Weakly monadic Boolean extension functors *

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Abstract. Let A be a set of power $|A|<\lambda$, where λ is a regular cardinal. Having first defined what we mean by a varietal theory K and a varietal category $K^{\#}$ of λ-complete Boolean algebras, we define a Boolean extension functor $A[-]\colon K^{\#}\to \mathbf{Set}$ and study its structure theory T. We show that $A[-]$ is represented by the power set algebra 2^A and that the canonical comparison functor $E\colon K^{\#}\to T^{\#}$, where $T^{\#}$ is the category of all T-algebras with homomorphisms, is a category equivalence. It follows that all coalgebra-representable Boolean extension functors (which we define) are weakly monadic. Our analysis of Boolean extensions as representable functors enables us to cope easily with infinitary operations and provides an alternative to the topological or sheaf-theoretic analysis of Boolean extensions.

0. Introduction. The notion of the Boolean extension of a finitary universal algebra $\mathscr{A}$ by a Boolean algebra was introduced by A. L. Foster [2, 3] as a device for investigating structural similarities between the variety of Boolean algebras and other varieties such as the p-rings. One of Foster's principal results was that, when $\mathscr{A}$ is primal (i.e., finite and nontrivial, with every finitary function $A^n\to A$ being a composite of the fundamental operations in $\mathscr{A}$), the variety generated by $\mathscr{A}$ is the class of all isomorphic copies of Boolean extensions of $\mathscr{A}$.

In the present paper, we use the category-theoretic apparatus of algebraic theories (see [5, 8]) to investigate Boolean extensions as functors. Having first defined what we mean by a varietal theory K and a varietal category $K^{\#}$ of λ-complete Boolean algebras, we define a Boolean extension functor $A[-]\colon K^{\#}\to\mathbf{Set}$, where A is a set with $1<|A|<\lambda$, and study the structure theory T of $A[-]$.

We prove that the canonical functor $E\colon K^{\#}\to T^{\#}$, given by the structure-semantics adjointness for algebraic theories, is an equivalence of categories. Foster's theorem about primally-generated varieties, cited above, corresponds to the case in which K is the theory $\mathbf{BA}$ of Boolean algebras. The universal nature of the functor E provides a means of defining Boolean extension functors from $K^{\#}$ to certain varietal categories $V^{\#}$ which may involve infinitary operations. The fact that E is an equivalence of categories implies that such Boolean extension functors are weakly monadic.

It is assumed that the reader is familiar with basic category theory on the level of Mac Lane [6] and with the rudiments of category-theoretic universal algebra

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as expounded in Linton [5] or Wraith [8]. A very incomplete and condensed exposition of basic notions related to algebraic theories is included here to establish notation.

All functors explicitly dealt with in this paper are defined so that they are covariant. The composite of $\varphi\colon X\to Y$ with $\psi\colon Y\to Z$ is written as $\psi\varphi$, and $M(X, Y)$ denotes the family of all arrows with domain X and codomain Y in the category M. It should be noted that our category **Set** of sets is assumed to be one in which the axiom of choice holds.

1. Varietal theories of Boolean algebras. An algebraic theory is a coproduct-preserving functor of the form $t\colon \mathbf{Set}\to T$, where for every object Z of T there is a set X such that $Z = t(X)$. If all hom-sets in T are actually sets, i.e., objects in **Set**, then t is called a *varietal theory*. Since the functor t only serves as a labeling device, with T carrying most of the information that interests us, we informally refer to T as a theory and speak of T-algebras, and so on.

The structure of T is such that every object $t(X)$ is an Xth copower of $t(1)$ (where $1 = \{0\}$ is the ordinal "one"), while each arrow $g\colon t(Y)\to t(X)$ in T is induced by a Y-sequence $\langle g_y\colon y\in Y\rangle$ of arrows $g_y\colon t(1)\to t(X)$. A T-arrow of the form $f\colon t(1)\to t(X)$ is called an *X-ary T-operation.* If f factors as $t(h)g$, where $g\colon t(1)\to t(Y)$ is Y-ary and $|Y|<|X|$, while $h\colon Y\to X$ is a function, or if $f = t(h)$ for some function $h\colon 1\to X$, then f is a trivial X-ary operation. If T has nontrivial X-ary operations for arbitrarily large sets X, then the rank of T is defined to be ∞; otherwise, the rank of T is the least cardinal λ such that, for all sets X with $|X|\geqslant\lambda$, all X-ary T-operations are trivial. For any cardinal λ, the λ-truncation of T is the theory obtained from T by eliminating all nontrivial X-ary operations, for sets X with $|X|\geqslant\lambda$.

A T-algebra is a functor $\mathscr{B}\colon T^{\mathrm{op}}\to\mathbf{Set}$ such that the composite $\mathbf{Set}^{\mathrm{op}}\xrightarrow{t^{\mathrm{op}}} T^{\mathrm{op}}\xrightarrow{\mathscr{B}}\mathbf{Set}$ preserves products. We write B for the underlying set $\mathscr{B}(t(1))$ of $\mathscr{B}$, and B^X for $\mathscr{B}(t(X))$. For T-operations f, we write $f_{\mathscr{B}}$ or sometimes just f for $\mathscr{B}(f)$. Note that $\mathscr{B}$ is determined by its underlying set B together with the functions $f_{\mathscr{B}}$. If $Y\subseteq B$, then the smallest T-subalgebra $\mathscr{B}'$ of $\mathscr{B}$ with $Y\subseteq B'$ is called the *subalgebra T-generated by Y*. If $\mathscr{B}' = \mathscr{B}$, then $\mathscr{B}$ is said to be *T-generated by Y*.

The varietal category $T^{\#}$ is the full subcategory of all T-algebras in $\mathrm{Set}^{T^{\mathrm{op}}}$. Note that each arrow $\varphi\colon \mathscr{B}_0\to\mathscr{B}_1$ in $T^{\#}$ (called a *T-homomorphism*) is determined by its underlying function $\varphi_{t(1)}\colon B_0\to B_1$. We shall occasionally refer to operation-preserving functions as homomorphisms. The underlying set functor on $T^{\#}$ is $U_T\colon T^{\#}\to\mathbf{Set}$; its left adjoint, the free T-algebra functor $F_T\colon\mathbf{Set}\to T^{\#}$, is defined as $F_T = J_T t$, where $J_T\colon T\to T^{\#}$ is the Yoneda embedding. The category $T^{\#}$ is complete and cocomplete; in particular, for each set X and T-algebra $\mathscr{B}$, there is an Xth copower $X\cdot\mathscr{B}$ in $T^{\#}$.

A mapping of theories from t_0 to t_1 is a functor $m\colon T_0\to T_1$ such that $mt_0 = t_1$. The induced functor $m^{\#}\colon T_1^{\#}\to T_0^{\#}$ is called an *algebraic functor*, or *reduct functor*.

Our definition of the notion of a varietal theory of Boolean algebras relies upon

some well-known examples of such theories which we shall now describe briefly.

The theory **CABA** of complete atomic Boolean algebras (equivalently, completely distributive complete Boolean algebras — see Sikorski [7], p. 105) has rank ∞. Its X-ary operations, for all sets X, correspond to the functions $2^X\to 2$. The category $\mathbf{CABA}^{\#}$ is equivalent to the category of all complete atomic Boolean algebras, with complete (i.e., sup- and inf-preserving) Boolean homomorphisms. A typical **CABA**-algebra is a power 2^I of a two-element **CABA**-algebra 2, and will be identified with the complete Boolean algebra of all subsets of I (the power set algebra of I).

The $\aleph_0$-truncation of **CABA** is the theory **BA** of Boolean algebras, with $\mathbf{BA}^{\#}$ being equivalent to the category of all Boolean algebras with Boolean homomorphisms.

The varietal theory $\mathbf{BA}_\lambda$ of λ-complete Boolean algebras is not quite so concretely describable. Let λ be an infinite regular cardinal. We say that a Boolean algebra $\mathscr{B}$ is λ-complete if every family of fewer than λ elements of $\mathscr{B}$ has a sup and an inf in $\mathscr{B}$ relative to the customary partial order ($x\leqslant y$ if and only if $x\wedge y = x$) in $\mathscr{B}$. A λ-complete Boolean homomorphism is a Boolean homomorphism which preserves the sups and infs of families of fewer than λ elements. We deviate from Sikorski's usage in [7] in order to avoid having to refer continually to algebras which are "$\varkappa$-complete for all $\varkappa<\lambda$". It follows from the material in Sikorski [7], p. 131 ff., that the category of all λ-complete Boolean algebras with λ-complete Boolean homomorphisms has a free algebra functor which — with appropriately restricted codomain — may be construed as the theory $ba_\lambda\colon\mathbf{Set}\to\mathbf{BA}_\lambda$ of λ-complete Boolean algebras, whose $\aleph_0$-truncation is **BA**. It is a theorem that $\mathbf{BA}_\lambda^{\#}$ is equivalent to the category of λ-complete Boolean algebras. One should note that all the nontrivial $\mathbf{BA}_\lambda$-operations are built up by composition from the complement operation $\neg$, the infinitary joins $\bigvee_{x\in X}$ (where $|X|<\lambda$), and the trivial operations. Also note that there are obvious mappings of theories $m\colon\mathbf{BA}\to\mathbf{BA}_\lambda$ and $n\colon\mathbf{BA}_\lambda\to\mathbf{CABA}$ such that $nm\colon\mathbf{BA}\to\mathbf{CABA}$ is the customary mapping, which says that joins are unions, and so on.

