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A measurable selection theorem

by

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Abstract. A family of subsets of a Polish space has the *partition-selection property* if the equivalence relation generated by countably many members of the family always admits a selector measurable with respect to the family. It is shown that the family of Baire-property sets enjoys the partition-selection property. The same is true of the Borel-programmable sets, the R-sets, the absolutely Δ^1_2 sets, and the Lebesgue measurable sets.

§ 1. Measurable selectors and transversals for countably-generated equivalence relations and partitions: A survey

1.1. Introduction. Throughout this section, let X be an uncountable Polish space (topological space admitting a countable basis and a complete metric). E.g. X might be the Baire space $J = \omega^\omega$ of infinite sequences of natural numbers under the topology having as basis the sets $U_s = \{y\colon y \text{ extends } s\}$ for s a finite sequence. Or X might be the Cantor space $I = 2^\omega$ of infinite $\{0, 1\}$-sequences, considered as a subspace of J. The letters Y, Z will also denote Polish spaces.

Let E be an equivalence relation on X, and identify E with its graph $\{(x, x')\colon xEx'\} \subseteq X^2$. Let x/E denote the E-equivalence class of $x \in X$. Associated with E we have the partition $Q = \{x/E\colon x \in X\}$ of X into disjoint classes; and conconversely, every such partition is associated with an equivalence relation. A *section* for E is a map $\sigma\colon Q \to X$ satisfying $\sigma(A) \in A$. A *selector* for E is a function $S\colon X \to X$ of form $S(x) = \sigma(x/E)$ for some section σ. Equivalently, S is a selector if we always have $S(x)Ex$, and have $S(x) = S(x')$ whenever xEx'. A *transversal* for E is a set $T \subseteq X$ consisting of exactly one representative from each E-equivalence class. Selector S and transversal T are *associated* if $T = \text{range } S$, or equivalently $S(x) =$ the unique $x' \in T$ with xEx'. $A \subseteq X$ is *E-invariant* if $x' \in A$ whenever $x \in A$ and xEx'. A countable family $\{A_n\colon n \in \omega\}$ of subsets of X *generates* E if

$$E = \{(x, x')\colon \forall n (n \in A_n \leftrightarrow x' \in A_n)\},$$

or equivalently if the A_n are invariant sets which *separate* distinct E-equivalence classes (so that whenever $x/E \neq x'/E$ there is an A_n with $x/E \subseteq A_n$ and $x'/E \cap A_n = \emptyset$

or *vice versa*). A function $f\colon X \to Y$ *induces* E if $E = \{(x, x')\colon f(x) = f(x')\}$. Thus f induces E iff $\{f^{-1}[V_n]\colon n \in \omega\}$ generates E, where the V_n are a basis for the Polish space Y; and $\{A_n\colon n \in \omega\}$ generates E iff $f\colon X \to I$ induces E, where f is defined by letting $f(x)(n) = 0$ iff $x \in A_n$.

Let $\mathscr{H}$ be a σ-field of subsets of X. A function $f\colon X \to Y$ is *$\mathscr{H}$-measurable* if $f^{-1}[V] \in \mathscr{H}$ for every open $V \subseteq Y$ (and hence for every Borel set). $\mathscr{H}$ is called a *tribe* if it contains all open sets (and hence all Borel sets). $\mathscr{H}$ will be called *uniform* if it is a tribe and in addition any composition of two $\mathscr{H}$-measurable functions $X \to X$ is again $\mathscr{H}$-measurable, or equivalently the inverse image of an element of $\mathscr{H}$ under an $\mathscr{H}$-measurable function is always itself an element of $\mathscr{H}$. In order of size, the most important uniform families are: the Borel sets, the C-sets (= smallest family containing the open sets and stable under complementation and operation $\mathscr{A}$ = smallest uniform family containing the analytic sets), the Borel-programmable sets of Blackwell [1], the R-sets of Kolmogorov (cf. [4]), the absolutely Δ^1_2 sets of Solovay (as in [5]), and the universally measurable sets (sets measurable w.r.t. every complete, σ-finite, Borel-regular measure). Non-uniform tribes include: the σ-field generated by analytic sets, the sets possessing the property of Baire, and the Lebesgue measurable sets (for $X = [0, 1]$). Note that if $\mathscr{H}$ is a tribe and S an $\mathscr{H}$-measurable selector for the equivalence relation E on X, then the associated transversal $T = (\text{identity} \times S)^{-1}[\text{diagonal of } X^2]$ belongs to $\mathscr{H}$. But even for uniform $\mathscr{H}$ the existence of a transversal in $\mathscr{H}$ need not imply the existence of an $\mathscr{H}$-measurable selector, except in the case $\mathscr{H}$ = Borel sets. E will be called *$\mathscr{H}$-generated* if it is generated by a countable subfamily of $\mathscr{H}$, or equivalently induced by some $\mathscr{H}$-measurable function $f\colon X \to Y$ to some other Polish space.

