Hereditarily indecomposable tree-like continua, II

by

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Abstract. In this paper is presented an example of an hereditarily indecomposable tree-like continuum such that each non-degenerate subcontinuum of it has positive span.

1. Introduction. In an abstract in 1951 Anderson [1] stated that there is an hereditarily indecomposable tree-like continuum which contains only degenerate chainable continua. In 1959 Anderson and Choquet [2] constructed a tree-like continuum in the plane such that no two of its non-degenerate subcontinua are homeomorphic. In 1967 Cook [3] constructed a continuum with the property that the identity is the only non-constant mapping of the continuum into itself. The purpose of this paper is to present an example of an hereditarily indecomposable tree-like continuum containing only degenerate subcontinua with span zero. Thus, the example of this paper contains only degenerate chainable continua. The construction of this continuum (in Section 5) is similar to that employed by Cook and the reader familiar with that paper will notice arguments here which are quite similar to those in that paper.

Throughout this paper the term mapping means continuous function and the term continuum means compact, connected metric space.

2. Atomic maps, span, and atriodic continua. In this section we establish the main lemmas for the constructions of this paper.

DEFINITION. Suppose f is a mapping of a continuum X into a continuum Y. The statement that f is *atomic* means if K is a subcontinuum of X such that $f(K)$ is non-degenerate then $K=f^{-1}(f(K))$. (In [3, p. 242] Cook called such a map preatomic, but in [5, Theorem 1] Emeryk and Horbanowicz showed that such mappings are monotone making the usage here consistent with that of Anderson and Choquet [2, p. 347].) The *span of* f, denoted σf, is the least upper bound of the set to which the number ε belongs if and only if there exists a subcontinuum Z of $X\times X$ such that $p_1(Z)=p_2(Z)$ (p_1 and p_2 denote the two projections of $X\times X$ onto X) and if (x,y) is in Z then $d(f(x),f(y))\geqslant\varepsilon$. The *span of* X, denoted σX, is the span of the identity mapping of X onto X.

THEOREM 1. *Suppose X is a continuum and f is an atomic mapping of X onto an atriodic continuum Y. If, for each point y of Y, $f^{-1}(y)$ is atriodic, then X is atriodic.*

Proof. Suppose H is a triod lying in X. Then H is the union of three continua H_1, H_2, and H_3 such that the common part of each two of them is the common part of all three of them and is a proper subcontinuum of each one of them. By hypothesis $f(H)$ is non-degenerate.

We now show $f(H_i)$ is not $f(H)$ for $i = 1, 2, 3$. If i is an integer such that $f(H_i)$ is $f(H)$ then $f(H_i)$ is non-degenerate and $f^{-1}(f(H_i)) = H_i$. However, in $H_j, j \neq i$, there is a point Q_j not in H_i. It is easy to see that that $f(Q_j)$ is not in $f(H_i)$.

Further, if H_i and H_j are two of H_1, H_2, and H_3, then $f(H_i) \cup f(H_j)$ is not $f(H)$. To see this, for convenience suppose $f(H_1) \cup f(H_2)$ is $f(H)$. Since $f(H_1 \cup H_2) = f(H)$ and $f(H)$ is non-degenerate, $f^{-1}f(H_1 \cup H_2) = H_1 \cup H_2$. But, there is a point Q_3 of H_3 not in $H_1 \cup H_2$ and $f(Q_3)$ is not in $f(H_1) \cup f(H_2)$.

Since $f(H)$ is the union of three continua, $f(H_1)$, $f(H_2)$, and $f(H_3)$, such that no one of them is a subset of the union of the other two, $f(H)$ contains a triod [12]. This involves a contradiction.

THEOREM 2. *If X is a continuum, f is a monotone mapping of X onto a continuum Y, and ε is a positive number such that $\sigma Y > \varepsilon$, then $\sigma X > 0$ and $\sigma f > \varepsilon$.*

Proof. Suppose ε is a positive number such that $\sigma Y > \varepsilon$. Since f is uniformly continuous, there is a positive number δ such that if (x, y) is a point of $X \times X$ and $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

We now show $\sigma X \geqslant \delta$. There is continuum Z in $Y \times Y$ such that $p_1(Z) = p_2(Z)$ and if (P, Q) is a point of Z, $d(P, Q) \geqslant \varepsilon$. Let $Z' = (f \times f)^{-1}(Z)$ and note that Z' is a continuum because f is monotone.

To see that $p_1(Z') = p_2(Z')$, suppose (x, y) is a point of Z'. Then $(f(x), f(y))$ is in Z. Since $p_1(Z) = p_2(Z)$, there are points P and Q of Y such that $(P, f(x))$ and $(f(y), Q)$ are in Z. If s and t are points of X such that $f(s) = P$ and $f(t) = Q$, then (s, x) and (y, t) are in Z'.

If (x, y) is in Z', then $d(x, y) \geqslant \delta$; for if $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$, but $(f(x), f(y))$ is in Z. Since $f \times f(Z') = Z$, $\sigma f > \varepsilon$.

3. A "spiral" to a simple triod. The crucial difference between the construction of this paper and that of Cook [3] is our need here to have a "spiral" to a simple triod such that the mapping f of T onto T (see below for a definition of f) will extend to a mapping of the closure of the "spiral" to itself. The mapping f does not extend to the usual (straight around) spiral to a simple triod. In this section we construct a "double spiral" to a simple triod which we will employ in the main construction of this paper.

We recall some definitions and notations from [7]. Denote by T the set of all points (ϱ, θ) in the plane in polar coordinates such that $0 \leqslant \varrho \leqslant 1$ and θ is in $\{0, \frac{1}{2}\pi, \pi\}$. Denote by A the point $(1, \frac{1}{2}\pi)$, by B the point $(1, \pi)$, by C the point $(1, 0)$, and by O the point $(0, 0)$.