1.1. Definition. A varietal theory of λ-complete Boolean algebras is a varietal theory $k\colon\mathbf{Set}\to K$ with a mapping of theories $m\colon\mathbf{BA}_\lambda\to K$ which is full on the λ-truncation of $\mathbf{BA}_\lambda$ and a mapping of theories $n\colon K\to\mathbf{CABA}$ such that $nm\colon\mathbf{BA}_\lambda\to\mathbf{CABA}$ is the customary mapping.

The definition above is intended to encompass varietal theories which describe λ-complete Boolean algebras enjoying some extra completeness or distributivity conditions. Intuitively, a K-algebra will be a λ-complete Boolean algebra whose extra operations, if any, are X-ary for $|X|\geqslant\lambda$ and sufficiently "Boolean" that every power set algebra is a K-algebra.

Throughout the remainder of this paper, $k\colon\mathbf{Set}\to K$ is a varietal theory of λ-complete Boolean algebras, with mappings m, n as specified in (1.1). The constant elements of any K-algebra are 0 and 1, and 2 is a two-element K-algebra. The free K-algebra $F_K(1)$ is written as 2^2 and has elements $0, 1, x, \neg x$, with x being our favourite free generator. The free K-algebra $F_K(X)$ is written as $X\cdot 2^2$, with coproduct injections $\tau_x\colon 2^2 \to X\cdot 2^2$, $x \in X$. (It will be clear from the context whether x is an element of 2^2 or of X). Given a set A, the K-algebra 2^A is referred to as a power set algebra, and its elements are identified with the corresponding subsets of A. We write 1 for the subset A and 0 for the empty subset. Note that, for $|A|<\lambda$, 2^A is K-generated by its atoms.

A regular epimorphism in a category M is an epimorphism φ which is a coequalizer of some pair of arrows in M. The regular epimorphisms in a varietal category $T^{\#}$ are precisely the surjective T-homomorphisms (i.e., the T-homomorphisms whose underlying functions are surjective). An object X of M is regular-projective in M if and only if, for every arrow $\gamma\colon X \to Y$ and regular epimorphism $\varphi\colon Z \to Y$ in M, there is an arrow $\gamma'\colon X \to Z$ such that $\gamma = \varphi\gamma'$. The free algebras in any varietal category are regular-projective in that category.

1.2. Proposition. *Let $\mathcal{B}$ be a K-algebra which is K-generated by a set $G \subseteq B$ with $|G|<\lambda$; let $\mathcal{B}_0$ be any K-algebra, and let $\gamma\colon m^{\#}(\mathcal{B}) \to m^{\#}(\mathcal{B}_0)$ be a BA_λ-homomorphism. Then γ is the BA_λ-reduct $m^{\#}(\gamma')$ of a T-homomorphism $\gamma'\colon \mathcal{B} \to \mathcal{B}_0$.*

Proof. Let $\psi_{\bar{g}}\colon G\cdot 2^2 \to \mathcal{B}$ be the K-homomorphism which picks out the G-sequence $\bar{g}$ of generators of $\mathcal{B}$, i.e., the unique K-homomorphism ψ such that $\psi\tau_g\colon 2^2 \to \mathcal{B}$ sends x to g, all $g \in G$. Since G K-generates $\mathcal{B}$, $\psi_{\bar{g}}$ is surjective. Let $\bar{z} \in B^X$, and let $\psi_{\bar{z}}\colon X\cdot 2^2 \to \mathcal{B}$ be the K-homomorphism which picks out $\bar{z}$. Because $X\cdot 2^2$ is regular-projective in $K^{\#}$, there is a K-homomorphism $\psi\colon X\cdot 2^2 \to G\cdot 2^2$ such that $\psi_{\bar{g}}\psi = \psi_{\bar{z}}$. Furthermore, there is some $h\colon k(X) \to k(G)$ in K, which may be throught of as an X-sequence of G-ary operations, such that $J_K(h) = \psi$. Now let f be an X-ary K-operation; we must show that γ preserves f. Since $|G|<\lambda$, the G-ary operations $h_{\mathcal{B}}\colon B^G \to B^X$ and $f_{\mathcal{B}}h_{\mathcal{B}}\colon B^G \to B$ are BA_λ-operations on $m^{\#}(\mathcal{B})$ and so are preserved by γ. Suppressing subscripts on γ, we have

$$\gamma\big(f_{\mathcal{B}}(\bar{z})\big) = \gamma\big(f_{\mathcal{B}}(h_{\mathcal{B}}(\bar{g}))\big) = f_{\mathcal{B}_0}h_{\mathcal{B}_0}(\gamma(\bar{g})) = f_{\mathcal{B}_0}\big(\gamma(h_{\mathcal{B}}(\bar{g}))\big) = f_{\mathcal{B}_0}\big(\gamma(\bar{z})\big),$$

so γ preserves the K-operation f. ∎

1.3. Corollary. *Let $\mathcal{B}$, $\mathcal{B}_0$, G be as in (1.2). Then any function $h\colon B \to B_0$ which preserves $\neg$ and the G-ary join $\bigvee_{g\in G}$ on $\mathcal{B}$ is the underlying function of a K-homomorphism.* ∎

1.4. Definition. Let A be a set with $1<|A|<\lambda$, and let $\mathcal{B}$ be any K-algebra. An A-indexed partition of unity in $\mathcal{B}$ is a function $p\colon A \to B$ such that $p(a)\wedge p(b) = 0$ for all distinct $a, b \in A$, while $\bigvee_{a\in A} p(a) = 1$.

1.5. Proposition. *Let $1<|A|<\lambda$, and let $j\colon A \to 2^A$ be the function which sends each $a \in A$ to $\{a\} \in 2^A$. Then for each A-indexed partition of unity p in $\mathcal{B}$ there is a unique K-homomorphism $\varphi\colon 2^A \to \mathcal{B}$ such that $U_K(\varphi)j = p$.*

Proof. For each $Z \in 2^A$, define $g(Z) = \bigvee_{a\in Z} p(a)$. Obviously, g is a function from 2^A to B, and $gj = p$. Since 2^A is generated by its atoms, of which there are fewer than λ, by (1.3) one need only verify that g preserves the A-ary join $\bigvee_{a\in A}$ and the complement operation $\neg$ in order to show that g is $U_K(\varphi)$, for some K-homomorphism $\varphi\colon 2^A \to \mathcal{B}$. The verification is easy, so we omit it. Since the atoms of 2^A K-generate 2^A in this case, any K-homomorphism $\gamma\colon 2^A \to \mathcal{B}$ is completely determined by its values on the atoms of 2^A, so there can be at most one $\varphi\colon 2^A \to \mathcal{B}$ such that $U_K(\varphi)j = p$. ∎

A generator in a category M is an object X such that, for any parallel pair $\alpha, \beta\colon Y \to Z$ of distinct arrows in M, there is $\varphi\colon X \to Y$ such that $\alpha\varphi \neq \beta\varphi$. Free T-algebras $F_T(X)$ for $|X|\geqslant 1$ are generators in any varietal category $T^{\#}$.

1.6. Proposition. *If $1<|A|<\lambda$, then 2^A and all its copowers $X\cdot 2^A$ (X nonempty) in $K^{\#}$ are regular-projective generators in $K^{\#}$.*

Proof. Since any coproduct of a nonempty family of regular-projective objects (or generators) is regular-projective (resp. a generator), it is sufficient to show that 2^A is a regular-projective generator in $K^{\#}$. Let $\varphi\colon \mathcal{B} \to \mathcal{B}_0$ be a surjective K-homomorphism, and let $\gamma\colon 2^A \to \mathcal{B}_0$ be any K-homomorphism. To show that 2^A is regular-projective, we must find a K-homomorphism $\gamma'\colon 2^A \to \mathcal{B}$ such that $\varphi\gamma' = \gamma$. By (1.5) it is enough to find a partition $p\colon A \to \mathcal{B}$ such that $\varphi(p(a)) = \gamma(\{a\})$ for each $a \in A$. Let $|A| = \varkappa$, and let $\langle a_i\colon i<\varkappa\rangle$ be a well-ordering of A. Take $p(a_1)$ to be any element b_1 of $\mathcal{B}$ such that $\varphi(b_1) = \gamma(\{a_1\})$. If $0<i<\varkappa$, and $p(a_j)$ is defined for each j, $0<j<i$, such that $\varphi(p(a_j)) = \gamma(\{a_j\})$, let $p(a_i) = b_i - \bigvee_{0<j<i} p(a_j)$, where b_i is an element of $\mathcal{B}$ chosen so that $\varphi(b_i) = \gamma(\{a_i\})$. This suffices to define $p(a_i)$ for all nonzero $i<\varkappa$. Let $p(a_0) = \neg\bigvee_{0<i<\varkappa} p(a_i)$.