$\mathscr{H}$ will be said to have the *partition-selection property* if every $\mathscr{H}$-generated equivalence relation admits an $\mathscr{H}$-measurable selector. We will survey the status of the partition-selection property for various families. The main new positive results have been listed in the Abstract above. Conversations with R. D. Mauldin, D. E. Miller, S. M. Srivastava, and D. H. Wagner have contributed materially to the development of these results in their present form.

1.2. Borel sets. The Borel sets lack the partition-selection property, as the following example from [9] shows: Let $f\colon X \to X$ be a continuous function whose range A is properly analytic. If there existed a Borel-measurable selector S and hence a Borel transversal T for the induced equivalence relation, then A would be the injective image under a continuous function f or a Borel set T and hence Borel, a contradiction! The failure of the partition-selection property for the Borel sets is the motivation for turning to larger families.

1.3. C-sets. The C-sets are the smallest reasonable family for which the partition-selection property is known to hold. From the treatment of this family in [3] we isolate four propositions (corresponding to the following items in [3]: proof of Lemme 1, proofs of Lemme 2(i) and of Proposition 3, Proposition 3 as stated, Théorème Principal) which will be used below in connection with other families.

First, a refinement of a theorem of Kaniewski [6]: Here *analytically measurable* means measurable w.r.t. the σ-field generated by the analytic sets.

PROPOSITION I. *Let Z be a Polish space, $P \subseteq Z$ co-analytic, $R \subseteq Z^2$ analytic, $D = R \cap P^2$. Suppose that D is an equivalence relation on P and that every D-equivalence class is relatively closed in P. Then D admits an analytically measurable selector whose associated transversal is co-analytic.*

As an immediate consequence we get what seems to be the sharpest general result available about Borel-generated equivalences:

COROLLARY. *A Borel-generated equivalence relation E on the Polish space X admits an analytically measurable selector S whose associated transversal T is co-analytic.*

Proof. Let $f\colon X \to Y$ be a Borel-measurable function inducing E. Apply Proposition I to $Z = X \times Y$, $P = \text{graph } f$, $R = \{((x, y), (x', y))\colon x, x' \in X \,\&\, y \in Y\}$, obtaining selector S_0 and transversal T_0 for $R \cap P^2$. It suffices to set $S(x)$ = 1st coordinate of $S_0(x, f(x))$, so $T = \{x\colon (x, f(x)) \in T_0\}$. ■

It is worth noting that in the situation of the corollary, to get just an analytically measurable selector (resp. co-analytic transversal) the celebrated theorem of Yankov-von Neumann (resp. of Kondô) on the uniformization of analytic (resp. of co-analytic) sets would have sufficed.

Second we have a consequence of Proposition I involving a technical notion which here will be called *presentability*. For $\mathscr{H}$ a σ-field of subsets of X and $A \subseteq X$, an *$\mathscr{H}$-presentation* of A is a quadruple (Y, B, P, G) where (i) Y is a Polish space, (ii) $B \subseteq X \times Y$ is clopen, (iii) $P \subseteq X \times Y$ is co-analytic, (iv) $G\colon X \to Y$ has graph $\subseteq P$, (v) A = projection to 1st coordinate of $B \cap P$, (vi) G is $\mathscr{H}$-measurable. $\mathscr{H}$ is *presentable* if it is uniform and every $A \in \mathscr{H}$ admits an $\mathscr{H}$-presentation.

PROPOSITION II. *Every presentable family has the partition-selection property.*

This result obviously provides a general criterion for the property that interests us. It will be applied below to several families.