Denote by f the mapping of T onto T defined as follows:

$$f(\varrho, \theta) = \begin{cases} (1-2\varrho, \pi) & \text{if} \quad 0 \leqslant \varrho \leqslant \frac{1}{2} \text{ and } \theta = 0, \\ (2\varrho-1, 0) & \text{if} \quad \frac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } \theta = 0, \\ (1-3\varrho, \pi) & \text{if} \quad 0 \leqslant \varrho \leqslant \frac{1}{3} \text{ and } \theta = \pi, \\ (3\varrho-1, \frac{1}{2}\pi) & \text{if} \quad \frac{1}{3} \leqslant \varrho \leqslant \frac{1}{2} \text{ and } \theta = \pi, \\ (2-3\varrho, \frac{1}{2}\pi) & \text{it} \quad \frac{1}{2} \leqslant \varrho \leqslant \frac{2}{3} \text{ and } \theta = \pi, \\ (3\varrho-2), 0) & \text{if} \quad \frac{2}{3} \leqslant \varrho \leqslant 1 \text{ and } \theta = \pi, \\ (1-4\varrho, \pi) & \text{if} \quad 0 \leqslant \varrho \leqslant \frac{1}{4} \text{ and } \theta = \frac{1}{2}\pi, \\ (4\varrho-1, \frac{1}{2}\pi) & \text{if} \quad \frac{1}{4} \leqslant \varrho \leqslant \frac{1}{2} \text{ and } \theta = \frac{1}{2}\pi, \\ (3-4\varrho, \frac{1}{2}\pi) & \text{if} \quad \frac{1}{2} \leqslant \varrho \leqslant \frac{3}{4} \text{ and } \theta = \frac{1}{2}\pi, \\ (4\varrho-3, 0) & \text{if} \quad \frac{3}{4} \leqslant \varrho \leqslant 1 \text{ and } \theta = \frac{1}{2}\pi. \end{cases}$$

Denote by M the inverse limit of the inverse limit sequence $\{T_i, f_i\}$ where, for each i, $T_i = T$ and $f_i = f$. In [7] the author showed that M is an atriodic tree-like continuum with positive span.

Denote by K the composant of M containing the point $(C, C, C, \ldots)$. Note that K is a topological ray, and let h denote a homeomorphism throwing $(0, 1]$ onto K. Then $h(1)$ is $(C, C, C, \ldots)$. Denote by R the subset of $M \times [-1, 1]$ to which the point (x, y) belongs if and only if it is true that if y is 0 then x is in M, if $y > 0$ then x is $h(y)$, and if $y < 0$ then x is $h(-y)$. This is a "double spiral" to M. Denote by F the mapping of M onto M induced by f, i.e. $F(x_1, x_2, \ldots) = (f(x_1), x_1, x_2, \ldots)$. Note that G, defined as follows is a mapping of R onto R:

$$G(x, y) = \begin{cases} (F(x), 0) & \text{if} \quad y = 0, \\ (F(x), h^{-1}Fh(y)) & \text{if} \quad y > 0, \\ (F(x), -h^{-1}Fh(-y)) & \text{if} \quad y < 0. \end{cases}$$

Denote by S the "spiral" to T, R/π_1 (π_1 is the projection of M onto T_1). The mapping G of R onto R induces a mapping φ of S onto S [4, p. 126] and $\varphi | T$ is f.

4. The main constructions. In this section we describe the inverse limit systems that produce the main examples of the paper.

DEFINITIONS. If f is a mapping of the continuum X onto the continuum Y, f is called an *A^*-map* if it is atomic and there do not exist infinitely many points y of Y such that $f^{-1}(y)$ is non-degenerate.

THEOREM 3. *There exists an inverse limit sequence $\{X_i, f_i\}$ such that (1) X_1 is T and, for each positive integer n, X_n is a continuum, (2) for each n, f_n is an A^*-map, (3) if x is in X_n and $f_n^{-1}(x)$ is non-degenerate then x is interior to an arc lying in X_n and $f_n^{-1}(x)$ is a simple triod, (4) if x is in X_n and $f_n^{-1}(x)$ is non-degenerate and α is an arc containing x in its interior such that α contains no other point with a non-degenerate inverse image, then $f_n^{-1}(\alpha)$ is homeomorphic to S, (5) if T is a maximal simple triod lying in X_n then $f_n^{-1}(T)$ is not a simple triod, and (6) if α is an arc lying in X_n then there exists an integer $m > n$ ruch that $(f_n^m)^{-1}(\alpha)$ is not an arc.*

We adopt the following notation used by Cook [3]. If α is an ordered pair (i,j) of positive integers, denote i by $n_1(\alpha)$, denote j by $n_2(\alpha)$, denote the ordered pair $(i+1,j)$ by α^*, and denote the ordered pair $(i,j+1)$ by α'. Let D denote the set of all ordered pairs of positive integers directed by the relation $<$ where $\alpha<\beta$ if and only if α and β are two elements of D such that either $n_1(\alpha)<n_1(\beta)$ or $n_1(\alpha)=n_1(\beta)$ and $n_2(\alpha)<n_2(\beta)$.

Recall from Section 3 that M denotes the atriodic tree-like continuum with positive span from [7].

THEOREM 4. *There exists an inverse limit sequence* $\{X_i, f_i\}$ *such that,* (1) X_1 *is* M *and for each* n, X_n *is an atriodic tree-like continuum with* $\sigma X_n>0$, (2) *for each* n *and* m *with* $n<m$, f_n^m *is an atomic mapping and if* x *is in* X_i *where* i *is a positive integer then* $(f_i)^{-1}(x)$ *is either degenerate or homeomorphic to* M, *and* (3) *if* n *is a positive integer and* I *is an arc lying in* X_n *then there exists an integer* $m>n$ *such that* $(f_n^m)^{-1}(I)$ *contains a homeomorphic copy of* M.