To show that 2^A is a generator in $K^{\#}$, let $\alpha, \beta\colon \mathcal{B} \to \mathcal{B}_0$ be distinct parallel K-homomorphisms, and let b be an element of $\mathcal{B}$ such that $\alpha(b) \neq \beta(b)$. Choose distinct $a_0, a_1 \in A$ and define $\varphi(\{a_0\}) = b$, $\varphi(\{a_1\}) = \neg b$, and $\varphi(\{a\}) = 0$ for all $a \in A-\{a_0, a_1\}$. By (1.5) this suffices to define a K-homomorphism $\varphi\colon 2^A \to \mathcal{B}$; obviously, $\alpha\varphi \neq \beta\varphi$. ∎

When $|X|<\lambda$ and $1<|A|<\lambda$, we can say a few useful things about the K-algebra $X\cdot 2^A$. For each $\bar{a} \in A^X$, let $s_{\bar{a}}$ be the element $\bigwedge_{x\in X} \sigma_x(\{a_x\})$ of $X\cdot 2^A$, where $\sigma_x\colon 2^A \to X\cdot 2^A$ is the xth coproduct injection.

1.7. Proposition. *Let $|X|<\lambda$ and $1<|A|<\lambda$. Then an element of the K-algebra $X\cdot 2^A$ is an atom if and only if it is $s_{\bar{a}}$ for some $\bar{a} \in A^X$.*

Proof. For each $\bar{a} \in A^X$, every K-homomorphism γ with domain $X\cdot 2^A$ such that $\gamma(s_{\bar{a}}) = 1$ factors through the unique K-homomorphism $\varphi_{\bar{a}}\colon X\cdot 2^A \to 2$ such that $\varphi_{\bar{a}}\sigma_x(\{a_x\}) = 1$, all $x \in X$. Furthermore, any K-homomorphism $\varphi\colon X\cdot 2^A \to 2$ is $\varphi_{\bar{a}}$ for some $\bar{a} \in A^X$; indeed, $\varphi\sigma_x\colon 2^A \to 2$ determines some unique $a_x \in A$ such that $\varphi\sigma_x(\{a_x\}) = 1$, so $\varphi = \varphi_{\bar{a}}$, where $\bar{a} = \langle a_x\colon x \in X\rangle$. If $K = \mathrm{BA}_\lambda$, the above is sufficient to characterize the atoms of $X\cdot 2^A$ as the elements $s_{\bar{a}}$, $\bar{a}\in A^X$. Otherwise, we note that $X\cdot 2^A$ is $\boldsymbol{K}$-generated by the set $\{\sigma_x(\{a\})\colon a\in A, x\in X\}$, which has fewer than λ elements. Appealing to (1.2), we see that the observations above concerning $\varphi_{\bar{a}}$ are sufficient to show that the elements $s_{\bar{a}}$, $\bar{a}\in A^X$, are the atoms of the $\mathbf{BA}_\lambda$-algebra $m^{\#}(X\cdot 2^A)$ and therefore also of the $\boldsymbol{K}$-algebra $X\cdot 2^A$. ∎

Let X, A be any sets, and for each $x\in X$ let $\varrho_x\colon A^X\to A$ be the x-projection, i.e., the function which sends each $\bar{a}\in A^X$ to its x-coordinate a_x. Then a $\boldsymbol{K}$-homomorphism (in fact, a **CABA**-homomorphism) $\varrho_x^*\colon 2^A\to 2^{A^X}$ is defined by $\varrho_x^*(Z)=\{\bar{a}\in A^X\colon a_x\in Z\}$ for each $Z\in 2^A$. Now let $\varrho_X\colon X\cdot 2^A\to 2^{A^X}$ be the unique $\boldsymbol{K}$-homomorphism φ such that $\varphi\sigma_x=\varrho_x^*$, all $x\in X$.

1.8. Proposition. *When $|A^X|<\lambda$, $\varrho_X\colon X\cdot 2^A\to 2^{A^X}$ is a retraction. If X is finite, ϱ_X is an isomorphism.*

Proof. Each atom $s_{\bar{a}}$ of $X\cdot 2^A$ is sent by ϱ_X to its counterpart $\{\bar{a}\}$ in 2^{A^X}; thus, the image of ϱ_X contains all the atoms of 2^{A^X}, which $\boldsymbol{K}$-generate 2^{A^X} since there are less than λ of them, so ϱ_X is surjective. Using the fact that 2^{A^X} is regular-projective when $|A^X|<\lambda$, we see that there is a $\boldsymbol{K}$-homomorphism $\varepsilon_X\colon 2^{A^X}\to X\cdot 2^A$ such that $\varrho_X\varepsilon_X$ is the identity homomorphism on 2^{A^X}, so ϱ_X is a retraction.

Now we prove by induction on the cardinality of X that $X\cdot 2^A$ is atomic when X is finite. We need only show that $\bigvee_{\bar{a}\in A^X} s_{\bar{a}}=1$, which certainly holds for $|X|\leqslant 1$. Now suppose that $Y\cdot 2^A$ is atomic, and $X=\{x\}\cup Y$, where $x\notin Y$. The atoms in $Y\cdot 2^A$ add up to 1, so we have $\bigvee_{\bar{b}\in A^Y}\bigwedge_{y\in Y}\sigma_y(\{b_y\})=1$ in $X\cdot 2^A\cong 2^A u(Y\cdot 2^A)$. But then for any $c\in A$ we have

$$\sigma_x(\{c\})=\bigvee_{\bar{b}\in A^Y}\Big(\sigma_x(\{c\})\wedge\bigwedge_{y\in Y}\sigma_y(\{b_y\})\Big)=\bigvee_{a_x=c}s_{\bar{a}},$$

So $1=\bigvee_{c\in A}\sigma_x(\{c\})=\bigvee_{\bar{c}\in A^X}s_{\bar{a}}$. Assigning $s_{\bar{a}}$ to each $\bar{a}\in A^X$ defines an A^X-indexed partition of unity in $X\cdot 2^A$ which, by (1.5), determines a $\boldsymbol{K}$-homomorphism $\gamma\colon 2^{A^X}\to X\cdot 2^A$ such that γ is a two-sided inverse for ϱ_X. ∎

It is worth noting that, if X is infinite, ϱ_X is not generally an isomorphism, since by (31.3) of Sikorski [7] it follows that, for $\lambda>2^{\aleph_0}$ and X infinite, the free $\mathbf{BA}_\lambda$-algebra $X\cdot 2^2$ is not isomorphic to 2^{2^X}.

2. Set-valued Boolean extension functors. Throughout the remainder of this paper, A is a set with $1<|A|<\lambda$.

2.1. Definition. The Boolean extension functor $A[-]\colon \boldsymbol{K}^{\#}\to\mathbf{Set}$ is defined as follows.

(i) For each $\boldsymbol{K}$-algebra $\mathscr{B}$, $A[\mathscr{B}]$ is the set of all A-indexed partitions of unity in $\mathscr{B}$.

(ii) For each $\boldsymbol{K}$-homomorphism $\varphi\colon\mathscr{B}\to\mathscr{B}_0$, $A[\varphi]\colon A[\mathscr{B}]\to A[\mathscr{B}_0]$ is the function which sends each $p\in A[\mathscr{B}]$ to $U_K(\varphi)p\in A[\mathscr{B}_0]$.

2.2. Proposition. *$A[-]$ is faithful, has a left adjoint, and is naturally isomorphic to $\boldsymbol{K}^{\#}(2^A,-)$.*

Proof. It is immediate from (1.5) that the function $1\to A[2^A]$ which picks out the partition j defined in (1.5) is a universal arrow from 1 to $A[-]$ in the sense of Mac Lane [6]; from this it follows that $A[-]$ is naturally isomorphic to $\boldsymbol{K}^{\#}(2^A,-)$. Faithfulness of $A[-]$ then follows from the fact that 2^A is a generator in $\boldsymbol{K}^{\#}$ (see (1.6)). It is easily verified that the obvious functor $L\colon\mathbf{Set}\to\boldsymbol{K}^{\#}$ which takes X to $X\cdot 2^A$ is left adjoint to $A[-]$. ∎

For any **Set**-valued functor U and any set X, U^X is the functor obtained by taking the Xth Cartesian power of the values of U.