Third in [3] comes the application of the above criterion to the C-sets:

PROPOSITION III. *The family of C-sets is presentable, and hence enjoys the partition-selection property.*

Fourth comes a consequence of an observation due to Miller [11] and independently to Srivastava. The *outer saturation* of $B \subseteq X$ for the equivalence E is the smallest invariant set containing B, *viz.* $B^+ = \{x\colon \exists x' \in B(xEx')\}$. Note that if (the graph of) E is analytic, B^+ is analytic for any analytic B.

LEMMA (Miller-Srivastava). *If E is an equivalence relation on the Polish space X for which every equivalence class is a G_δ set, and if $\{V_n\colon n \in \omega\}$ is a basis for X, then E is generated by the outer saturations $\{V_n^+\colon n \in \omega\}$.*

PROPOSITION IV. *An analytic equivalence relation E on the Polish space X for which every equivalence class is a G_δ set admits a C-measurable selector.*

In this connection a question arises naturally: Let E be as in Proposition IV. Does E admit a co-analytic transversal? We will see in the next section that this cannot be proved from the usual ZF axioms of set theory; whether it can be refuted is open.

1.4. Borel-programmable sets. The *Borel-programmable* or *BP*-sets were introduced by Blackwell in [1]. While assuming a nodding acquaintance with this paper (which contains e.g. the proof that the *BP*-sets form a uniform family containing the *C*-sets), we recall the basic definitions. The *termwise partial order* $\preceq$ on J is

$$\{(x, y)\colon \forall n(x(n) \leqslant y(n))\}.$$

A Borel-measurable function $p\colon I \to I$ satisfying $p(x) \preceq x$ for all x will be called a *program.* Its αth *iterate* for ordinal $\alpha \leqslant \Omega$ is defined inductively: (i) $p^0(x) = x$, (ii) $p^{\beta+1}(x) = p(p^\beta(x))$, (iii) at limits $p^\gamma(x) = \preceq\text{-inf}\{p^\beta(x)\colon \beta < \gamma\}$. An *encoder* is a Borel-measurable function from a Polish space to I; a *decoder* is a Borel-measurable function from I to a Polish space, e.g. the discrete space $\{0, 1\}$. A *BP-function* Polish spaces is a composition of form $d \circ p^\Omega \circ e$ for encoder e, program p, decoder d. The *characteristic function* χ_A of a set A satisfies $\chi_A(x) = 0$ for $x \in A$ and $= 1$ for $x \notin A$. A *BP-set* is one whose characteristic function is *BP*.

THEOREM I. *The family of Borel-programmable sets is presentable, and hence enjoys the partition-selection property.*

Proof. Let $A \subseteq X$ have $\chi_A = d \circ p^\Omega \circ e$ where $e\colon X \to I$ is an encoder, $p\colon I \to I$ a program, $d\colon I \to \{0, 1\}$ a decoder. We must produce a quadruple (Y, B, P, G) satisfying clauses (i)–(vi) of the definition of presentation. We will verify (vi) by showing that G is a *BP*-function, $G = h \circ q^\Omega \circ f$ for some encoder f, program q, decoder h. Before defining all these items we need some technical devices.

Let $\pi\colon \omega \times \omega \to \omega$ be the bijection $\pi(m, n) = 2^m(2n+1) - 1$. Let $\Lambda\colon I \to$ power set $(\omega \times \omega)$ be the bijection $\Lambda(y) = \{(m, n)\colon y(\pi(m, n)) = 0\}$ Let $\Theta\colon I \to I^\omega$ be the bijection sending y to the sequence whose mth term $y_m = \Theta(y)(m)$ is determined by $y_m(n) = y(\pi(m, n))$. For $y \in I$, let $\nu(y)$ be $1+$the least n for which $y(n) = 1$ and $p(y)(n) = 0$ any such exists, and 0 otherwise. Now to define Y, B, P, G, f, q, h.

Set $Y = I \times I \times \{0, 1\}$, $B = X \times I \times I \times \{0\}$. Let P be the set of $(x, y, z, k) \in X \times Y$ such that $\Lambda(y)$ is a wellordering of its field and: (i) if i is the $\Lambda(y)$-least element, then $\Theta(z)(i) = e(x)$; (ii) if i is the immediate $\Lambda(y)$-successor of j, then $\Theta(z)(i) = p(\Theta(z)(j))$; (iii) if i is a limit point in $\Lambda(y)$, then

$$\Theta(z)(i) = \preceq\text{-inf}\{\Theta(z)(j)\colon j \neq i \,\&\, (j, i) \in \Lambda(y)\};$$

(iv) 0 is the $\Lambda(y)$-greatest element, and $\Theta(z)(0) = p(\Theta(z)(0))$ and $k = d(\Theta(z)(0))$; (v) if $i \notin$ field of $\Lambda(y)$, then $\Theta(z)(i) =$ constant sequence with value one.