Proof. There exists an inverse mapping system $\{T_\alpha, \pi_\alpha^\beta\}$ over D such that, (1) for each α in D, T_α is a continuum, (2) if $\alpha_1, \alpha_2, \ldots, \alpha_n$ is a sequence of elements of D such that, for each $i<n$, α_{i+1} is either α_i^* or α_i' then $\pi_{\alpha_1}^{\alpha_n}$ is the composite $\pi_{\alpha_1}^{\alpha_2}\circ\pi_{\alpha_2}^{\alpha_3}\circ\ldots\circ\pi_{\alpha_{n-1}}^{\alpha_n}$, (3) if α is in D, T_α contains only finitely many mutually exclusive simple triods, (4) if α is in D and $n_1(\alpha)=1$ then T_α is a simple triod and $\pi_\alpha^{\alpha'}$ is f, (5) for each positive integer i, then inverse limit sequence $\{T_{(n,i)}, \pi_{(n,i)}^{(m,i)}\}$ satisfies all the conditions of Theorem 3, and (6) if n is a positive integer and K is a maximal simple triod in $T_{(n,j)}$ which is thrown by $\pi_{(n,i)}^{(n,j)}$ onto a simple triod in $T_{(n,i)}$ then $\pi_{(n,i)}^{(n,j)}|K$ is f^{j-i}.

For each n let X_n be the inverse limit of the inverse limit sequence $\{T_{(n,i)}, \pi_{(n,i)}^{(n,j)}\}$ and let f_n be the mapping of X_{n+1} onto X_n induced by $\pi_{(n,1)}^{(n+1,1)}$, $\pi_{(n,2)}^{(n+1,2)}$, $\pi_{(n,3)}^{(n+1,3)}, \ldots$ Cook [3, Theorem 1, p. 242] has shown that f_n is atomic, thus f_n^m is atomic if $m>n$. Each X_n is atriodic by Theorem 1. By (6), for each n, X_n contains a copy of M. Thus, for each n, X_n has positive span.

THEOREM 5. *The inverse limit of an inverse limit sequence satisfying the conditions of Theorem 4 is an atriodic tree-like continuum containing only degenerate subcontinua with span zero.*

Proof. Such an inverse limit is atriodic because an inverse limit with each factor space an atriodic continuum is atriodic. Suppose K is a non-degenerate subcontinuum of the inverse limit. By condition (3) of Theorem 4, some projection of K contains a copy of M. This projection is atomic [3, Theorem 2, p. 242] so by Theorem 2 K contains a subcontinuum with positive span. Thus K has positive span.

In the final section of this paper it is shown that each member of the collection H defined in [10] has positive span. Denote by N the member of the collection H which is obtained by using an inverse limit on simple triods with odd numbered bonding maps "crooked" and even numbered bonding maps the mapping f of section 3 of this paper.

THEOREM 6. *There exists an inverse mapping sequence* $\{Y_i, g_i\}$ *such that,* (1) Y_1 *is* N *and for each* i, Y_i *is an hereditarily indecomposable tree-like continuum with positive span,* (2) *for each* i, g_i *is an atomic mapping of* Y_{i+1} *onto* Y_i, *and, if* y *is a point of* Y_i, $g_i^{-1}(y)$ *is either degenerate or is homeomorphic to* N, (3) *if* i *is a positive integer and* P *is a pseudo-arc lying in* Y_i *then there is an integer* $j>i$ *such that* $(g_i^j)^{-1}(P)$ *contains an hereditarily indecomposable tree-like continuum with positive span.*

Proof. We wish to construct an inverse mapping system $\{T_\alpha, \pi_\alpha^\beta\}$ over A as in the proof of Theorem 4 except that if we restrict our attention to an inverse limit sequence $\{T_\gamma, \pi_\gamma^\delta\}$ such that $n_1(\gamma)=n_1(\delta)=i$ then the inverse limit of this sequence is hereditarily indecomposable. This can be achieved by inserting sufficiently crooked mappings into the inverse mapping system of Theorem 4, so that each mapping of a maximal simple triod onto a simple triod is crooked in the manner described in [10]. It should be noted that such a crooked mapping of a simple triod extends to a crooked mapping of S onto S. Let Y_i be the inverse limit of the sequence $\{T_\gamma, \pi_\gamma^\delta\}$ where $n_1(\gamma)=n_1(\delta)=i$ and g_i the mapping of Y_{i+1} onto Y_i induced by $\pi_{(i,1)}^{(i+1,1)}$, $\pi_{(i,2)}^{(i+1,2)}$, $\pi_{(i,3)}^{(i+1,3)}, \ldots$ Using again the result of Cook [3, Theorem 1, p. 242] g_i is atomic. Since for each i, Y_i contains a copy of N, Y_i has positive span.

THEOREM 7. *The inverse limit of an inverse limit sequence satisfying the conditions of Theorem 6 is an hereditarily indecomposable tree-like continuum containing only degenerate continua with span zero.*

Proof. Such an inverse limit is hereditarily indecomposable because any inverse limit on hereditarily indecomposable continua is hereditarily indecomposable. That each non-degenerate subcontinuum has positive span may be seen using condition (3) of Theorem 6, the fact that the projections are atomic [3, Theorem 2, p. 242] and Theorem 2 as in the proof of Theorem 5.

5. Span. In this section we prove that each of the continua in the collection H defined in [10] has positive span. The proof is similar in nature to the proof that the continuum M of [7] has positive span. However, so much modification of the argument has been made that a complete argument is given here.

The function f was defined in Section 3 of this paper. The homeomorphism r of T onto T [8] is defined as follows:

$$r(\varrho,\theta)=\begin{cases}(\varrho,\tfrac{1}{2}\pi) & \text{if}\quad \theta=\tfrac{1}{2}\pi,\\ (\varrho,\pi) & \text{if}\quad \theta=0,\\ (\varrho,0) & \text{if}\quad \theta=\pi.\end{cases}$$

Notation. If p and q are positive integers and $p<q$, the point $\frac{pA}{q}$ denotes $\left(\frac{p}{q},\frac{\pi}{2}\right)$, $\frac{pB}{q}$ denotes $\left(\frac{p}{q},\pi\right)$ and $\frac{pC}{q}$ denotes $\left(\frac{p}{q},0\right)$. The notation $\langle\alpha,\beta\rangle$ implies that $\langle\alpha,\beta\rangle$ is a subcontinuum of $T\times T$ such that $p_1(\langle\alpha,\beta\rangle)$ is α, $p_2(\langle\alpha,\beta\rangle)$ is β and α and β are arcs in T. If X and Y are points of T, the arc from X to Y in T is denoted by XY.

