The assertion that this formula is indeed true in this model is a Σ_1 statement about Ω in Levy's hierarchy, by routine computations. Being true for Ω, it must therefore be true for all countable ordinals in some closed unbounded set $C \subseteq \Omega$. But for $\alpha \in C$ the truth of this statement about α is readily seen to imply that for some/any $x \in W$ with order type $\Lambda(x) = \alpha$ we have $\exists y \varphi^+(t, x, y)$. It follows that $C \subseteq S$, so $\Omega - S$ is not stationary after all, a contradiction! ■

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Accepté par la Rédaction le 19. 7. 1978

Analytic sets with countable sections

by

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Abstract. This article contains a new proof of Lusin's theorem that an analytic set in the product of two Polish spaces, having countable (vertical) sections, is a countable union of analytic graphs.

1. Introduction. Suppose X, Y are Polish spaces. If $E \subseteq X \times Y$ and $x \in X$, we denote by E^x the set $\{y \in Y: (x, y) \in E\}$. A set $G \subseteq X \times Y$ is said to be a *graph* if G^x contains at most one point for each $x \in X$. We denote the family of Borel graphs in $X \times Y$ by $\mathcal{G}$. Let $\pi_1(\pi_2)$ be the projection of $X \times Y$ to the first (second) coordinate.

Lusin proved the following fundamental results on Borel and analytic sets with countable (vertical) sections in his celebrated monograph [2]:

(i) *If E is a Borel set in $X \times Y$ such that $(\forall x \in X)$ (E^x is countable), then $\pi_1(E)$ is Borel in X* ([2], p. 178).

(ii) *If E is a Borel set in $X \times Y$ such that $(\forall x \in X)$ (E^x is countable), then $E \in \mathcal{G}_\sigma$* ([2], p. 244).

(iii) *If E is an analytic set in $X \times Y$ such that $(\forall x \in X)$ (E^x is countable), then there is a Borel set B in $X \times Y$ such that $E \subseteq B$ and $(\forall x \in X)$ (B^x is countable)* ([2], p. 247).

Finally, combining (ii) and (iii), Lusin obtained

(iv) *If E is an analytic set in $X \times Y$ such that $(\forall x \in X)$ (E^x is countable), then E is a countable union of analytic graphs* ([2], p. 252).

We shall prove the following:

THEOREM. *If A is analytic in $X \times Y$ such that $(\forall x \in X)$ (A^x is countable), then there is $H \in \mathcal{G}_\sigma$ such that $A \subseteq H$.*

A notable feature of our proof of the above theorem is that we do not use results (i)-(iv), so that these results fall out as easy consequences of our theorem. Our proof, though quite different from Lusin's proof of (iii), is based on ideas contained in Lusin's proof of (ii) and also on certain ideas in a recent article of Saint Raymond [3].

2. Proof of theorem. We will prove a sequence of lemmas from which the theorem will be deduced.

Let, then, X, Y be Polish spaces. If $E \subseteq X \times Y$, we define

$$M(E) = \{x \in \pi_1(E)\colon E^x \text{ contains at least two points}\}.$$

LEMMA 1. *If $E \subseteq X \times Y$ is analytic, then $M(E)$ is analytic.*

Proof. Let $\{P_n\}$ be an open base for Y. Then

$$M(E) = U[\pi_1(E \cap (X \times P_m)) \cap \pi_1(E \cap (X \times P_n))],$$

where the union extends over all ordered pairs (m, n) such that $P_m \cap P_n = \emptyset$. The sets within square brackets being analytic, so is $M(E)$.

For the rest of the proof, A will denote a fixed analytic subset of $X \times Y$. (To start with we do *not* assume that A has countable sections). Fix a continuous function f on Σ, the space of irrationals, onto A. Let $\{W_n\}$ be an open base for Σ. For $Z \subseteq \Sigma$, we define

$$D(Z) = \{\sigma \in Z\colon \pi_1(f(\sigma)) \in M(f(V \cap Z)) \quad \text{for every open neighbourhood } V \text{ of } \sigma\}.$$

It is easy to see that

$$D(Z) = \bigcap_{n \geq 1} E_n(Z),$$

where $E_n(Z) = \{\sigma \in Z\colon \sigma \in W_n \to \pi_1(f(\sigma)) \in M(f(W_n \cap Z))\}$, $n \geq 1$. Next we define sets Z_α, $\alpha < \omega_1$, by transfinite induction as follows. We put

$$Z_0 = \Sigma,$$

$$Z_\alpha = \begin{cases} D(Z_\beta) & \text{if } \alpha = \beta + 1, \\ \bigcap_{\beta < \alpha} Z_\beta & \text{if } \alpha \text{ is a limit ordinal}. \end{cases}$$

