \ldots, n$, $\langle a_0, \ldots, \hat{a}_i, \ldots, a_n\rangle \subset X\setminus A_i$. Hence we have

$$\bigcap_{i=0}^{n} \langle a_0, \ldots, \hat{a}_i, \ldots, a_n\rangle = \emptyset .$$

Let φ be the structure mapping on $\langle a_0, \ldots, a_n\rangle$, i.e. $\varphi = \Phi(\langle a_0, \ldots, a_n\rangle)$ and $\tilde{a}_i = \varphi(a_i)$ for $i = 1, \ldots, n$. Since $\varphi(\langle a_0, \ldots, \hat{a}_i, \ldots, a_n\rangle) = \langle \tilde{a}_0, \ldots, \hat{\tilde{a}}_i, \ldots, \tilde{a}_n\rangle$,

$$\bigcap_{i=0}^{n} \langle \tilde{a}_0, \ldots, \hat{\tilde{a}}_i, \ldots, \tilde{a}_n\rangle = \emptyset .$$

Hence, by Lemma 2, $\langle \tilde{a}_0, \ldots, \tilde{a}_n\rangle$ is a n-dimensional simplex. If we put $\tilde{A}_i = \varphi(A_i \cap \langle a_0, \ldots, a_n\rangle)$ for $i = 0, \ldots, n$, then $\{\tilde{A}_1, \ldots, \tilde{A}_n\}$ is an open covering of $\langle \tilde{a}_0, \ldots, \tilde{a}_n\rangle$ and $\tilde{a}_i \notin \tilde{A}_j$ for $i \neq j$. Since

$$\begin{aligned}\langle \tilde{a}_0, \ldots, \tilde{a}_n\rangle\setminus\tilde{A}_i &= \varphi(\langle a_0, \ldots, a_n\rangle)\setminus\varphi(A_i \cap \langle a_0, \ldots, a_n\rangle)\\ &= \varphi(\langle a_0, \ldots, a_n\rangle \cap (X\setminus A_i))\end{aligned}$$

and $\langle a_0, \ldots, a_n\rangle \cap (X\setminus A_i)$ is convex, $\langle \tilde{a}_0, \ldots, \tilde{a}_n\rangle/\tilde{A}_i$ is convex for $i = 0, \ldots, n$. Hence, by Lemma 1, there exists $\tilde{x}_0$ such that $\tilde{x}_0 \in \bigcap_{i=0}^{n} \tilde{A}_i$. Putting $x_0 = \varphi^{-1}(\tilde{x}_0)$, we have $x_0 \in \bigcap_{i=0}^{n} A_i$, i.e. $f(x_0, y_i)<c$ for $i = 0, \ldots, n$.

LEMMA 3′. *Let X be a finite set and $(Y, [\cdot], \Psi)$ be a convex space and f be a real-valued function on $X\times Y$ such that $f(x, \cdot)$ is quasi-convex and lower semicontinuous on Y for each $x \in X$. Suppose, in addition, that X is minimal with respect to the property: for each $y \in Y$ there exists an $x \in X$ such that $f(x, y)>c$. Then there exists $y_0 \in Y$ such that $f(x, y_0)>c$ for all $x \in X$.*

The proof of Lemma 3′ is same as the proof of Lemma 3.

Proof of Theorem 4. It is easily seen that

$$\max_{x\in X}\min_{y\in Y} f(x, y) \leqslant \min_{y\in Y}\max_{x\in X} f(x, y) .$$

Suppose

$$\max_{x\in X}\min_{y\in Y} f(x, y) < c < \min_{y\in Y}\max_{x\in X} f(x, y) .$$

Let $A_x = \{y \in Y\colon f(x, y)>c\}$ and $B_y = \{x \in X\colon f(x, y)<c\}$. Since the family $\{A_x\}_{x\in X}$ is an open covering of Y, there exists a finite subcovering $\{A_{x_1}, \ldots, A_{x_n}\}$ of $\{A_x\}_{x\in X}$. Similarly, since the family $\{B_y\}_{y\in Y}$ is an open covering of X, there exists a finite subcovering $\{B_{y_1}, \ldots, B_{y_m}\}$ of $\{B_y\}_{y\in Y}$. If we put $X_1 = \{x_1, \ldots, x_n\}$ and $Y_1 = \{y_1, \ldots, y_m\}$, then for each $y \in [Y_1]$ there exists an $x \in X_1$ such that $f(x, y)>c$ and for each $x \in \langle X_1\rangle$ there exists a $y \in Y_1$ such that $f(x, y)<c$. Let X_2 be a minimal subset of X_1 such that for each $y \in [Y_1]$ there exists an $x \in X_2$ such that $f(x, y)>c$. Let Y_2 be a minimal subset of Y_1 such that for each $x \in \langle X_2\rangle$ there exists a $y \in Y_2$ such that $f(x, y)<c$. By repeating this process of alternately reducing the X_i and Y_i, after a finite number of steps, we can choose a finite subset X_0 of X and a finite subset Y_0 of Y such that X_0 is minimal with respect to the property: for each $y \in [Y_0]$ there exists an $x \in X_0$ such that $f(x, y)>c$; and Y_0 is minimal with respect to the property: for each $x \in \langle X_0\rangle$ there exists a $y \in Y_0$ such that $f(x, y)<c$. An application of Lemma 3 to the subspace $\langle X_0\rangle$ yields that there exists an $x_0 \in \langle X_0\rangle$ such that $f(x_0, y)<c$ for all $y \in Y_0$. Since the function $f(x_0, \cdot)$ on Y is quasi-convex, $f(x_0, y)<c$ for all $y \in [Y_0]$. Similarly, by Lemma 3′, there exists a $y_0 \in [Y_0]$ such that $f(x, y_0)>c$ for all $x \in \langle X_0\rangle$. Then $c<f(x_0, y_0)<c$, which is a contradiction.

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