DEFINITION. A subset Z of $T\times T$ is said to have *property* L' provided Z is the union of twenty-four continua $\langle OA, BC\rangle$, $\langle BC, OA\rangle$, $\langle OA, OB\rangle$, $\langle OB, OA\rangle$, $\langle OB, AC\rangle$, $\langle AC, OB\rangle$, $\langle OC, AB\rangle$, $\langle AB, OC\rangle$, $\langle OA, OC\rangle$, $\langle OC, OA\rangle$, $\left\langle O\frac{A}{2}, BC\right\rangle$, $\left\langle BC, O\frac{A}{2}\right\rangle$, $\left\langle OB, \frac{A}{2}C\right\rangle$, $\left\langle \frac{A}{2}C, OB\right\rangle$, $\langle OB, OC\rangle$, $\langle OC, OB\rangle$, $\left\langle OC, \frac{A}{2}B\right\rangle$, $\left\langle \frac{A}{2}B, OC\right\rangle$, $\left\langle \frac{A}{2}A, \frac{A}{2}B\right\rangle$, $\left\langle \frac{A}{2}B, \frac{A}{2}A\right\rangle$, $\left\langle O\frac{A}{2}, \frac{A}{2}A\right\rangle$, $\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle$, $\left\langle \frac{A}{2}A, \frac{A}{2}C\right\rangle$, and $\left\langle \frac{A}{2}C, \frac{A}{2}A\right\rangle$.

LEMMA 1. *Suppose* $[a, b]$ *and* $[c, d]$ *are arcs and* H *and* K *are subcontinua of* $[a, b]\times[c, d]$. *If both* a *and* b *belong to* $p_1(H)$ *and both* c *and* d *belong to* $p_2(K)$, *then* H *and* K *contain a common point.*

Lemma 1 is used often in the proofs of Lemmas 2, 3, 4, and 5, below. It was also a useful tool in some of the proofs of [9].

LEMMA 2. *If* Z *is a subset of* $T\times T$ *with property* L', *then* Z *is a continuum.*

Proof. Each of $\langle AC, OB\rangle$, $\langle AB, OC\rangle$, $\langle OA, OB\rangle$, $\langle OA, OC\rangle$, and $\left\langle O\frac{A}{2}, BC\right\rangle$ intersects $\langle OA, BC\rangle$ (use Lemma 1) so the union of these six continua is a continuum, C_1. Each of $\langle BC, OA\rangle$, $\langle AB, OC\rangle$, $\langle OB, OA\rangle$, $\langle OB, OC\rangle$, $\left\langle BC, O\frac{A}{2}\right\rangle$, $\left\langle \frac{A}{2}B, \frac{A}{2}A\right\rangle$, and $\left\langle OB, \frac{A}{2}C\right\rangle$ intersects $\langle OB, AC\rangle$ so the union of these eight continua is a continuum, C_2. Each of $\langle BC, OA\rangle$, $\langle AC, OB\rangle$, $\langle OC, OA\rangle$, $\langle OC, OB\rangle$, $\left\langle \frac{A}{2}C, \frac{A}{2}A\right\rangle$, and $\left\langle OC, \frac{A}{2}B\right\rangle$ intersects $\langle OC, AB\rangle$ so the union of these seven continua is a continuum, C_3. Since C_1 intersects both C_2 and C_3, the union of these three continua is a continuum. Further, denote by C_i' $(i = 1, 2, 3)$ the continuum resulting from the union of all continua $\langle\alpha, \beta\rangle$ where $\langle\beta, \alpha\rangle$ is a subset of C_i. Then, C_1' intersects C_2, C_2' intersects C_3, and C_3' intersects C_1, so the union of C_1, C_2, C_3, C_1', C_2', and C_3' is a continuum. The continuum $\left\langle O\frac{A}{2}, \frac{A}{2}A\right\rangle$ intersects C_2 at a point of $\left\langle\frac{A}{2}B, \frac{A}{2}A\right\rangle$ while $\left\langle\frac{A}{2}A, O\frac{A}{2}\right\rangle$ intersects C_2' at a point of $\left\langle\frac{A}{2}A, \frac{A}{2}B\right\rangle$, so Z is a continuum.

LEMMA 3. *If* Z *is a subcontinuum of* $T\times T$ *with property* L', *then there exists a subcontinuum* Z_r *with property* L' *such that* $r\times r(Z_r) = Z$.

LEMMA 4. *Suppose* k *is a mapping of* T *onto* T *such that* $k^{-1}(x) = \{x\}$ *for* x *in* $\left\{O, \frac{A}{2}, A, B, C\right\}$, *and* Z *is a subcontinuum of* $T\times T$ *with property* L'. *Then there exists a subcontinuum* Z_k *of* $T\times T$ *with property* L' *such that* $k\times k(Z_k) = Z$.

Proof. If α and β are in $\left\{OA, OB, OC, O\frac{A}{2}, \frac{A}{2}A, AB, AC, BC, \frac{A}{2}B, \frac{A}{2}C\right\}$ then $k|\alpha\times k|\beta$ is a mapping of $\alpha\times\beta$ onto $\alpha\times\beta$ which is essential on the boundary, S^1, of $\alpha\times\beta$. Thus $k|\alpha\times k|\beta$ is not homotopic to a mapping g of $\alpha\times\beta$ to S^1 which has the property that $g|S^1 = (k|\alpha\times k|\beta)|S^1$. By a theorem of Mazurkiewicz [11, Theorem I, p. 328] $k|\alpha\times k|\beta$ is weakly confluent. For each continuum $\langle\alpha, \beta\rangle$ of the twenty-four whose union is Z let $\langle\alpha, \beta\rangle'$ be the component of $(k|\alpha\times k|\beta)^{-1}(\langle\alpha, \beta\rangle)$ which is thrown by $k|\alpha\times k|\beta$ onto $\langle\alpha, \beta\rangle$. The union of the twenty-four continua thus obtained is the desired continuum Z_k.