LEMMA 2. *If $Z \subseteq \Sigma$ is analytic, then $E_n(Z)$ is analytic for each $n \geq 1$. Consequently, if Z is analytic, then so is $D(Z)$.*

Proof. Observe that

$$E_n(Z) = Z \cap \left[(\Sigma - W_n) \cup f^{-1}\left(\pi_1^{-1}\left(M(f(W_n \cap Z))\right)\right)\right].$$

That $E_n(Z)$ is analytic follows from Lemma 1. The second assertion of Lemma 2 is now obvious.

LEMMA 3. *$(\forall \alpha < \omega_1) Z_\alpha$ is analytic.*

Proof. Use transfinite induction on α and the previous lemma.

Before stating the key lemma (Lemma 6), we state two results which will be used in its proof. The first result is due to V. I. Glivenko, the second is the generalized first separation principle for analytic sets due to P. S. Novikov.

LEMMA 4. *If $G \subseteq X \times Y$ is an analytic graph, then there is $H \in \mathcal{G}$ such that $G \subseteq H$.*

Proof (by S. M. Srivastava). Let $g\colon \pi_1(G) \to Y$ be the function whose graph is G. Since G is analytic, g is Borel measurable ([1], p. 398). So, by an extension theorem of Kuratowski ([1], p. 341), there is a Borel subset B of X and a Borel measurable function $h\colon B \to Y$ such that $\pi_1(G) \subseteq B$ and $h = g$ on $\pi_1(G)$. Let H be the graph of h. Then, as is easy to check, $H \in \mathcal{G}$ and $G \subseteq H$.

LEMMA 5. *If A_n, $n \geq 1$, are analytic subsets of a Polish space P such that $\bigcap_{n \geq 1} A_n = \emptyset$, then there exist Borel subsets B_n, $n \geq 1$, of P such that $(\forall n \geq 1)(A_n \subseteq B_n)$ and $\bigcap_{n \geq 1} B_n = \emptyset$.*

We omit the proof. The reader is referred to ([1], p. 418) or [3] for a proof.

LEMMA 6. *If $Z \subseteq \Sigma$ is analytic and $D(Z) \subseteq B$ for some Borel subset B of Σ, then there is $H \in \mathcal{G}_\sigma$ such that $f(Z - B) \subseteq H$.*

Proof. Since Z is a fixed set in this proof, we will suppress the dependence of $E_n(Z)$ on Z and write E_n for $E_n(Z)$.

By Lemma 2, the sets $E_n \cap B^c$, $n \geq 1$, are analytic. Moreover, $\bigcap_{n \geq 1} (E_n \cap B^c) = \emptyset$ since $\bigcap_{n \geq 1} E_n = D(Z) \subseteq B$. So, by Lemma 5, there exist Borel subsets B_n, $n \geq 1$, of Σ such that $E_n \cap B^c \subseteq B_n$, $n \geq 1$, and $\bigcap_{n \geq 1} B_n = \emptyset$. Plainly, we have:

$$E_n \subseteq B \cup B_n \tag{1}$$

so that

$$Z - (B \cup B_n) \subseteq Z - E_n \subseteq W_n \cap Z \tag{2}$$

and also, because $\bigcap_{n \geq 1} B_n = \emptyset$,

$$Z - B = \bigcup_{n \geq 1} (Z - (B \cup B_n)). \tag{3}$$

If we prove that the sets $f(Z - (B \cup B_n))$ are analytic graphs, we will be done. For then, by virtue of Lemma 4, there will exist sets $H_n \in \mathcal{G}$ such that

$$f(Z - (B \cup B_n)) \subseteq H_n,$$

from which and (3) it will follow that

$$f(Z - B) \subseteq \bigcup_{n \geq 1} f(Z - (B \cup B_n)) \subseteq \bigcup_{n \geq 1} H_n \in \mathcal{G}_\sigma.$$

Clearly, the set $f(Z - (B \cup B_n))$ is analytic. To show that $f(Z - (B \cup B_n))$ is a graph, let $x \in \pi_1(f(Z - (B \cup B_n)))$. Then $x = \pi_1(f(\sigma))$ for some $\sigma \in Z - (B \cup B_n)$, so that $\sigma \in W_n \cap Z$ and $\sigma \notin E_n$ (by (2)). It follows that $x = \pi_1(f(\sigma)) \notin M(f(W_n \cap Z))$. Hence, the x-section of $f(W_n \cap Z)$ is at most a singleton, and since $f(Z - (B \cup B_n)) \subseteq f(W_n \cap Z)$ (by (2)), this implies that the x-section of $f(Z - (B \cup B_n))$ is exactly a singleton. Consequently, $f(Z - (B \cup B_n))$ is a graph.