Towards defining G we introduce some auxiliaries: For $x \in X$ let $\alpha(x)$ be the least ordinal $\alpha < \Omega$ with $p^{\alpha+1}(e(x)) = p^\alpha(e(x))$. Let

$$\lambda(x) = \{(\nu(p^\gamma(e(x))), \nu(p^\beta(e(x))))\colon \gamma \leqslant \beta \leqslant \alpha(x)\}.$$

Let $\vartheta(x) \in I^\omega$ be the sequence with $\vartheta(x)(\nu(p^\beta(e(x)))) = p^\beta(e(x))$ and $\vartheta(x)(n) =$ constant sequence with value one, for other n. Finally let

$$G(x) = (\Lambda^{-1}(\lambda(x)), \Theta^{-1}(\vartheta(x)), \chi_A(x)).$$

In defining f, q, h we will cheat slightly, doing our programming on I^3 rather than I itself. Thus we will have $f\colon X \to I^3$, $q\colon I^3 \to I^3$, $h\colon I^3 \to Y$. We define $f(x) = (e(x), y, z)$ where $\Lambda(y) = \varnothing$ and for all i $\Theta(z)(i) =$ constant sequence with value one. We define $h(w, y, z) = (y, z, d(w))$.

We define $q(w, y, z)$ to be (w', y', z') where: (i) $w' = p(w) \preceq w$; (ii)

$$\Lambda(y') = \Lambda(y) \cup \{(n, \nu(w))\colon n \in \Lambda(y) \cup \{\nu(w)\}\},$$

so $y' \preceq y$; (iii) $\Theta(z')(\nu(w)) = w$ and $\Theta(z')(n) = \Theta(z)(n)$ for other n, so $z' \preceq z$.

The verification that all these items do what they are supposed to do will be left to the interested reader. Very roughly, the idea is that $G(x)$ keeps a complete record of how $\chi_A(x) = d(p^{\alpha(x)}(e(x)))$ was computed. ∎

1.5. ***R*-sets.** Traditionally the R-sets are defined as the union of the Kolmogorov hierarchy R^α for $\alpha < \Omega$, where $R^0 =$ Borel sets, $R^1 = C$-sets, etc. In [4] an alternative characterization is provided, according to which the R-sets are the union of the Black-well hierarchy B^α, where $B^0 =$ Borel sets, $B^1 = BP$-sets, etc. By laborious argumentation, presentability and hence the partition-selection property could be established for each R^α. To establish presentability and hence the partition-selection property for each B^α is much less work. Indeed, everything difficult is already contained in the proof of Theorem I. Without entering into any further details we announce:

COROLLARY. *The family of R-sets is presentable, and hence enjoys the partition-selection property.*

According to [4] we have: C-sets $\subsetneq$ BP-sets $\subsetneq$ R-sets $\subsetneq$ absolutely Δ^1_2 sets. We have treated the first three families; we defer treatment of the fourth to the next section.

1.6. Baire-property sets. This is the logical point at which to take up our main result.

Recall that $A \subseteq X$ is said to have the *property of Baire* if there exist Borel B and C such that the symmetric difference $A \triangle B$ is contained in C, and C is meager (1st category). Given Borel-measurable $f\colon Y \to X$, will be called *f-Baire* if there exist Borel B and C with $A \triangle B \subseteq C$ and $f^{-1}[C]$ meager. $g\colon X \to Z$ will be called *Baire-measurable* if $g^{-1}[V]$ has the Baire property for all open V, Z, and f-Baire measurability will be similarly defined. A set which is f-Baire for all relevant f is called *universally Baire*.