LEMMA 5. *If* Z *is a subcontinuum of* $T\times T$ *with property* L', *then there exists a subcontinuum* Z_f *of* $T\times T$ *with property* L' *such that* $f\times f(Z_f) = Z$.

Proof. We adopt the notation used in [8] as follows: if $\langle t, u\rangle$ is a subcontinuum of Z and v and w are arcs in T such that $f|v$ is a homeomorphism throwing v onto t and $f|w$ is a homeomorphism throwing w onto u then $L = (f|v)^{-1}\times(f|w)^{-1}(\langle t, u\rangle)$ is a continuum thrown by $f\times f$ onto $\langle t, u\rangle$ such that $p_1(L) = v$ and $p_2(L) = w$. This continuum will be denoted by $L(\langle t, u\rangle, v, w)$.

Let

$$\begin{aligned}\alpha_1 = {} & L_1^1\left(\langle AB, OC\rangle, O\frac{A}{2}, \frac{2B}{3}B\right)\cup L_2^1\left(\left\langle\frac{A}{2}A, \frac{A}{2}C\right\rangle, \frac{3A}{8}\frac{A}{2}, \frac{B}{2}B\right)\cup\\ & \cup L_3^1\left(\left\langle\frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{3A}{8}\frac{A}{2}, \frac{B}{2}\frac{2B}{3}\right)\cup L_4^1\left(\left\langle\frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{B}{2}\frac{2B}{3}\right)\cup\\ & \cup L_5^1\left(\left\langle\frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{B}{3}\frac{B}{2}\right)\cup L_6^1\left(\left\langle\frac{A}{2}A, \frac{A}{2}B\right\rangle, \frac{A}{2}\frac{5A}{8}, O\frac{B}{2}\right)\cup\\ & \cup L_7^1\left(\langle AC, OB\rangle, \frac{A}{2}A, O\frac{B}{3}\right)\cup L_8^1\left(\langle AC, OB\rangle, \frac{A}{2}A, O\frac{C}{2}\right)\cup\\ & \cup L_9^1\left(\langle OA, OB\rangle, \frac{A}{2}\frac{3A}{4}, O\frac{C}{2}\right)\cup L_{10}^1\left(\langle OA, OB\rangle, \frac{A}{4}\frac{A}{2}, O\frac{C}{2}\right)\cup\\ & \cup L_{11}^1\left(\langle OA, BC\rangle, \frac{A}{4}\frac{A}{2}, OC\right)\cup L_{12}^1\left(\langle AB, OC\rangle, O\frac{A}{2}, \frac{C}{2}C\right).\end{aligned}$$

With four exceptions, successive terms of the sequence $L_1^1, L_2^1, \ldots, L_{12}^1$ may be seen to intersect with the aid of Lemma 1. For example L_1^1 and L_2^1 are continua lying in $O\frac{A}{2}\times\frac{B}{2}B$, $p_1L_1^1$ contains both O and $\frac{A}{2}$ while $p_2L_2^1$ contains both $\frac{B}{2}$ and B. The exceptions to using Lemma 1 here are to see that L_3^1 intersects L_4^1, L_4^1 intersects L_5^1, L_7^1 intersect L_8^1, and L_9^1 intersects L_{10}^1. There is a point (A, P_1) in $\left\langle\frac{A}{2}A, O\frac{A}{2}\right\rangle$ so $\left(\frac{A}{2}, \left(f|\frac{B}{2}\frac{2B}{3}\right)^{-1}(P_1)\right)$ is in both L_3^1 and L_4^1. There is a point $\left(P_2, \frac{A}{2}\right)$ in

$\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle$ so $\left(\left(f\frac{A}{2}\frac{5A}{8}\right)^{-1}(P_2), \frac{B}{2}\right)$ is in both L_7^1 and L_8^1 and there is a point (P_3, B) in $\langle OA, OB\rangle$ which lifts to a point of both L_9^1 and L_{10}^1.

Let $\alpha_2 = \alpha_1^{-1}$, $\alpha_3 = L_1^1 \cup L_2^1 \cup \ldots \cup L_7^1$ and $\alpha_4 = \alpha_3^{-1}$. Let

$$\alpha_5 = \alpha_4 \cup L_1^5\left(\langle OC, AB\rangle, \frac{2B}{3}B, O\frac{A}{2}\right) \cup L_2^5\left(\langle OC, OB\rangle, \frac{2B}{3}B, O\frac{A}{4}\right) \cup$$
$$\cup L_3^5\left(\langle OC, OB\rangle, \frac{2B}{3}B, O\frac{C}{3}\right) \cup L_4^5\left(\left\langle \frac{A}{2}C, OB\right\rangle, \frac{B}{2}B, O\frac{C}{2}\right) \cup$$
$$\cup L_5^5\left(\left\langle O\frac{A}{2}, BC\right\rangle, \frac{B}{2}\frac{2B}{3}, OC\right).$$

By using Lemma 1 it can be seen that L_1^5 intersects L_2^5, L_3^5 intersects L_4^5, and L_4^5 intersects L_5^5. A point (P_5, B) of $\langle OC, OB\rangle$ lifts to a point lying in both L_2^5 and L_3^5. Now, α_4 contains $(L_1^1)^{-1} = L\left(\langle OC, AB\rangle, \frac{2B}{3}B, O\frac{A}{2}\right)$ and since this is L_1^5, α_5 is a continuum. Let $\alpha_6 = \alpha_5^{-1}$.