LEMMA 7. *If $Z_\alpha \subseteq B$ for some Borel set B in Σ, then there is $H \in \mathcal{G}_\sigma$ such that $f(\Sigma - B) \subseteq H$.*

Proof. We prove the result by induction on α. The result is obviously true for $\alpha = 0$. Suppose it is true for all $\beta < \alpha$.

Case 1. α is a limit ordinal. Since $\bigcap_{\beta<\alpha} Z_\beta = Z_\alpha \subseteq B$, it follows that $\bigcap_{\beta<\alpha}(Z_\beta \cap B^c) = \emptyset$. Moreover, by Lemma 3, the sets $Z_\beta \cap B^c$ are analytic. So, according to Lemma 5, there exist Borel subsets B_β of Σ such that $Z_\beta \cap B^c \subseteq B_\beta$, $\beta < \alpha$, and $\bigcap_{\beta<\alpha} B_\beta = \emptyset$. Note that $Z_\beta \subseteq B \cup B_\beta$, $\beta < \alpha$. By induction hypothesis, there exist $H_\beta \in \mathcal{G}_\sigma$ such that $f(\Sigma - (B \cup B_\beta)) \subseteq H_\beta$, $\beta < \alpha$. But $\Sigma - B = \bigcup_{\beta<\alpha}(\Sigma - (B \cup B_\beta))$ since $\bigcap_{\beta<\alpha} B_\beta = \emptyset$. Hence

$$f(\Sigma - B) = \bigcup_{\beta<\alpha} f(\Sigma - (B \cup B_\beta)) \subseteq \bigcup_{\beta<\alpha} H_\beta \in \mathcal{G}_\sigma .$$

Case 2. $\alpha = \beta + 1$. Since $D(Z_\beta) = Z_\alpha \subseteq B$, we use Lemma 6 to get $H_1 \in \mathcal{G}_\sigma$ such that $f(Z_\beta - B) \subseteq H_1$, so that $Z_\beta \subseteq f^{-1}(H_1) \cup B$. As the set $f^{-1}(H_1) \cup B$ is a Borel subset of Σ, the induction hypothesis yields a set $H_2 \in \mathcal{G}_\sigma$ such that

$$f\big(\Sigma - (f^{-1}(H_1) \cup B)\big) \subseteq H_2 .$$

Hence

$$\begin{aligned} f(\Sigma - B) &\subseteq f\big(\Sigma - (f^{-1}(H_1) \cup B)\big) \cup f\big(f^{-1}(H_1)\big) \\ &\subseteq H_2 \cup H_1 \in \mathcal{G}_\sigma . \end{aligned}$$

We show next that, if $Z_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, then some vertical section of A is uncountable. Towards this end, we introduce some terminology.

Say that a system of sets $\{V_1, \ldots, V_k, Q_1, \ldots, Q_k\}$ is *admissible* if

(a) V_i, Q_i are open in Σ, Y, respectively, $i = 1, \ldots, k$,

(b) $\pi_2(f(V_i)) \subseteq Q_i$, $i = 1, \ldots, k$, and

(c) $\bigcap_{i=1}^{k} \pi_1(f(V_i \cap Z_\alpha)) \neq \emptyset$ for all $\alpha < \omega_1$.

LEMMA 8. *Suppose $\{V_1, \ldots, V_k; Q_1, \ldots, Q_k\}$ is an admissible system. Let $\mathcal{B}(V_i)$, $\mathcal{B}(Q_i)$ be countable open bases for V_i, Q_i, respectively, $i = 1, \ldots, k$. Then there exists an admissible system $\{V_{10}, V_{11}, \ldots, V_{k0}, V_{k1}; Q_{10}, Q_{11}, \ldots, Q_{k0}, Q_{k1}\}$ such that $V_{ij} \in \mathcal{B}(V_i)$, $Q_{ij} \in \mathcal{B}(Q_i)$ and $Q_{i0} \cap Q_{i1} = \emptyset$, $j = 0, 1$, $i = 1, \ldots, k$.*