2.3. Corollary. *For any set X, $A[-]^X$ is naturally isomorphic to $\boldsymbol{K}^{\#}(X\cdot 2^A,-)$.* ∎

From here on we identify $A[-]$ with $\boldsymbol{K}^{\#}(2^A,-)$ and $A[-]^X$ with $\boldsymbol{K}^{\#}(X\cdot 2^A,-)$.

2.4. Proposition. *The functor $A[-]$ preserves and reflects surjections, i.e., for any $\boldsymbol{K}$-homomorphism $\varphi\colon\mathscr{B}\to\mathscr{B}_0$, $A[\varphi]$ is a surjective function if any only if $U_K(\varphi)$ is.*

Proof. It follows from the fact that 2^A is regular-projective in $\boldsymbol{K}^{\#}$ (see (1.6)) that $A[-]$ preserves surjections. If φ is not surjective, let b be an element of $\mathscr{B}_0$ which is not in the image of φ. Let a_0, a_1 be distinct elements of A, and let $\gamma\colon 2^A\to\mathscr{B}_0$ be the unique $\boldsymbol{K}$-homomorphism (by 1.5) which sends $\{a_0\}$ to b and $\{a_1\}$ to $\neg b$. Then γ is not $A[\varphi](\gamma')$ for any $\boldsymbol{K}$-homomorphism $\gamma'\colon 2^A\to\mathscr{B}$. ∎

3. The structure theory of the functor $A[-]$. Any covariant functor of the form $U\colon \boldsymbol{M}\to\mathbf{Set}$, where $\boldsymbol{M}$ is an arbitrary category, determines an algebraic theory (called the *structure theory of U*) whose X-ary operations correspond to the natural transformations $U^X\to U$. For the remainder of this paper, $t\colon\mathbf{Set}\to\boldsymbol{T}$ is the structure theory of $A[-]\colon\boldsymbol{K}^{\#}\to\mathbf{Set}$. The set A itself is associated with a varietal theory $t_A\colon\mathbf{Set}\to\boldsymbol{T}_A$, where $\boldsymbol{T}_A$ is the opposite of the full subcategory of the Cartesian powers of A in Set. In this section we investigate some of the ways in which the $\boldsymbol{T}$-operations are related to corresponding $\boldsymbol{T}_A$-operations.

3.1. Proposition. *$\boldsymbol{T}$ is a varietal theory and is equivalent to the full subcategory of copowers of 2^A in $\boldsymbol{K}^{\#}$.*

Proof. The Yoneda lemma and (2.3) give us a natural bijective correspondence between $\boldsymbol{T}(t(X),t(Y))$ and $\boldsymbol{K}^{\#}(X\cdot 2^A, Y\cdot 2^A)$ for all sets X and Y, so $\boldsymbol{T}$ is varietal. By setting $G(t(X))=X\cdot 2^A$ and defining $G(f)$ to be the $\boldsymbol{K}$-homomorphism $\varphi_f\colon X\cdot 2^A\to Y\cdot 2^A$ (given by the Yoneda lemma) corresponding to $f\colon t(X)\to t(Y)$ in $\boldsymbol{T}$, we obtain a full embedding $G\colon\boldsymbol{T}\to\boldsymbol{K}^{\#}$ which establishes the equivalence claimed in (3.1). ∎

The functor $A[-]$ is called a Boolean *extension* functor because each Boolean extension $A[\mathscr{B}]$ contains a copy of the set A, namely $A[2]$. Applying $A[-]$ to the embedding $2\to\mathscr{B}$, we have an embedding $A[2]\to A[\mathscr{B}]$. An element $a\in A$ is represented in $A[\mathscr{B}]$ by the unique $\boldsymbol{K}$-homomorphism $\varphi_a\colon 2^A\to\mathscr{B}$ which sends $\{a\}$ to 1. Evidently, each natural transformation $f\colon A[-]^X\to A[-]$ induces a function

$f_A\colon A^X \to A$ corresponding to the 2-component $f_2\colon A[2]^X \to A[2]$ of f. We shall characterize those functions $h\colon A^X \to A$ such that $h = f_A$ for some T-operation f and derive a formula which expresses certain T-operations f (such as finitary ones) in terms of f_A.

First we show how f_A works. Let an X-ary T-operation f be given, and let $\varphi_f\colon 2^A \to X\cdot 2^A$ be the K-homomorphism which represents f. For any $\bar{a} \in A^X$, $\varphi_{\bar{a}}\colon X\cdot 2^A \to 2$ is the unique K-homomorphism which sends $s_{\bar{a}}$ to 1. Since $|A| < \lambda$, every K-homomorphism $\varphi\colon 2^A \to 2$ sends exactly one atom to 1, so $f_A(\bar{a})$ is the unique $b \in A$ such that $\varphi_{\bar{a}}\varphi_f$ sends $\{b\}$ to 1.

3.2. Proposition. *Let $R\colon T \to T_A$ be the covariant functor which takes $t(X)$ to A^X and f to f_A. Then R is a mapping of theories.* ∎

Given any X and any X-ary function $g\colon A^X \to A$, a K-homomorphism $g^*\colon 2^A \to 2^{A^X}$ is defined by $g^*(Z) = \{\bar{a} \in A^X\colon g(\bar{a}) \in Z\}$ (g^* is the K-reduct of a CABA-homomorphism). Given any K-homomorphism $\varphi\colon 2^A \to 2^{A^X}$, there is a function $g\colon A^X \to A$ such that $\varphi = g^*$; in fact, for each $\bar{a} \in A^X$, $g(\bar{a})$ is the unique $b \in A$ such that $\psi_{\bar{a}}\varphi$ sends $\{b\}$ to 1, where $\psi_{\bar{a}}\colon 2^{A^X} \to 2$ is the $\bar{a}$-projection. It follows that the correspondence between K-homomorphisms $2^A \to 2^{A^X}$ and functions $A^X \to A$ is bijective.

3.3. Proposition. *For any X-ary T-operation f, $f_A^* = \varrho_X \varphi_f$, where $\varphi_f\colon 2^A \to X\cdot 2^A$ represents f in $K^\#$, and $\varrho_X\colon X\cdot 2^A \to 2^{A^X}$ is as defined immediately before* (1.8).

Proof. It is sufficient to prove that, for each $\bar{a} \in A^X$, $\psi_{\bar{a}} f_A^* = \psi_{\bar{a}}\varrho_X\varphi_f$, where $\psi_{\bar{a}}\colon 2^{A^X} \to 2$ is the $\bar{a}$-projection, which is characterized by the fact that it sends $\{\bar{a}\}$ to 1. It is easy to check that $\psi_{\bar{a}}\varrho_X$ sends the atom $s_{\bar{a}}$ of $X\cdot 2^A$ to 1, so $\psi_{\bar{a}}\varrho_X = \varphi_{\bar{a}}$. Then $\psi_{\bar{a}}\varrho_X\varphi_f = \varphi_{\bar{a}}\varphi_f$, so to prove (3.3) we must show that $\psi_{\bar{a}} f_A^*$ sends $\{f_A(\bar{a})\}$ to 1. But

$$\psi_{\bar{a}} f_A^*(\{f_A(\bar{a})\}) = \psi_{\bar{a}}(\{\bar{b} \in A^X\colon f_A(\bar{b}) = f_A(\bar{a})\}) \geqslant \psi_{\bar{a}}(\{\bar{a}\}) = 1. \;∎$$

3.4. Corollary. *The $\aleph_0$-truncations of T and T_A are isomorphic.*

Proof. Apply (1.8) and the remarks immediately preceeding (3.3). ∎

3.5. Proposition. *A function $g\colon A^X \to A$ is f_A for some X-ary T-operation f if and only if, for each $a \in A$, $g^*(\{a\})$ belongs to the subalgebra of 2^{A^X} which is K-generated by the family of all subsets of A^X of the form $\{\bar{a} \in A^X\colon a_x = b\}$, where $x \in X$ and $b \in A$.*

Proof. The subalgebra of 2^{A^X} specified in (3.5) is the image of ϱ_X, so necessity of the condition is clear by (3.3). Suppose $g\colon A^X \to A$ satisfies the condition, i.e., $g^*\colon 2^A \to 2^{A^X}$ factors through the image of ϱ_X. Applying (1.6), we lift g^* through the surjective component of ϱ_X to obtain $\varphi\colon 2^A \to X\cdot 2^A$ such that $g^* = \varrho_X\varphi$. Then the T-operation f which is represented in $K^\#$ by φ is such that $g = f_A$. ∎

3.6. Proposition. *Let f be an X-ary T-operation, where $|X| < \lambda$. If $\mathscr{B}$ is a K-algebra whose $\mathbf{BA}_\varkappa$-reduct has a $\varkappa$-complete embedding into a power set algebra,*

for some $\varkappa$ such that $|X| < \varkappa \leqslant \lambda$, then the $\mathscr{B}$-component $f_{\mathscr{B}}\colon A[\mathscr{B}]^X \to A[\mathscr{B}]$ of f is given by

$$[f_{\mathscr{B}}(\bar{p})](\{c\}) = \sup_{f_A(\bar{a}) = c} \Big(\bigwedge_{x \in X} p_x(\{a_x\})\Big).$$

for all $\bar{p} \in A[\mathscr{B}]^X$ and $c \in A$, where $p_x = \bar{p}\sigma_x$.