Let

$$\alpha_7 = L_1^7\left(\left\langle OB, \frac{A}{2}C\right\rangle, O\frac{C}{2}, \frac{B}{2}B\right) \cup L_2^7\left(\left\langle BC, O\frac{A}{2}\right\rangle, OC, \frac{B}{2}\frac{2B}{3}\right) \cup$$
$$\cup L_3^7\left(\left\langle BC, O\frac{A}{2}\right\rangle, OC, \frac{B}{3}\frac{B}{2}\right) \cup L_4^7\left(\left\langle OC, \frac{A}{2}B\right\rangle, \frac{C}{2}C, O\frac{B}{2}\right) \cup$$
$$\cup L_5^7\left(\langle OC, OB\rangle, \frac{C}{2}C, O\frac{B}{2}\right) \cup L_6^7\left(\langle OC, OB\rangle, \frac{C}{2}C, O\frac{A}{4}\right) \cup$$
$$\cup L_7^7\left(\langle OC, AB\rangle, \frac{C}{2}C, O\frac{A}{2}\right) \cup L_8^7\left(\langle BC, OA\rangle, OC, \frac{A}{4}\frac{A}{2}\right) \cup$$
$$\cup L_9^7\left(\langle OB, OA\rangle, O\frac{C}{2}, \frac{A}{4}\frac{A}{2}\right) \cup L_{10}^7\left(\langle OB, OA\rangle, O\frac{C}{2}, \frac{A}{2}\frac{3A}{4}\right) \cup$$
$$\cup L_{11}^7\left(\langle OB, AC\rangle, O\frac{C}{2}, \frac{A}{2}A\right).$$

By using Lemma 1 and the facts that $\left\langle BC, O\frac{A}{2}\right\rangle$ contains a point with second coordinate $\frac{A}{2}$, $\langle OC, OB\rangle$ contains a point with second coordinate B, and $\langle OB, OA\rangle$ contains a point with second coordinate A, as before it can be seen that α_7 is a continuum. Let $\alpha_8 = \alpha_7^{-1}$ and let $\alpha_9 = L_7^7 \cup L_8^7 \cup \ldots \cup L_{11}^7$ and $\alpha_{10} = \alpha_9^{-1}$.

Let

$$\alpha_{11} = L_1^{11}\left(\langle AB, OC\rangle, O\frac{A}{2}, \frac{2B}{3}B\right) \cup L_2^{11}\left(\left\langle \frac{A}{2}A, \frac{A}{2}C\right\rangle, \frac{3A}{8}\frac{A}{2}, \frac{B}{2}B\right) \cup$$
$$\cup L_3^{11}\left(\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{3A}{8}\frac{A}{2}, \frac{B}{2}\frac{2B}{3}\right) \cup L_4^{11}\left(\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{3A}{8}\frac{A}{2}, \frac{B}{3}\frac{B}{2}\right) \cup$$
$$\cup L_5^{11}\left(\left\langle \frac{A}{2}A, \frac{A}{2}B\right\rangle, \frac{3A}{8}\frac{A}{2}, O\frac{B}{2}\right) \cup L_6^{11}\left(\langle OA, OB\rangle, \frac{A}{4}\frac{A}{2}, O\frac{B}{3}\right) \cup$$
$$\cup L_7^{11}\left(\langle OA, OB\rangle, \frac{A}{4}\frac{A}{2}, O\frac{C}{2}\right) \cup L_8^{11}\left(\langle OA, BC\rangle, \frac{A}{4}\frac{A}{2}, OC\right).$$

Here Lemma 1 and the facts that $\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle$ contains a point with second coordinate $\frac{A}{2}$ and $\langle OA, OB\rangle$ contains a point with second coordinate B may be used to see that α_{11} is a continuum. Let $\alpha_{12} = \alpha_{11}^{-1}$.

Let

$$\alpha_{13} = L_1^{13}\left(\left\langle \frac{A}{2}B, OC\right\rangle, O\frac{B}{2}, \frac{C}{2}C\right) \cup L_2^{13}\left(\left\langle O\frac{A}{2}, BC\right\rangle, \frac{B}{3}\frac{B}{2}, OC\right) \cup$$
$$\cup L_3^{13}\left(\left\langle O\frac{A}{2}, BC\right\rangle, \frac{B}{2}\frac{2B}{3}, OC\right) \cup L_4^{13}\left(\left\langle \frac{A}{2}C, OB\right\rangle, \frac{B}{2}B, O\frac{C}{2}\right) \cup$$
$$\cup L_5^{13}\left(\left\langle \frac{A}{2}C, OB\right\rangle, \frac{B}{2}B, O\frac{A}{4}\right) \cup L_6^{13}\left(\langle OC, AB\rangle, \frac{2B}{3}B, O\frac{A}{2}\right).$$

That α_{13} is a continuum may be seen with the aid of Lemma 1 and the facts that $\left\langle O\frac{A}{2}, BC\right\rangle$ contains a point with first coordinate $\frac{A}{2}$ and $\left\langle \frac{A}{2}C, OB\right\rangle$ contains a point with second coordinate B. Let $\alpha_{14} = \alpha_{13}^{-1}$ and $\alpha_{15} = L_1^{13} \cup L_2^{13} \cup L_3^{13} \cup L_4^{13}$ and $\alpha_{16} = \alpha_{15}^{-1}$.

Let

$$\alpha_{17} = \alpha_{16} \cup L_1^{17}\left(\left\langle OC, \frac{A}{2}B\right\rangle, \frac{C}{2}C, O\frac{3A}{8}\right) \cup L_2^{17}\left(\langle OC, AB\rangle, \frac{C}{2}C, O\frac{A}{2}\right).$$

That L_1^{17} intersects L_2^{17} may be seen using Lemma 1. Since $(L_1^{13})^{-1}$ is a subset of α_{16}, $(L_1^{13})^{-1} = L\left(\left\langle OC, \frac{A}{2}B\right\rangle, \frac{C}{2}C, O\frac{B}{2}\right)$, and $\left\langle OC, \frac{A}{2}B\right\rangle$ contains a point with second coordinate B, L_1^{17} may be seen to intersect $(L_1^{13})^{-1}$. Thus α_{17} is a continuum. Let $\alpha_{18} = \alpha_{17}^{-1}$.