A classical theorem tells us that if $g\colon X \to Z$ is Baire-measurable, then there exist a meager Borel M and a Borel-measurable $h\colon X \to Z$ such that the restrictions

$g|X-M$ and $h|X-M$ agree. The usual proof applies to f-Baire-measurable g as well, producing M with $f^{-1}[M]$ meager. Another classical theorem tells us that operation $\mathscr{A}$ preserves the Baire property, so that all analytic sets possess that property. One of the usual proofs (involving a covering by an Ω-sequence of outer approximations) applies equally well to the f-Baire property. These observations made we proceed to:

THEOREM II. *Let $\mathscr{H}$ be a σ-field of subsets of the Polish space X, all possessing the Baire property. If the equivalence relation E on X is $\mathscr{H}$-generated, then there exist an E-invariant meager $N \in \mathscr{H}$ and an $\mathscr{H}$-measurable selector S for the restriction of E to $X-N$.*

Proof. Fix an $\mathscr{H}$-measurable function f: $X \to Y$ inducing E. Fix a meager Borel M and a Borel-measurable g with $f|X-M = g|X-M$, and let A be the analytic set $g[X-M]$. Apply the Yankov-von Neumann theorem to graph $g^{-1} \cap (A \times (X-M))$ to obtain an analytically measurable h: $A \to X-M$ satisfying $fh = gh =$ identity. Apply our observations above to obtain a Borel $L \subseteq Y$ and a Borel-measurable k: $Y \to X$ such that $g^{-1}[L]$ is meager, $B = A \cap (Y-L)$ is Borel, and $k|B = h|B$. It suffices to set $N = X-f^{-1}[B] \subseteq M \cup g^{-1}[L]$ and to define S on $X-N$ by $S(x) = k(f(x))$. ∎

COROLLARY. *The family of Baire-property sets enjoys the partition-selection property.*

Proof. Given a Baire-property-generated equivalence E on X, apply the foregoing theorem, obtaining N and S. By the Axiom of Choice there exists some selector S' for the restriction of E to N. Then $S'' = S \cup S'$ is a Baire-measurable selector for E. ∎

1.6. Measurable sets. Let μ be a complete, σ-finite, Borel-regular measure on X, e.g. Lebesgue measure for $X = [0, 1]$. Reasoning almost identical to that of the last section establishes the partition-selection property for the family of μ-measurable sets. Indeed, a measure-theoretic analogue of Theorem II for $\mathscr{H} =$ Borel sets is obtained (in a rather special group-theoretic setting, which is inessential) by Mackey [8]. It is unknown whether the universally measurable sets enjoy the partition-selection property, and the corresponding problem for category is also open.

§ 2. Classical hierarchies from a modern standpoint, Part IV: Absolutely Δ_2^1 sets

In this section, which can be viewed as a continuation of [4], we study equivalences and selectors connected with a metamathematically defined family of sets introduced by Solovay. We assume familiarity with the analytical hierarchy, constructibility, and forcing. For the space J we use logical notation: analytic $= \Sigma_1^1$, co-analytic $= \Pi_1^1$, etc.

A famous result of Shoenfield tells us that a truth-functional compound $\vartheta(t)$ of Σ_2^1 statements about an element $t \in J$ is absolute. This means: (i) if $\vartheta(t)$ is true in the "real world" V, then $\vartheta(t)$ is true in the inner model $L[x]$ of sets constructible from $x \in J$, provided of course $t \in L[x]$; (ii) if $M = V$ or $M = L[x]$ and $t \in M$, and if $\mathscr{P} \in M$ is a set of forcing conditions, then if $\vartheta(t)$ is true in M, it remains true in the Boolean-valued extension $M^{\mathscr{P}}$ of M obtained from the Boolean algebra associated with $\mathscr{P}$. Actually, we will only be interested in forcing conditions $\mathscr{P} \in L[x]$ which are x-*accessible*, i.e. such that inside the model $L[x]$ there are no inaccessible cardinals $\leqslant$ the least α with $\mathscr{P} \in L_\alpha[x]$. We call a Π_3^1 statement $\vartheta(t)$ *absolutely* true if it is true in every $L[x]^{\mathscr{P}}$ with $t \in L[x]$ and $\mathscr{P}$ x-accessible. (By a short argument using Shoenfield's theorem, this implies that $\vartheta(t)$ is really true.)

A triple $(t, \varphi^+, \varphi^-)$ consisting of a parameter $t \in J$ and two Π_1^1 formulas is said to provide a Δ_2^1 (resp. *absolutely* Δ_2^1) definition of a set $A \subseteq J$ if

$$A = \{x\colon \exists y \varphi^+(t, x, y)\}$$

and the following is true (resp. absolutely true):

$$\forall x(\exists y \varphi^+(t, x, y) \leftrightarrow \neg \exists y \varphi^-(t, x, y)). \quad (0)$$

A is Δ_2^1 (resp. absolutely Δ_2^1) if it possesses such a definition. Ours is not quite the absoluteness notion used in [5], but the arguments there do suffice to show that our absolutely Δ_2^1 sets form a uniform family of universally measurable sets.