Proof. Without loss of generality, we assume that the $\mathbf{BA}_\varkappa$-reduct of $\mathscr{B}$ is a $\mathbf{BA}_\varkappa$-subalgebra of 2^I for some set I, so we identify elements of $\mathscr{B}$ with subsets of I. For each $x \in X$ and $i \in I$, there is exactly one $a_{x,i} \in A$ such that $i \in p_x(\{a_{x,i}\})$. Writing $\bar{a}_i$ for $\langle a_{x,i}\colon x \in X\rangle$, we have

$$i \in \bigcap_{x \in X} p_x(\{a_{x,i}\}) = \bigcap_{x \in X} \bar{p}\sigma_x(\{a_{x,i}\}) = \bar{p}(s_{\bar{a}_i}), \quad \text{all } i \in I.$$

It follows that $\sup_{\bar{a} \in A^X} (\bar{p}(s_{\bar{a}})) = 1$ in $\mathscr{B}$. Now we have

$$[f_{\mathscr{B}}(\bar{p})](\{c\}) = [\bar{p}\varphi_f](\{c\}) = \sup_{\bar{a} \in A^X} \big(\bar{p}(\varphi_f(\{c\}) \wedge s_{\bar{a}})\big).$$

Recall that $\varphi_f(\{c\}) \wedge s_{\bar{a}} \neq 0$ if and only if $\varphi_f(\{c\}) \geqslant s_{\bar{a}}$. But the latter is equivalent to $[\varphi_{\bar{a}}\varphi_f](\{c\}) = 1$, i.e., $f_A(\bar{a}) = c$. Hence, we have

$$[f_{\mathscr{B}}(\bar{p})](\{c\}) = \sup_{f_A(\bar{a}) = c} (\bar{p}(s_{\bar{a}})) = \sup_{f_A(\bar{a}) = c} \Big(\bigwedge_{x \in X} p_x(\{a_x\})\Big). \;∎$$

3.7. Corollary. *Every finitary T-operation f is determined by the corresponding function f_A according to the formula in* (3.6). ∎

3.8. Corollary. *If every K-algebra admits a $\varkappa$-complete embedding into a power set algebra, then the $\varkappa$-truncations of T and T_A are isomorphic.* ∎

The formula given in (3.6) is the one A. L. Foster used in [2] to extend the finitary operations on a universal algebra $\mathscr{A}$ to its Boolean extensions $\mathscr{A}[\mathscr{B}]$. Foster's formula (for finitary operations) is the subject of Theorem 2, p. 148 of Grätzer [4]; Section 3 of the present paper may be understood as an enlargement of that theorem.

4. Equivalence of $K^\#$ and $T^\#$. The structure-semantics adjointness for varietal theories, as discussed in Linton [5] or Wraith [8], provides a canonical functor $E\colon K^\# \to T^\#$ such that $U_T E = A[-]$. In this section we prove that E is an equivalence functor, generalizing Foster's result [3] that every algebra in the variety generated by a primal algebra $\mathscr{A}$ is isomorphic to a Boolean extension of $\mathscr{A}$.

For any K-algebra $\mathscr{B}$, $E(\mathscr{B})$ is the T-algebra $\mathscr{C}$ such that $\mathscr{C}(t(X)) = A[\mathscr{B}]^X$ for all sets X, while $\mathscr{C}(f) = f_{\mathscr{B}}$ for all T-operations f. For any K-homomorphism φ, $E(\varphi)$ is the T-homomorphism whose underlying function is $A[\varphi]$.

Recall that $L\colon \mathbf{Set} \to K^\#$ is the left adjoint of $A[-]$.

4.1. Proposition. *E is faithful and preserves all copowers of the K-algebra 2^2. Furthermore, EL is naturally isomorphic to F_T.*

Proof. Faithfulness of E is an immediate consequence of the faithfulness of $A[-$. That $EL \cong F_T$ follows from Lemma 2 of Linton [5], and may be proved directly by verifying that EL is left adjoint to U_T.

Now let X be a set, and let $X\cdot 2^2$ be an Xth copower of 2^2 in $\boldsymbol{K}^{\#}$, with coproduct injections τ_x: $2^2 \to X\cdot 2^2$, $x \in X$. We wish to show that $E(X\cdot 2^2)$, with the maps $E(\tau_x)$, $x \in X$, is an Xth copower in $\boldsymbol{T}^{\#}$ of $E(2^2)$. Recall that "taking the Xth copower" may be construed as a functor $X\cdot(-)$: $\boldsymbol{K}^{\#} \to \boldsymbol{K}^{\#}$, with the coproduct injections being induced by natural transformations $\mathrm{Id}_{\boldsymbol{K}^{\#}} \to X\cdot(-)$. Also note that $EL \cong F_T$ implies that E preserves all copowers of the $\boldsymbol{K}$-algebra 2^A. The coproduct injections for $X\cdot 2^A$ are σ_x: $2^A \to X\cdot 2^A$, $x \in X$.

Let a_0, a_1 be distinct elements of A, and let β: $2^A \to 2^2$ be the $\boldsymbol{K}$-homomorphism which sends $\{a_1\}$ to x and $\{a_0\}$ to $\neg x$. Define α: $2^2 \to 2^A$ to be the unique $\boldsymbol{K}$-homomorphism which sends x to $\{a_1\}$. Then $\beta\alpha$ is the identity homomorphism on 2^2, so β is a retraction and α is a coretraction.

Now let $\mathscr{B}$ be any $\boldsymbol{T}$-algebra and $\langle \varphi_x\colon x \in X\rangle$ a sequence of $\boldsymbol{T}$-homomorphisms of the form φ_x: $E(2^2) \to \mathscr{B}$. We claim that a unique $\boldsymbol{T}$-homomorphism φ: $E(X\cdot 2^2) \to \mathscr{B}$ such that $\varphi E(\tau_x) = \varphi_x$, all $x \in X$, is given by $\varphi = \psi E(X\cdot\alpha)$, where ψ: $E(X\cdot 2^A) \to \mathscr{B}$ is induced by the $\boldsymbol{T}$-homomorphisms $\varphi_x E(\beta)$: $E(2^A) \to \mathscr{B}$, $x \in X$.

To prove the claim, compute

$$\begin{aligned}[\psi E(X\cdot\alpha)]E(\tau_x) &= \psi E([X\cdot\alpha]\tau_x) = \psi E(\sigma_x\alpha) = [\psi E(\sigma_x)]E(\alpha)\\ &= [\varphi_x E(\beta)]E(\alpha) = \varphi_x\end{aligned}$$

as required. Now suppose φ': $E(X\cdot 2^2) \to \mathscr{B}$ is some $\boldsymbol{T}$-homomorphism such that $\varphi' E(\tau_x) = \varphi_x$, all $x \in X$. Then

$$[\varphi' E(X\cdot\beta)]E(\sigma_x) = \varphi' E([X\cdot\beta]\sigma_x) = \varphi' E(\tau_x\beta) = [\varphi' E(\tau_x)]E(\beta) = \varphi_x E(\beta),$$

all $x \in X$, so by the uniqueness of ψ we have $\varphi' E(X\cdot\beta) = \psi$. Then $\varphi' = \varphi' E(X\cdot\beta)E(X\cdot\alpha) = \psi E(X\cdot\alpha)$. ■

Let v: **Set** $\to \boldsymbol{V}$ and w: **Set** $\to \boldsymbol{W}$ be varietal theories. A $\boldsymbol{V}$-coalgebra in $\boldsymbol{W}^{\#}$ (or a $(\boldsymbol{V}, \boldsymbol{W})$-bimodel, in Wraith [8]) is a functor H: $\boldsymbol{V} \to \boldsymbol{W}^{\#}$ such that Hv: **Set** $\to \boldsymbol{W}^{\#}$ preserves coproducts. The $\boldsymbol{W}$-algebra $H(v(1))$ is called the *underlying $\boldsymbol{W}$-algebra of H*, and the $\boldsymbol{W}$-homomorphism of the form $H(f)$: $H(v(1)) \to H(v(X))$, where f is an X-ary $\boldsymbol{V}$-operation, are called *X-ary co-operations of H*. Associated with the coalgebra H is a so-called coalgebra-representable functor H^*: $\boldsymbol{W}^{\#} \to \boldsymbol{V}^{\#}$ (written as $\mathrm{Hom}_{\boldsymbol{W}}(H, -)$ in Wraith's notation) defined in the following way. For each $\boldsymbol{W}$-algebra $\mathscr{B}$, $H^*(\mathscr{B})$ is the composite of H^{op}: $\boldsymbol{V}^{\mathrm{op}} \to \boldsymbol{W}^{\#\mathrm{op}}$ and $\boldsymbol{W}^{\#}(-, \mathscr{B})$: $\boldsymbol{W}^{\#\mathrm{op}} \to$ **Set**; for each $\boldsymbol{W}$-homomorphism φ, $H^*(\varphi)$ is ϕH, where ϕ is the natural transformation $\boldsymbol{W}^{\#}(-, \varphi)$. In practical terms, $H^*(\mathscr{B})$ is the $\boldsymbol{V}$-algebra whose underlying set is $\boldsymbol{W}^{\#}(H(v(1)).\mathscr{B})$, with each X-ary $\boldsymbol{V}$-operation f operating on $\bar{p} \in H^*(\mathscr{B})^X = \boldsymbol{W}^{\#}(H(v(X)), \mathscr{B})$ by $f(\bar{p}) = \bar{p}H(f)$. If φ: $\mathscr{B} \to \mathscr{B}_0$ is a $\boldsymbol{W}$-homomorphism, then $[H^*(\varphi)](p) = \varphi p$ for each element p: $H(v(1)) \to \mathscr{B}$ of $H^*(\mathscr{B})$.