Let

$$\alpha_{19} = L_1^{19}\left(\langle OC, AB\rangle, \frac{3A}{4}A, O\frac{A}{2}\right) \cup L_2^{19}\left(\langle AC, OB\rangle, \frac{A}{2}A, O\frac{A}{4}\right) \cup$$
$$\cup L_3^{19}\left(\langle AC, OB\rangle, \frac{A}{2}A, O\frac{B}{3}\right) \cup L_4^{19}\left(\left\langle \frac{A}{2}A, \frac{A}{2}B\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{OB}{2}\right) \cup$$
$$\cup L_5^{19}\left(\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{B}{3}\frac{B}{2}\right) \cup L_6^{19}\left(\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{B}{2}\frac{2B}{3}\right) \cup$$
$$\cup L_7^{19}\left(\left\langle \frac{A}{2}A, \frac{A}{2}C\right\rangle, \frac{A}{2}\frac{5A}{8}, \frac{B}{2}B\right).$$

Since $\langle AC, OB\rangle$ contains a point with second coordinate B and $\left\langle \frac{A}{2}A, O\frac{A}{2}\right\rangle$ contains a point with second coordinate $\frac{A}{2}$, it may be seen that L_2^{19} intersects L_3^{19} and L_5^{19} intersects L_6^{19}. Thus, by using Lemma 1, α_{19} is a continuum. Let $\alpha_{20} = \alpha_{19}^{-1}$. Let $\alpha_{21} = L_1^{19} \cup L_2^{19}$ and $\alpha_{22} = \alpha_{21}^{-1}$.

Let

$$\alpha_{23} = \alpha_{21} \cup L_1^{23}\left(\langle AC, OB\rangle, \frac{A}{2}A, O\frac{C}{2}\right) \cup L_2^{23}\left(\langle OA, BC\rangle, \frac{A}{2}\frac{3A}{4}, OC\right).$$

Using Lemma 1, L_1^{23} intersects L_2^{23}. Since $\langle AC, OB\rangle$ contains a point with second coordinate B, it may be seen that L_2^{19} intersects L_1^{23} and so α_{23} is a continuum. Let $\alpha_{24} = \alpha_{23}^{-1}$.

Let $Z_f = \alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_{24}$. By Lemma 2 Z_f is a continuum. The continua $\alpha_1, \alpha_2, \ldots, \alpha_{24}$ are, in order, the continua required so that Z_f have property L'. By construction $f\times f(Z_f) = Z$.

Let $\mathscr{K}$ denote the set to which k belongs if and only if k is a mapping of T onto T such that $k^{-1}(x) = \{x\}$ for each point x of $\left\{A, B, C, O, \frac{A}{2}\right\}$.

LEMMA 6. *Suppose, for each n, f_n is in $\mathscr{K} \cup \{f, r\}$. Then, if $n>1$, $\sigma f_1^n \geqslant \frac{1}{2}$ where $f_1^n = f_1 \cdot f_2 \cdot \ldots \cdot f_{n-1}$.*

Proof. Let

$$Z_1 = ((OA\times\{B\}) \cup (\{A\}\times BC)) \cup ((BC\times\{A\}) \cup (\{B\}\times OA)) \cup$$
$$\cup ((OA\times\{B\} \cup (\{A\}\times OB)) \cup ((OB\times\{A\}) \cup (\{B\}\times OA)) \cup$$
$$\cup ((OB\times\{A\}) \cup (\{B\}\times AC)) \cup ((\{A\}\times OB) \cup (AC\times\{B\})) \cup$$
$$\cup ((OC\times\{A\}) \cup (\{C\}\times AB)) \cup ((\{A\}\times OC) \cup (AB\times\{C\})) \cup$$
$$\cup ((OC\times\{A\}) \cup (\{C\}\times OA)) \cup ((\{A\}\times OC) \cup (OA\times\{C\})) \cup$$
$$\cup \left(\left(O\frac{A}{2}\times\{B\}\right) \cup \left(\left\{\frac{A}{2}\right\}\times BC\right)\right) \cup \left(\left(\{B\}\times O\frac{A}{2}\right) \cup \left(BC\times\left\{\frac{A}{2}\right\}\right)\right) \cup$$

$$\cup \left((OB\times\{C\}) \cup \left(\{B\}\times\frac{A}{2}C\right)\right) \cup \left((\{C\}\times OB) \cup \left(\frac{A}{2}C\times\{B\}\right)\right) \cup$$
$$\cup ((OB\times\{C\}) \cup (\{B\}\times OC)) \cup ((\{C\}\times OB) \cup (OC\times\{B\})) \cup$$
$$\cup \left((OC\times\{B\}) \cup \left(\{C\}\times\frac{A}{2}B\right)\right) \cup \left((\{B\}\times OC) \cup \left(\frac{A}{2}B\times\{C\}\right)\right) \cup$$
$$\cup \left(\left(\frac{A}{2}A\times\{B\}\right) \cup \{A\}\times\frac{A}{2}B\right) \cup \left(\left(\{B\}\times\frac{A}{2}A\right) \cup \left(\frac{A}{2}B\times\{A\}\right)\right) \cup$$
$$\cup \left(\left(\frac{A}{2}A\times\{0\}\right) \cup \left(\{A\}\times O\frac{A}{2}\right)\right) \cup \left(\left(\{O\}\times\frac{A}{2}A\right) \cup \left(O\frac{A}{2}\times\{A\}\right)\right) \cup$$
$$\cup \left(\left(\frac{A}{2}A\times\{C\}\right) \cup \left(\{A\}\times\frac{A}{2}C\right) \cup \left(\left(\{C\}\times\frac{A}{2}A\right) \cup \left(\frac{A}{2}C\times\{A\}\right)\right).$$

If (p, q) is in Z_1 then $d(p, q) \geqslant \frac{1}{2}$ and Z_1 has property L'. The lemma follows by induction using Lemmas 3, 4, and 5.