Proof. Fix $\alpha < \omega_1$. Let

$$x \in \bigcap_{i=1}^{k} \pi_1\big(f(V_i \cap Z_{\alpha+1})\big) .$$

Then

$$x = \pi_1(f(\sigma_i)) \quad \text{for some } \sigma_i \in V_i \cap Z_{\alpha+1} = V_i \cap D(Z_\alpha),\ i = 1, \ldots, k .$$

It follows that $x = \pi_1(f(\sigma_i)) \in M(f(V_i \cap Z_\alpha))$, $i = 1, \ldots, k$. So there exist $y_{i0} \neq y_{i1}$ such that $(x, y_{ij}) \in f(V_i \cap Z_\alpha)$, $j = 0, 1$, $i = 1, \ldots, k$. Hence we can find $\sigma_{ij} \in V_i \cap Z_\alpha$ such that $f(\sigma_{ij}) = (x, y_{ij})$, $j = 0, 1$, $i = 1, \ldots, k$. Condition (b) now implies that $y_{ij} \in Q_i$, $j = 0, 1$, $i = 1, \ldots, k$. So we can choose sets $Q_{i0}(\alpha)$, $Q_{i1}(\alpha) \in \mathcal{B}(Q_i)$ such that $y_{ij} \in Q_{ij}(\alpha)$ and $Q_{i0}(\alpha) \cap Q_{i1}(\alpha) = \emptyset$, $j = 0, 1$, $i = 1, \ldots, k$. Next we use continuity of the function $\pi_2 \circ f$ to choose sets $V_{ij}(\alpha) \in \mathcal{B}(V_i)$ such that

$$\sigma_{ij} \in V_{ij}(\alpha) \quad \text{and} \quad \pi_2\big(f(V_{ij}(\alpha)) \subseteq Q_{ij}(\alpha),\ j = 0, 1,\ i = 1, \ldots, k .$$

It is now easy to see that

$$x \in \bigcap_{i=1}^{k} \bigcap_{j=0}^{1} \pi_1\big(f(V_{ij}(\alpha) \cap Z_\alpha)\big).$$

Thus, for each $\alpha < \omega_1$, there is a system of sets

$$\{V_{10}(\alpha), V_{11}(\alpha), \ldots, V_{k0}(\alpha); V_{k1}(\alpha), Q_{10}(\alpha), Q_{11}(\alpha), \ldots, Q_{k1}(\alpha)\}$$

such that $V_{ij}(\alpha) \in \mathcal{B}(V_i)$, $Q_{ij}(\alpha) \in \mathcal{B}(Q_i)$, $Q_{i0}(\alpha) \cap Q_{i1}(\alpha) = \emptyset$, $j = 0, 1$, $i = 1, \ldots, k$, and satisfying conditions (a), (b) and the condition $\bigcap_{i=1}^{k} \bigcap_{j=0}^{1} \pi_1\big(f(V_{ij}(\alpha) \cap Z_\alpha)\big) \neq \emptyset$. Since $\mathcal{B}(V_i)$, $\mathcal{B}(Q_i)$ are countable for $i = 1, \ldots, k$, it follows that there are only countably many such systems. Consequently, there is a system

$$\{V_{10}, V_{11}, \ldots, V_{k0}, V_{k1}; Q_{10}, Q_{11}, \ldots, Q_{k0}, Q_{k1}\}$$

such that

$$\begin{aligned} \{V_{10}(\alpha), V_{11}(\alpha), \ldots, V_{k0}(\alpha), V_{k1}(\alpha); Q_{10}(\alpha), Q_{11}(\alpha), \ldots, Q_{k0}(\alpha), Q_{k1}(\alpha)\} \\ = \{V_{10}, V_{11}, \ldots, V_{k0}, V_{k1}; Q_{10}, Q_{11}, \ldots, Q_{k0}, Q_{k1}\} \end{aligned}$$

for uncountably many α. Using the fact that the Z_α's are nonincreasing, one verifies easily that the system $\{V_{10}, V_{11}, \ldots, V_{k0}, V_{k1}; Q_{10}, Q_{11}, \ldots, Q_{k0}, Q_{k1}\}$ has all the required properties.