As Wraith indicates in [8], a $\boldsymbol{V}$-coalgebra H in $\boldsymbol{W}^{\#}$ may be regarded as a generalized mapping of theories, with H^* corresponding to the associated algebraic functor.

For any varietal theory $\boldsymbol{V}$, the Yoneda embedding $J_{\boldsymbol{V}}$: $\boldsymbol{V} \to \boldsymbol{V}^{\#}$ is a $\boldsymbol{V}$-coalgebra in $\boldsymbol{V}^{\#}$, with $J_{\boldsymbol{V}}^{*}$ being naturally isomorphic to the identity functor $\mathrm{Id}_{\boldsymbol{V}^{\#}}$. The natural isomorphism matches each element p of a $\boldsymbol{V}$-algebra $\mathscr{B}$ with the $\boldsymbol{V}$-homomorphism θ_p: $F_{\boldsymbol{V}}(1) \to \mathscr{B}$ which sends the free generator of $F_{\boldsymbol{V}}(1) = J_{\boldsymbol{V}}(v(1))$ to p. The X-ary co-operation of $J_{\boldsymbol{V}}$ corresponding to an X-ary $\boldsymbol{V}$-operation f is the $\boldsymbol{V}$-homomorphism $J_{\boldsymbol{V}}(f) = \psi_f$: $F_{\boldsymbol{V}}(1) \to F_{\boldsymbol{V}}(X)$ which represents f in $\boldsymbol{V}^{\#}$. If φ: $\mathscr{B} \to \mathscr{B}_0$ is a $\boldsymbol{V}$-homomorphism, then $[J_{\boldsymbol{V}}^{\#}(\varphi)](\theta_p) = \varphi\theta_p = \theta_{\varphi(p)}$ for each element p of $\mathscr{B}$.

Noting that the functor G: $\boldsymbol{T} \to \boldsymbol{K}^{\#}$ defined in the proof of (3.1) is a $\boldsymbol{T}$-coalgebra in $\boldsymbol{K}^{\#}$, we allow the reader to verify the following.

4.2. Proposition. $E = G^*$. ■

From (4.1) it follows that $EJ_{\boldsymbol{K}}$ is a $\boldsymbol{K}$-coalgebra in $\boldsymbol{T}^{\#}$. Define

$$C = (EJ_{\boldsymbol{K}})^*\colon \boldsymbol{T}^{\#} \to \boldsymbol{K}^{\#}\,.$$

We shall now apply our knowledge of $\boldsymbol{T}$ to obtain a useful characterization of C.

By well-ordering the set A appropriately, we can find lattice operations $\wedge$ and $\vee$ on A relative to which A is a complete linearly-ordered lattice with a least element a_0 and a greatest element a_1. This lattice admits a pseudocomplement operation $\neg$, where

$$\neg a = \begin{cases} a_0 & \text{if} \quad a \neq a_0\,,\\ a_1 & \text{if} \quad a = a_0\end{cases}$$

for all $a \in A$. A binary operation d: $A^2 \to A$ is defined by

$$d(a, b) = \begin{cases} a_1 & \text{if} \quad a = b\,,\\ a_0 & \text{if} \quad a \neq b\end{cases}$$

for all $a, b \in A$.

The operations defined above correspond, by (3.7), to uniquely determined $\boldsymbol{T}$-operations which we shall respectively call $\wedge$, $\vee$, $\neg$, and d. Each element $a \in A$ determines a constant (0-ary) $\boldsymbol{T}$-operation e_a; in particular, write e_{a_0} as e_0 and e_{a_1} as e_1. By (3.4) each $\boldsymbol{T}$-algebra is, relative to the operations just given, a pseudo-complemented distributive lattice with least element e_0 and greatest element e_1, in which there is a complete linearly-ordered sublattice of constants and an extra binary operation d.

4.3. Proposition. *The A-ary join $\bigvee_{a\in A}$ in the complete linearly-ordered lattice A described above is induced by an A-ary $\boldsymbol{T}$-operation $\sum_{a\in A}$.*

Proof. The join $\bigvee_{a\in A}$ in A meets the criterion given in (3.5) for a function on A to be induced by a $\boldsymbol{T}$-operation. To see this, note that, for any $c \in A$,

$$\{\bar{a} \in A^A\colon \bigvee \bar{a} > c\} = \bigcup_{b\in A}\ \bigcup_{c'>c} \{\bar{a} \in A^A\colon a_b = c'\}$$

belongs to the image of ϱ_A, and we can write

$$\{\bar{a} \in A^A: \bigvee \bar{a} = c\} = \big(\bigcap_{b<c} \{\bar{a}: \bigvee \bar{a} > b\}\big) - \{\bar{a}: \bigvee \bar{a} > c\}\,. \blacksquare$$

4.4. Proposition. *The T-operations defined above have the following properties.*

(i) *For any T-algebra $\mathscr{B}$, the set $R(\mathscr{B})$ of all regular elements of $\mathscr{B}$ (i.e., elements p such that $\neg\neg p = p$) is a Boolean algebra in which the T-operations $\wedge$ and $\neg$ respectively induce the Boolean meet and complement, and where e_0 and e_1 are, respectively, the Boolean zero and unit. Furthermore, the restriction of any T-homomorphism $\varphi\colon \mathscr{B} \to \mathscr{B}_0$ to $R(\mathscr{B})$ is a Boolean homomorphism $R(\varphi)\colon R(\mathscr{B}) \to R(\mathscr{B}_0)$.*

(ii) *For any T-algebra $\mathscr{B}$ and any element p of $\mathscr{B}$, we have $d(e_a, p) \wedge d(e_b, p) = e_0$ for all distinct $a, b \in A$, while $p = \sum_{a\in A} (e_a \wedge d(e_a, p))$.*

(iii) *For any regular element p of a T-algebra $\mathscr{B}$,*

$$d(e_a, p) = \begin{cases} p & \text{if} \quad a = a_1\,, \\ e_0 & \text{if } a_0 < a < a_1\,, \\ \neg p & \text{if} \quad a = a_0\,. \end{cases}$$

Proof. Item (i) expresses a well-known fact about pseudocomplemented distributive lattices. Items (ii) and (iii) follow from (3.4) together with the fact that the operations defined on A satisfy the corresponding identities. For example, $p = \sum_{a\in A} (e_a \wedge d(e_a, p))$ holds for T-algebras because the composite $\bigvee_{a\in A} (a \wedge d(a, -))$ is a unary operation on A which coincides with the identity operation on A. $\blacksquare$

4.5. Proposition. *For any T-algebra $\mathscr{B}$, $R(\mathscr{B})$ is a K-algebra, and for any T-homomorphism φ, $R(\varphi)$ is a K-homomorphism. Furthermore, R, construed as a functor from $T^\#$ to $K^\#$, is naturally isomorphic to C.*

Proof. Let $\alpha\colon 2^2 \to 2^A$ and $\beta\colon 2^A \to 2^2$ be the K-homomorphisms defined in the proof of (4.1), with a_0, a_1 respectively taken to be the least and the greatest elements of A as specified above. Note that $\alpha\beta\colon 2^A \to 2^A$ is the K-homomorphism $\varphi_{d(e_1,-)}$ which represents the T-operation $d(e_1, -)$. In the discussion immediately before (4.2) it was indicated that there is a natural isomorphism from $\mathrm{Id}_{T^\#}$ to J_T^* which matches each element p of a T-algebra $\mathscr{B}$ with the T-homomorphism $\theta_p\colon F_T(1) \to \mathscr{B}$ sending the free generator id_{2^A} of $F_T(1) = EL(1) = E(2^A)$ to p.