THEOREM 8. *Suppose, for each n, T_n is T and f_n is in $\mathscr{K} \cup \{r, f\}$ and M is the inverse limit of the inverse limit sequence $\{T_n, f_n\}$. Then $\sigma M > 0$.*

Proof. Apply Lemma 6 and Theorem 4 of [7].

THEOREM 9. *Each continuum in the collection H has positive span.*

THEOREM 10. *No continuum in the collection H is homogeneous.*

Proof. Suppose X is a member of H which is homogeneous. In the proof that X has positive span we show that for each positive integer n there exists a subcontinuum V_n of $T\times T$ such that $p_1(V_n) = OB$ and $p_2(V_n) = OC$ and $f_1^n\times f_1^n(V_n)$ is a subset of a continuum Z_1 in $T_1\times T_1$ with the property that $d(x, y) \geqslant \frac{1}{2}$ if (x, y) is in Z_1. The continuum X is the inverse limit of $\{T_i, \pi_i^j\}$ where $T_i = T$ and π_i^{i+1} is in $\{k_i\circ f, k_i\circ r\circ f\}$ with k_i "crooked".

Since X is homogeneous, by a theorem of Hagopian [6] there exists a positive number δ such that if P and Q are points of X and $d(P, Q)<\delta$ then there is a homeomorphism h of X onto X such that $h(P) = Q$ and if x is a point of X then $d(x, h(x))<\frac{1}{4}$. There exists a positive integer N such that if x and y are points of X such that $x_{N-1} = y_{N-1}$ then $d(x, y)<\delta$. Let P and Q be points of X such that P_N is B and Q_N is C. Then $P_{N-1} = Q_{N-1}$ so $d(P, Q)<\delta$. Let h be a homeomorphism of M onto M such that $h(P) = Q$ and if x is in M, $d(x, h(x))<\frac{1}{4}$. Consider a subcontinuum K of X containing P such that $\pi_N K = [OB]$. Since $h(P) = Q$, $\pi_N h(K)$ contains C.

Case 1. If $\pi_N h(K)$ is a subset of $[OC]$, then $L = \{(x, y)|\ (x, y)$ is in $T\times T$ and there is a point z of K such that $x = \pi_N(z)$ and $y = \pi_N h(z)\}$ is a continuum in $[OB]\times[OC]$ such that $p_1 L = OB$ and $p_2 L$ is a subset of OC. Thus, L contains a point of V_N. This gives a point z of K such that $(\pi_N(z), \pi_N h(z))$ is in V_N, so $(\pi_1(z), \pi_1 h(z))$ is in Z_1. This means $d(z, h(z)) \geqslant \frac{1}{4}$, a contradiction.

Case 2. If OC is a subset of $\pi_N h(K)$, there is a subcontinuum K_0 of K such that $\pi_N h(K_0) = OC$. Then $L = \{(x, y) |\ (x, y)$ is in $T \times T$ and there is a point z of K_0 such that $x = \pi_N(z)$ and $y = \pi_N h(z)\}$ is a subcontinuum of $T \times T$ such that $P_1(L)$ is a subset of OB and $P_2(L) = OC$. Thus, L contains a point of V_N. As before, this involves a contradiction.

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Convexity on a topological space

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Abstract. Although convexity is an attribute of subsets of linear spaces in general, we define convexity on topological spaces without linear structures paying attention to the concept of convex hull. Then some theorems which have been obtained in linear topological spaces are given in these spaces.

Takahashi [5] discussed a convexity on a metric space. In this paper, we discuss a convexity on a topological space without linear space structure. We introduce a convexity on a topological space and several concepts concerning the convexity, and obtain some theorems which generalize the theorems proved by Browder [1], Fan [2] and Sion [4]. All topological structures are implicitly assumed to satisfy Hausdorff separation axiom.

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1. Definitions and some elementary properties. Let X be a topological space, $\mathscr{A}(X)$ the family of all subsets of X and $\mathscr{F}(X)$ the family of all finite subsets of X. An *H-operator* on X is a mapping $\langle\cdot\rangle$ from $\mathscr{A}(X)$ into $\mathscr{A}(X)$ satisfying the following conditions:

(a) $\langle\emptyset\rangle = \emptyset$, where $\emptyset$ is the empty set;

(b) $\langle\{x\}\rangle = \{x\}$, $x \in X$;

(c) $\langle\langle A\rangle\rangle = \langle A\rangle$, $A \in \mathscr{A}(X)$;

(d) $\langle A\rangle = \bigcup\{\langle F\rangle\colon F \subset A, F \in \mathscr{F}(X)\}$.

The image $\langle A\rangle$ of A is said to be the *convex hull* of A. A *convex set* in X is a subset of X which is equal to its convex hull.

PROPOSITION 1. (i) *An H-operator is monotone, i.e. if* $A \subset B$, *then* $\langle A\rangle \subset \langle B\rangle$.

(ii) *The convex hull* $\langle A\rangle$ *of* $A \in \mathscr{A}(X)$ *is the smallest convex set containing* A.

(iii) *The entire space* X *and the empty set* $\emptyset$ *are convex sets.*

(iv) *If* $\{C_\nu\}_{\nu\in I}$ *is a family of convex sets, then* $\bigcap_{\nu\in I} C_\nu$ *is a convex set.*

(v) *If* $\{C_\nu\}_{\nu\in I}$ *is a family of convex sets such that for any two indices* ν_1 *and* ν_2 *there exists an index* μ *with* $C_\mu \subset C_{\nu_1} \cap C_{\nu_2}$, *then* $\bigcup_{\nu\in I} C_\nu$ *is a convex set.*