THEOREM III. *The families of Δ_2^1 and of absolutely Δ_2^1 sets are presentable, and hence enjoy the partition-selection property.*

Proof. Consider the plain Δ_2^1 case first. Let $(t, \varphi^+, \varphi^-)$ be a Δ_2^1 definition of a set $A \subseteq J$. We must produce (Y, B, P, G) as in the definition of presentation. Let $Y = J \times \{0, 1\}$, $B = J \times J \times \{0\}$. Let $P = \{(x, y, k)\colon \psi(t, x, y, k)\}$ where $\psi(u, v, w, k)$ is the Π_1^1 formula $[(\varphi^+(u, v, w) \,\&\, k = 0) \vee (\varphi^-(u, v, w) \,\&\, k = 1)]$.

Apply Addison's effective version of Kondô's theorem, to obtain a Π_1^1 formula $\vartheta(u, v, w, k)$ for which the following are provable in ZF:

$$\forall u, v, k \forall w(\vartheta(u, v, w, k) \to \psi(u, v, w, k)), \quad (1)$$

$$\forall u, v, k(\exists w \psi(u, v, w, k) \to \exists! w \vartheta(u, v, w, k)). \quad (2)$$

Let $G(x) =$ the unique pair (y, k) such that $\vartheta(t, x, y, k)$. Clauses (i)–(v) of the definition of presentability should be evident.

To get (vi), we must consider a basic open subset $U_s \times \{k\}$ of Y for s a finite sequence of natural numbers and $k \in \{0, 1\}$, and show that $G^{-1}[U_s \times \{k\}]$ possesses a Δ_2^1 definition (t, σ^+, σ^-). It suffices to let $\sigma^+(u, v, w) = [\vartheta(u, v, w, k) \,\&\, w \text{ extends } s]$, and to let $\sigma^-(u, v, w) = \{\vartheta(u, v, w, 1-k) \vee [\vartheta(u, v, w, k) \,\&\, \neg\, (w \text{ extends } s)]\}$. For we do indeed then have:

$$\forall x(\exists y \sigma^+(t, x, y) \leftrightarrow \neg \exists y \sigma^-(t, x, y)). \quad (4)$$

This completes the treatment of the plain case. For the absolute case it suffices to note that (1) and (2) are always absolutely true, so that if (0) is absolutely true, so is (4). ∎

This is the result promised in the preceding section. We now take up a question mentioned there. For $x \in J$ let Ω^x denote the ordinal (necessarily $\leqslant \Omega$) which inside the model $L[x]$ plays the role of the least uncountable cardinal. The following hypothesis is known to be independent of ZF, and equiconsistent with the existence of an inaccessible cardinal:

$$(+) \qquad \forall x \in J(\Omega^x < \Omega).$$

EXAMPLE. Assume (+). Then there exists an analytic equivalence relation E on a Polish space X such that every E-equivalence class is a G_δ set, but E does not admit a co-analytic transversal.

Proof. Kechris [7] and independently Sacks have shown that (+) implies the existence, for each $t \in J$, of a largest countable Π_1^1-in-t set C_t, and indeed of a single Π_1^1 set $P \subseteq J \times J$ such that for each t, the cross-section $\{x\colon (t, x) \in P\}$ is precisely C_t. Let E be the Σ_1^1 equivalence relation on $J \times J$ defined by

$$E = \{((t, x), (t, x'))\colon x = x' \vee ((t, x) \notin P \;\&\; (t, x') \notin P)\}.$$

Each E-equivalence class is either a singleton or a set of form $\{t\} \times (J - C_t)$, and hence is a G_δ set. If there were a transversal T for E Π_1^1-in-t for some t, then for that t the cross-section $\{x\colon (t, x) \in T\}$ would be a Π_1^1-in-t set consisting of C_t together with one additional element $x \notin C_t$, a contradiction! ∎

By contrast we have a result which has already appeared in [2]: An analytic equivalence relation E on a Polish space X for which every equivalence class is simultaneously G_δ and F_δ admits a co-analytic transversal T. In view of the virtual inaccessability of this reference, it may be well to sketch the proof: We take $X = J$ and let $\lhd$ be the lexicographic linear order.