For each T-algebra $\mathscr{B}$, define a function $h_{\mathscr{B}}\colon R(\mathscr{B}) \to U_K C(\mathscr{B})$ by $h_{\mathscr{B}}(p) = \theta_p E(\alpha)$, each $p \in R(\mathscr{B})$. Then $h_{\mathscr{B}}$ is a bijection, its inverse $h_{\mathscr{B}}^{-1}$ being given by $h_{\mathscr{B}}^{-1}(q) = [qE(\beta)](\mathrm{id}_{2^A})$ for each element q of $C(\mathscr{B})$. Note in particular that

$$[\theta_p E(\alpha) E(\beta)](\mathrm{id}_{2^A}) = [\theta_p E(\alpha\beta)](\mathrm{id}_{2^A}) = \theta_{d(e_1,p)}(\mathrm{id}_{2^A}) = d(e_1, p) = p$$

for all $p \in R(\mathscr{B})$.

Since U_K creates isomorphisms, there are a unique K-algebra $\mathscr{B}_0$ and a unique K-isomorphism $\gamma_{\mathscr{B}}\colon \mathscr{B}_0 \to C(\mathscr{B})$ such that $h_{\mathscr{B}} = U_K(\gamma_{\mathscr{B}})$. The K-operations on $C(\mathscr{B})$ are simply pulled back through $h_{\mathscr{B}}$ to define corresponding operations on $R(\mathscr{B})$ to create $\mathscr{B}_0$. For each X-ary K-operation f (represented by $\psi_f\colon 2^2 \to X\cdot 2^2$),

the K-homomorphism $[X\cdot\alpha]\psi_f\beta\colon 2^A \to X\cdot 2^A$ represents a T-operation which induces f on $R(\mathscr{B})$ in such a way that $h_{\mathscr{B}}$ preserves it. In particular, the Boolean meet and complement are induced on $R(\mathscr{B})$ by the T-operations $\wedge$ and $\neg$. To see this, it is sufficient to check that $\varphi_\wedge\alpha = [2\cdot\alpha]\psi_\wedge$ and $\varphi_\neg\alpha = \alpha\psi_\neg$, where the T-operations are represented by $\varphi_\wedge$ and $\varphi_\neg$, while $\psi_\wedge$ and $\psi_\neg$ represent the corresponding Boolean ones. The checking is done for $\wedge$ by using the isomorphism $\varrho_2\colon 2\cdot 2^A \to 2^{A^2}$ provided by (1.8) to figure out how $\varphi_\wedge$ works and then chasing the element x of 2^2. We have

$$\varphi_\wedge\alpha(x) = \varphi_\wedge(\{a_1\}) = \sigma_0(\{a_1\}) \wedge \sigma_1(\{a_1\}) = [2\cdot\alpha](x_0 \wedge x_1) = [2\cdot\alpha]\psi_\wedge(x)\,.$$

The procedure for $\neg$ is similar. The computation showing that $\gamma\colon R \to C$ is a natural transformation is routine, so we omit it. $\blacksquare$

4.6. Proposition. *CE is naturally isomorphic to* $\mathrm{Id}_{K^\#}$.

Proof. Using (3.6) one can verify that, for each $a \in A$ and $p \in E(\mathscr{B})$, where $\mathscr{B}$ is any K-algebra, we have $[d(e_a, p)](\{a_1\}) = p(\{a\})$. Then by part (iii) of (4.4) we see that the regular elements of $E(\mathscr{B})$ are the K-homomorphisms $p\colon 2^A \to \mathscr{B}$ such that $p(\{a\}) = 0$ for all $a \in A - \{a_0, a_1\}$. Let $\hat{\delta}_{\mathscr{B}}$ be the function which maps each regular element p of $E(\mathscr{B})$ to $p(\{a_1\})$ in $\mathscr{B}$. If f is any X-ary K-operation and $\bar{p}\colon x\cdot 2^A \to \mathscr{B}$ is induced in $K^\#$ by a sequence $\langle p_x\colon x \in X\rangle$ of K-homomorphisms $p_x\colon 2^A \to \mathscr{B}$ which are regular elements of $E(\mathscr{B})$, then (noting that $\bar{p}[X\cdot\alpha]\colon X\cdot 2^2 \to \mathscr{B}$ sends the xth free generator of $X\cdot 2^2$ to $p_x(\{a_1\})$, all $x \in X$) we have

$$\begin{aligned}[\bar{p}[X\cdot\alpha]\psi_f\beta](\{a_1\}) &= [\bar{p}[X\cdot\alpha]\psi_f](\beta(\{a_1\})) = [\bar{p}[X\cdot\alpha]\psi_f](x) \\ &= f_{\mathscr{B}}(\langle p_x(\{a_1\})\colon x \in X\rangle)\,,\end{aligned}$$

so $\hat{\delta}_{\mathscr{B}}$ is an isomorphism of K-algebras. We leave it to the reader to verify that $\hat{\delta}$ is a natural transformation. A natural isomorphism $\delta\colon CE \to \mathrm{Id}_{K^\#}$ is then given by $\delta = \hat{\delta}\gamma^{-1}$. $\blacksquare$

4.6. Proposition. *There is a natural transformation $\eta\colon \mathrm{Id}_{T^\#} \to EC$.*

Proof. Given a T-algebra and an element p of $\mathscr{B}$, we define $\eta_{\mathscr{B}}(p) = C(\theta_p)\delta_{2^A}^{-1}$, where $\theta_p\colon E(2^A) \to \mathscr{B}$ is the T-homomorphism which sends the free generator id_{2^A} of $E(2^A)$ to p.

To show that $\eta_{\mathscr{B}}\colon \mathscr{B} \to EC(\mathscr{B})$ is a T-homomorphism, let $\langle p_x\colon x \in X\rangle$ be a sequence of elements of $\mathscr{B}$, and let f be an X-ary T-operation, represented by the T-homomorphism $E(\varphi_f)\colon E(2^A) \to E(X\cdot 2^A)$. The T-homomorphism

$$\theta_{\bar{p}}\colon E(X\cdot 2^A) \to \mathscr{B}$$

is such that $\theta_{\bar{p}}E(\sigma_x) = \theta_{p_x}$, all $x \in X$. For each $x \in X$, let $q_x = \eta_{\mathscr{B}}(p_x)$; then $\theta_{\bar{q}}\colon E(x\cdot 2^A) \to EC(\mathscr{B})$ is the T-homomorphism induced by $\langle \theta_{q_x}\colon x \in X\rangle$. We must show that $\eta_{\mathscr{B}}\theta_{\bar{p}}E(\varphi_f) = \theta_{\bar{q}}E(\varphi_f)$, so clearly it suffices to show $\eta_{\mathscr{B}}\theta_{\bar{p}} = \theta_{\bar{q}}$. First, note that

$$[\eta_{\mathscr{B}}\theta_{p_x}](\mathrm{id}_{2^A}) = \eta_{\mathscr{B}}(\theta_{p_x}(\mathrm{id}_{2^A})) = \eta_{\mathscr{B}}(p_x) = q_x = \theta_{q_x}(\mathrm{id}_{2^A})\,,$$

so $\eta_{\mathscr{B}}\theta_{p_x} = \theta_{q_x}$, all $x \in X$. Then $\eta_{\mathscr{B}}\theta_{\bar{p}}E(\sigma_x) = \theta_{\bar{q}}E(\sigma_x)$, all $x \in X$, so $\eta_{\mathscr{B}}\theta_{\bar{p}} = \theta_{\bar{q}}$ as required. The computation which shows that η: $\mathrm{Id}_{T^{\#}} \to EC$ is a natural transformation is straightforward, so we omit it. ∎

4.7. Proposition. *E and C are adjoint equivalence functors.*

Proof. It is sufficient to prove that η is a natural isomorphism, i.e., that each component $\eta_{\mathscr{B}}$ of η is bijective. Let $\mathscr{B}$ be a T-algebra, p an element of $\mathscr{B}$, and $a \in A$. We claim that $[\eta_{\mathscr{B}}(p)](\{a\}) = \gamma_{\mathscr{B}}(d(e_a, p))$. This implies that $\eta_{\mathscr{B}}$ is an injection, since if p_0, p_1 are elements of $\mathscr{B}$ such that $\eta_{\mathscr{B}}(p_0) = \eta_{\mathscr{B}}(p_1)$ we have

$$\gamma_{\mathscr{B}}(d(e_a, p_0)) = \gamma_{\mathscr{B}}(d(e_a, p_1)) \quad \text{for all } a \in A .$$

But then the second identity of (4.4) part (ii) tells us that $p_0 = p_1$.