LEMMA 9. *If $Z_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, then there is $x_0 \in X$ such that A^{x_0} contains a homeomorph of the Cantor set.*

Proof. Fix a metric d in Σ. Since $Z_\alpha \neq \emptyset$ for each $\alpha < \omega_1$, it follows that the system $\{\Sigma, Y\}$ is admissible. By repeatedly using Lemma 8, we get, for each finite sequence $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$ of 0's and 1's, an open set $V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}$ in Σ and an open set $Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}$ in Y such that, for each $k \geq 1$,

(i) $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \neq (\delta_1, \delta_2, \ldots, \delta_k) \to Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k} \cap Q_{\delta_1\delta_2\ldots\delta_k} = \emptyset$,

(ii) $\mathrm{cl}(V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_{k+1}}) \subseteq V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}$, where cl is the closure operator,

(iii) d-diameter $(V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}) < 1/k$, and

(iv) $\{V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}; Q_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}: (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in S_k\}$ is admissible, where S_k is the set of finite sequences of 0's and 1's of length k.

Define $C = \bigcup \bigcap_{k=1}^{\infty} V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k}$, where the union extends over all infinite sequences $(\varepsilon_1, \varepsilon_2, \ldots)$ of 0's and 1's. Using conditions (i)-(iv), one checks that C is a homeomorph of the Cantor set. Next observe that (iv) implies that

$$\bigcap\{\pi_1(f(V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k})): (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in S_k\} \neq \emptyset \quad \text{for each } k \geq 1 .$$

It now follows that the set $\pi_1(f(C))$ contains exactly one point, say, x_0. Finally, conditions (i) and (iv) imply that $\pi_2 \circ f$, restricted to C, is one-one. So A^{x_0} contains a homeomorph of the Cantor set.

We now turn to the proof of the theorem. Assume that A^x is countable for each $x \in X$. Then, by Lemma 9, there is $\lambda < \omega_1$ such that $Z_\lambda = \emptyset$. Now Lemma 7 with $\alpha = \lambda$ and $B = \emptyset$ yields a set $H \in \mathcal{G}_\sigma$ such that $A = f(\Sigma) \subseteq H$. This completes the proof of the theorem.

Discussions with H. Sarbadhikari and S. M. Srivastava are gratefully acknowledged.

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Accepté par la Rédaction le 19. 7. 1978

On pure semi-simple Grothendieck categories II

by

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Abstract. Given a pure semi-simple Grothendieck category $\mathcal{A}$, we construct a new pure-semi-simple functor category $I(\mathcal{A})$ such that gl. dim$\mathcal{A}$ = gl. dim$I(\mathcal{A})$. The map $\mathcal{A} \to I(\mathcal{A})$ defines a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. Applications of this result are given.

Introduction. In [13] the notion of a pure semi-simple Grothendieck category is introduced as a "pure" counterpart of semi-simple categories (cf. [9]). We recall that a Grothendieck category is pure semi-simple if each of its objects is a direct sum of finitely presented objects. Pure semi-simple Grothendieck categories are investigated in [11]–[17].

In the present paper we give two constructions of new pure semi-simple Grothendieck categories from a given pure semi-simple one. Given a pure semi-simple Grothendieck category $\mathcal{A}$ a pure semi-simple functor category $I(\mathcal{A})$ is constructed in a such a way that gl.dim$\mathcal{A}$ = gl.dim$I(\mathcal{A})$ and the category of all noetherian injective objects in $\mathcal{A}$ is equivalent to the category of all finitely generated projective objects in $I(\mathcal{A})$. Further, given a skeletally small additive category $\mathcal{C}$ such that the functor category $\mathcal{C}$-Mod is locally coherent and $\mathcal{C}^{\mathrm{op}}$-Mod is pure semi-simple, a pure semi-simple Grothendieck category $\tilde{\mathcal{C}}$ is constructed. The map $\mathcal{A} \mapsto I(\mathcal{A})$ is the inverse (with respect to an equivalence) of the map $\mathcal{C}^{\mathrm{op}}$-Mod $\mapsto \tilde{\mathcal{C}}$, and conversely. These maps define a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. In Section 2 we illustrate our constructions by simple examples.

In Section 1 we recall from [12]–[14] some background material on functor categories and pure semi-simple Grothendieck categories. An extension of Theorem A in [3] is given.

Section 2 contains the constructions and main results mentioned above. As a consequence of our general considerations we get the following two corollaries. Any injective noetherian object of a pure semi-simple Grothendieck category has a right pure semi-simple endomorphism ring. If $\mathcal{C}$ is a skeletally small abelian category such that the category $\mathcal{C}$-Mod is perfect, then $\mathcal{C}$-Mod is locally noetherian.