$$T_s = \{x \in U_s\colon \neg \exists y \in U_s(xEy \;\; x \neq y \;\&\; y \lhd x)\}$$

is, for each basic clopen set U_s, a Π_1^1 set containing at most one representative of any equivalence class, and containing one from any class whose intersection with U_s is nonempty and closed. Now any set which is both G_δ and F_σ meets some U_s in a nonempty closed set. Let $P = \bigcup_s(\{\bar{s}\} \times T_s) \subseteq \omega \times J$, where $\bar{s}$ denotes the code number of s. Impose a Π_1^1 norm on P, i.e. a map $\varrho\colon P \to \Omega$ such that the relation

$$R^{\leqslant} = \{(x, y)\colon y \notin P \vee (x \in P \;\&\; y \in P \;\&\; \varrho(x) \leqslant \varrho(y))\}$$

as well as the similar relation $R^{<}$ for strict inequality are both Σ_1^1. It suffices to set

$$T = \{x\colon \exists n[(n, x) \in P \;\&\; \neg \exists m < n \exists y(xEy \;\&\; y \neq x \;\&\; (m, y) R^{\leqslant}(n, x)) \;\&$$
$$\& \; \neg \exists m > n \exists y(xEy \;\&\; (m, y) R^{<}(n, x))]\}.$$

Before closing, we cite some equivalents of (+) connected with matters studied in this section.

Remark. The following are all equivalent to (+):

(a) $\forall x \in J(\Omega$ is an inaccessible cardinal in $L[x])$.

(b) Every Π_1^1 set is either countable or has a perfect subset.

(c) Every Σ_2^1 set is either countable or has a perfect subset.

(d) Every Σ_1^1-generated equivalence relation has either countably many equivalence classes or else a perfect set of pairwise inequivalent elements.

(e) Every Δ_2^1-generated equivalence relation has either countably many equivalence classes or else a perfect set of pairwise inequivalent elements.

(f) Every Δ_2^1 set is absolutely Δ_2^1.

(g) Every Π_3^1 truth is absolutely true.

Proof. The equivalence of (a) with (+) is folklore. Mansfield [10] and independently Solovay gave (b) and (c). Trivially (e) implies (d) and (g), (f). (c) immediately implies (e): given an equivalence as in (e), we apply (c) to the transversal given us by Theorem III. $\neg$(b) immediately implies $\neg$(d): if P is a counter-example to (b), then the equivalence E generated by the sets $U_s \cup (X - P)$, i.e. the equivalence $E = \{(x, y)\colon x = y \vee (x \notin P \;\&\; y \notin P)\}$, is a counterexample to (d). (a) immediately implies (f): for it implies that for any x-accessible set $\mathscr{P}$ of forcing conditions, the power set of $\mathscr{P}$ as computed in $L[x]$ is merely countable, so that a $\mathscr{P}$-generic set G exists in the "real world" V; then any Π_3^1 statement that is true in V will be true in $L[x][G]$ and hence in $L[x]^{\mathscr{P}}$. It only remains to show that $\neg$(+) implies $\neg$(f).

To this end, suppose we have a t_0 with $\Omega^{t_0} = \Omega$. Let Λ be as in the proof of Theorem I; let W be the Π_1^1 set $\{y \in I\colon \Lambda(y)$ is a wellordering of $\omega\}$; and let $\cong$ be the Σ_1^1 equivalence $\{(y, y')\colon \Lambda(y)$ is isomorphic to $\Lambda(y')\}$. Using the canonical wellordering of $L[t_0]$, the usual construction of a stationary set $S \subseteq \Omega$ whose complement is also stationary, can be carried out in such a way that $D = \{y \in W\colon$ order type $\Lambda(y) \in S\}$ is Δ_2^1-in-t_0. Suppose for contradiction that D possesses an *absolutely* Δ_2^1 definition $(t, \varphi^+, \varphi^-)$. The following are then absolutely true, (5) by hypothesis and (6) since it is a Π_2^1 statement and is true.

$$(5) \qquad \forall x(\exists y \varphi^+(t, x, y) \vee \exists y \varphi^-(t, x, y)),$$

$$(6) \qquad \forall x, x', y, y'(x \cong x' \;\&\; \varphi^+(t, x, y) \to \neg \varphi^-(t, x', y')).$$

For any ordinal $\alpha \leqslant \Omega$ let $\mathscr{R}(\alpha)$ be the usual set of forcing conditions for adjoining a map of ω onto α, and hence an $x \in W$ with order type $\Lambda(x) = \alpha$. Let ξ be the canonical term of the forcing language for this x. Assuming as we may that t_0 is recursive in t $\mathscr{R}(\Omega)$ is certainly t-accessible.