To prove the claim, first note that

$$\begin{aligned}\delta_{2^A}(E(\varphi_{d(e_a,-)})E(\alpha)) &= [\hat{\delta}_{2^A}\gamma_{2^A}^{-1}](E(\varphi_{d(e_a,-)})E(\alpha)) = [[E(\varphi_{d(e_a,-)})E(\alpha\beta)](\mathrm{id}_{2^A})](\{a_1\}) \\ &= [E(\varphi_{d(e_a,-)})(\mathrm{id}_{2^A})](\{a_1\}) = \varphi_{d(e_a,-)}(\{a_1\}) = \{a\} .\end{aligned}$$

Then we have

$$\begin{aligned}[\eta_{\mathscr{B}}(p)](\{a\}) &= [C(\theta_p)\delta_{2^A}^{-1}](\{a\}) = [C(\theta_p)](E(\varphi_{d(e_a,-)})E(\alpha)) = \theta_p E(\varphi_{d(e_a,-)})E(\alpha) \\ &= \theta_{d(e_a,p)}E(\alpha) = \gamma_{\mathscr{B}}(d(e_{a,p})) .\end{aligned}$$

Our proof that $\eta_{\mathscr{B}}$ is surjective is based on two claims — first, that C preserves surjections, and second, that ηE is a natural isomorphism. Given these two facts, let ε: $E(X\cdot 2^A) \to \mathscr{B}$ be a surjective T-homomorphism from a sufficiently large free T-algebra $E(X\cdot 2^A)$ onto $\mathscr{B}$; then by the naturality of η we have $\eta_{\mathscr{B}}\varepsilon = EC(\varepsilon)\eta_{E(X\cdot 2^A)}$. But $EC(\varepsilon)$ is surjective, by (2.4) and the definition of E together with the first claim, and $\eta_{E(X\cdot 2^A)}$ is surjective by the second claim. Then $\eta_{\mathscr{B}}\varepsilon$ is surjective, so $\eta_{\mathscr{B}}$ is.

The C preserves surjections follows easily from the fact that $E(2^2)$ is regular-projective in $T^{\#}$, since it is a retract of the free T-algebra $E(2^A)$.

To prove that η_E is a natural isomorphism, we show that it is the inverse of the natural isomorphism $E\delta$. Let $\mathscr{B}$ be any K-algebra and p: $2^A \to \mathscr{B}$ an element of $E(\mathscr{B})$. Note that θ_p: $E(2^A) \to E(\mathscr{B})$ is actually $E(p)$. Then

$$[E(\delta_{\mathscr{B}})\eta_{E(\mathscr{B})}](p) = [E(\delta_{\mathscr{B}})](C(\theta_p)\delta_{2^A}^{-1}) = \delta_{\mathscr{B}}CE(p)\delta_{2^A}^{-1} = p . \ ∎$$

It is noteworthy that the main ideas of our proof that E is an equivalence functor are present in Foster's first paper [2] on Boolean extensions. The use of the "Post algebra" structure on T-algebras, and particularly of the identity $p = \sum_{a\in A}(e_a \wedge d(e_a, p))$ is quite clear in that paper, while the K-homomorphisms α, β used in our proof to tie various coalgebras together correspond to Foster's device of looking at a two-element "subframe" of his "kernel" algebra $\mathscr{A}$ to recover the "core" Boolean algebra $\mathscr{B}$ from $\mathscr{A}[\mathscr{B}]$.

The category-theoretic analysis of set-valued extension functors which we have carried out above makes it possible to define algebra-valued Boolean extension

functors in a way which is independent of the Foster formula given in (3.6), avoiding the completeness and distributivity conditions on the K-algebras required by that formula.

5. Algebra-valued Boolean extension functors. Because E: $K^{\#} \to T^{\#}$ is a component of the unit of the structure-semantics adjunction for varietal theories (see [5]), it has the following universal property.

5.1. Proposition. *Let V be any varietal theory, and let H: $K^{\#} \to V^{\#}$ be any functor such that $U_V H = A[-]$. Then there is a unique mapping of theories h: $V \to T$ such that $H = h^{\#}E$.* ∎

5.2. Definition. Let K be a varietal theory of λ-complete Boolean algebras, and let E: $K^{\#} \to T^{\#}$ be the canonical functor defined in Section 4, with T being the varietal theory determined by a Boolean extension functor $A[-]$: $K^{\#} \to$ **Set**. Let h: $V \to T$ be a mapping of theories. Then the *Boolean extension functor induced by h* is $h^{\#}E$: $K^{\#} \to V^{\#}$.

The functor $A[-]$ is the Boolean extension functor induced by t: **Set** $\to T$ if we regard t as a mapping of theories, while E is the Boolean extension functor induced by the identity mapping Id_T: $T \to T$. The classical definition of the extension $\mathscr{A}[\mathscr{B}]$ of a universal algebra $\mathscr{A}$ by a Boolean algebra $\mathscr{B}$ (as given in Foster [2] or Burris [1]) requires that $\mathscr{A}$ have only finitary operations and that $\mathscr{B}$ be at least $|A|^+$-complete, relying on Foster's formula (3.6) to extend the operations of $\mathscr{A}$ to $\mathscr{A}[\mathscr{B}]$. The Boolean extension functors corresponding to the classical construction are those induced by mappings of theories of the form h: $V \to T$ where V is a theory of rank $\leqslant \aleph_0$. The so-called bounded Boolean extensions, which are constructed using partitions which have only finitely many nonzero values, are not covered by (5.2).

A functor U: $M \to N$ is (weakly) monadic if and only if it has a left adjoint and the canonical comparison functor D: $M \to N^{\mathscr{T}}$ is an isomorphism (resp. equivalence) of categories, where $N^{\mathscr{T}}$ is the category of Eilenberg–Moore algebras over the monad $\mathscr{T}$ induced by U (see [6], [8] for details). In particular, it is proved in Wraith [8], Appendix C that every algebraic functor is monadic, and it is not difficult to see that the composition of a monadic functor with an equivalence functor is weakly monadic. From these considerations we obtain the following result, which accounts for many of the known properties of classical Boolean extensions.

5.3. Proposition. *Every Boolean extension functor $h^{\#}E$ (of type defined in (5.2)) is weakly monadic.*

Our concluding result is a characterization of the Boolean extension functors defined in (5.2) which eliminates the definition's reference to the functor E: $K^{\#} \to T^{\#}$. We omit the simple proof.

5.4. Proposition. *A functor H: $K^{\#} \to V^{\#}$ is a Boolean extension functor in the sense of (5.2) if and only if H is represented by a V-coalgebra in $K^{\#}$ whose underlying K-algebra is isomorphic to 2^A, for some set A with $1 < |A| < \lambda$.* ∎

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A measurable selection theorem

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Abstract. A family of subsets of a Polish space has the *partition-selection property* if the equivalence relation generated by countably many members of the family always admits a selector measurable with respect to the family. It is shown that the family of Baire-property sets enjoys the partition-selection property. The same is true of the Borel-programmable sets, the R-sets, the absolutely Δ_2^1 sets, and the Lebesgue measurable sets.

§ 1. Measurable selectors and transversals for countably-generated equivalence relations and partitions: A survey

1.1. Introduction. Throughout this section, let X be an uncountable Polish space (topological space admitting a countable basis and a complete metric). E.g. X might be the Baire space $J = \omega^\omega$ of infinite sequences of natural numbers under the topology having as basis the sets $U_s = \{y\colon y \text{ extends } s\}$ for s a finite sequence. Or X might be the Cantor space $I = 2^\omega$ of infinite $\{0, 1\}$-sequences, considered as a subspace of J. The letters Y, Z will also denote Polish spaces.

Let E be an equivalence relation on X, and identify E with its graph $\{(x, x')\colon xEx'\} \subseteq X^2$. Let x/E denote the E-equivalence class of $x \in X$. Associated with E we have the partition $Q = \{x/E\colon x \in X\}$ of X into disjoint classes; and conconversely, every such partition is associated with an equivalence relation. A *section* for E is a map $\sigma\colon Q \to X$ satisfying $\sigma(A) \in A$. A *selector* for E is a function $S\colon X \to X$ of form $S(x) = \sigma(x/E)$ for some section σ. Equivalently, S is a selector if we always have $S(x)Ex$, and have $S(x) = S(x')$ whenever xEx'. A *transversal* for E is a set $T \subseteq X$ consisting of exactly one representative from each E-equivalence class. Selector S and transversal T are *associated* if $T = \text{range } S$, or equivalently $S(x) =$ the unique $x' \in T$ with xEx'. $A \subseteq X$ is E-*invariant* if $x' \in A$ whenever $x \in A$ and xEx'. A countable family $\{A_n\colon n \in \omega\}$ of subsets of X *generates* E if

$$E = \{(x, x')\colon \forall n (n \in A_n \leftrightarrow x' \in A_n)\},$$

or equivalently if the A_n are invariant sets which *separate* distinct E-equivalence classes (so that whenever $x/E \neq x'/E$ there is an A_n with $x/E \subseteq A_n$ and $x'/E \cap A_n = \emptyset$