We claim that, forcing over $L[t]$, either all $p \in \mathscr{R}(\Omega)$ force $\exists y \varphi^+(t, \xi, y)$ or all force $\exists y \varphi^-(t, \xi, y)$. Indeed, suppose p^- does *not* force the former, and p^+ does *not* force the latter, so some $q^- \leqslant p^-$ forces the negation of the former, and some $q^+ \leqslant p^+$ the negation of the latter. By the absolute truth of (5), q^+ forces $\exists y \varphi^+(t, \xi, y)$ and q^- forces $\exists y \varphi^-(t, \xi, y)$. Then forcing with $\mathscr{R}(\Omega) \times \mathscr{R}(\Omega)$, the pair (q^+, q^-) forces the existence of two elements $x^+, x^- \in W$ with order type $\Lambda(x^\pm) = \Omega$ constituting a counterexample to (6). But (6) is absolutely true, a contradiction which establishes our claim. Suppose for definiteness it is $\exists y \varphi^+(t, \xi, y)$ that is forced by all p and hence true in $L[t]^{\mathscr{R}(\Omega)}$.

The assertion that this formula is indeed true in this model is a Σ_1 statement about Ω in Levy's hierarchy, by routine computations. Being true for Ω, it must therefore be true for all countable ordinals in some closed unbounded set $C \subseteq \Omega$. But for $\alpha \in C$ the truth of this statement about α is readily seen to imply that for some/any $x \in W$ with order type $\Lambda(x) = \alpha$ we have $\exists y \varphi^+(t, x, y)$. It follows that $C \subseteq S$, so $\Omega - S$ is not stationary after all, a contradiction! ■

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Analytic sets with countable sections

by

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Abstract. This article contains a new proof of Lusin's theorem that an analytic set in the product of two Polish spaces, having countable (vertical) sections, is a countable union of analytic graphs.

1. Introduction. Suppose X, Y are Polish spaces. If $E \subseteq X \times Y$ and $x \in X$, we denote by E^x the set $\{y \in Y: (x, y) \in E\}$. A set $G \subseteq X \times Y$ is said to be a *graph* if G^x contains at most one point for each $x \in X$. We denote the family of Borel graphs in $X \times Y$ by $\mathcal{G}$. Let $\pi_1(\pi_2)$ be the projection of $X \times Y$ to the first (second) coordinate.

Lusin proved the following fundamental results on Borel and analytic sets with countable (vertical) sections in his celebrated monograph [2]:

(i) *If E is a Borel set in $X \times Y$ such that* $(\forall x \in X)$ *(E^x is countable), then $\pi_1(E)$ is Borel in X* ([2], p. 178).

(ii) *If E is a Borel set in $X \times Y$ such that* $(\forall x \in X)$ *(E^x is countable), then* $E \in \mathcal{G}_\sigma$ ([2], p. 244).

(iii) *If E is an analytic set in $X \times Y$ such that* $(\forall x \in X)$ *(E^x is countable), then there is a Borel set B in $X \times Y$ such that $E \subseteq B$ and* $(\forall x \in X)$ *(B^x is countable)* ([2], p. 247).

Finally, combining (ii) and (iii), Lusin obtained

(iv) *If E is an analytic set in $X \times Y$ such that* $(\forall x \in X)$ *(E^x is countable), then E is a countable union of analytic graphs* ([2], p. 252).

We shall prove the following:

THEOREM. *If A is analytic in $X \times Y$ such that* $(\forall x \in X)$ *(A^x is countable), then there is $H \in \mathcal{G}_\sigma$ such that $A \subseteq H$.*

A notable feature of our proof of the above theorem is that we do not use results (i)-(iv), so that these results fall out as easy consequences of our theorem. Our proof, though quite different from Lusin's proof of (iii), is based on ideas contained in Lusin's proof of (ii) and also on certain ideas in a recent article of Saint Raymond [3].