Cauchy's functional equation on semigroups

by

Luigi Paganoni and **Stefania Paganoni Marzegalli** (Milano)

Abstract. Consider two semigroups $(G, \cdot)$ and $(H, \cdot)$, a subset $\Omega \subset G\times G$ and a function $f\colon G \to H$ such that $f(x\cdot y) = f(x)\cdot f(y)$ for all pairs $(x, y) \in \Omega$. In this paper we find some conditions on Ω under which f is a homomorphism.

1. Let G and H be semigroups and consider Cauchy's functional equation on a restricted domain

$$(*) \qquad f(x\cdot y) = f(x)\cdot f(y)$$

where $(x, y) \in \Omega \subset G\times G$, $f\colon G \to H$ and $\cdot$ denotes the composition law in G and H.

Several previous papers (see, for instance, [1]–[4], [6], [8]–[11], [13]) give some conditions under which a solution of Cauchy's equation is a homomorphism; in this paper we study the same problem and find some new conditions.

Throughout this paper the following notations are used:

$$N = (G\times G)\setminus\Omega, \quad \Omega_x = \{t\in G\colon (x, t)\in\Omega\}, \quad \Omega^x = \{t\in G\colon (t, x)\in\Omega\},$$

$$T_x = \{t\in G\colon \text{there exists } y\in G \text{ such that } (t, y)\in\Omega \text{ and } t\cdot y = x\},$$

$$T^x = \{t\in G\colon \text{there exists } y\in G \text{ such that } (y, t)\in\Omega \text{ and } y\cdot t = x\}.$$

We always suppose $\Omega_x \cup \Omega^x \cup T_x \neq \emptyset$, for every $x\in G$, so that the functional equation $(*)$ binds all the elements of G in some way.

Let $\tilde{\Omega} = \{(x, y)\colon f(x\cdot y) = f(x)\cdot f(y)\}$. In this paper we look for some conditions under which $\tilde{\Omega} = G\times G$, that is f is a homomorphism of G into H (¹).

We can prove the following theorem:

(¹) When G and H are groups it is easy to see that $\Omega^{x_0} = G$ ($\Omega_{x_0} = G$) implies

$$\{(x, x_0^{-1}), x\in G\}\subset \tilde{\Omega} \quad (\{(x_0^{-1}, x), x\in G\}\subset\tilde{\Omega}).$$

Therefore, in these cases, we always suppose, whithout loss of generality, that Ω has the following properties:

$$\text{if } \Omega^{x_0} = G \text{ then } \Omega^{x_0^{-1}} = G, \quad \text{if } \Omega_{x_0} = G \text{ then } \Omega_{x_0^{-1}} = G.$$

Furthermore, if $\Omega \supset \{(x, x^{-1}), x\in G\}$, we deduce:

$$(x, y)\in\tilde{\Omega} \text{ iff } (x\cdot y, y^{-1})\in\tilde{\Omega} \quad\text{and}\quad (x, y)\in\tilde{\Omega} \text{ iff } (x^{-1}, x\cdot y)\in\tilde{\Omega}$$

and so, in this case, we suppose, without loss of generality, that Ω is invariant under the maps:

$$\varphi_1\colon (x, y)\mapsto (x\cdot y, y^{-1}), \quad \varphi_2\colon (x, y)\to (x^{-1}, x\cdot y).$$

When G and H are abelian semigroups, we assume, without loss of generality, that Ω is symmetric with respect to the diagonal of $G\times G$.

THEOREM 1. *Consider the functional equation* (*) *and suppose* $(a, b) \in N$. *If at least one of the following conditions* (i)–(iv) *is satisfied*:

(i) *H is a right regular semigroup* [2] *and* $\{b\cdot(\Omega_b \cap \Omega_{a\cdot b})\} \cap \Omega_a \neq \emptyset$,

(ii) *H is a left regular semigroup and* $\{(\Omega^a \cap \Omega^{a\cdot b})\cdot a\} \cap \Omega^b \neq \emptyset$,

(iii) *G is a left regular semigroup and* $\{a\cdot(\Omega_a \cap T_b)\} \cap T_{a\cdot b} \neq \emptyset$,

(iv) *G is a right regular semigroup and* $\{(\Omega^b \cap T^a)\cdot b\} \cap T^{a\cdot b} \neq \emptyset$,

then $(a, b) \in \tilde{\Omega}$.

Proof. If (i) is satisfied, there exist x_0 and y_0 such that: $(a, x_0) \in \Omega$ and $x_0 = b\cdot y_0$ with $(b, y_0) \in \Omega$ and $(a\cdot b, y_0) \in \Omega$. This implies $f(a\cdot b)\cdot f(y_0) = f(a\cdot b\cdot y_0) = f(a)\cdot f(b\cdot y_0) = f(a)\cdot f(b)\cdot f(y_0)$ and, as H is a right regular semigroup, $f(a\cdot b) = f(a)\cdot f(b)$.

If (iii) is satisfied, there exist x_0 and y_0 with the following properties: $x_0 \in T_{a\cdot b}$ and $x_0 = a\cdot y_0$ with $y_0 \in T_b$ and $(a, y_0) \in \Omega$. As $x_0 \in T_{a\cdot b}$, there exists y_1 such that $(x_0, y_1) \in \Omega$ and $x_0\cdot y_1 = a\cdot b$; as $y_0 \in T_b$, there exists y_2 such that $(y_0, y_2) \in \Omega$ and $y_0\cdot y_2 = b$. But now $x_0\cdot y_2 = a\cdot y_0\cdot y_2 = a\cdot b$ and therefore $x_0\cdot y_2 = x_0\cdot y_1$; as G is a left regular semigroup we have $y_2 = y_1$. It follows:

$$f(a\cdot b) = f(x_0\cdot y_1) = f(x_0)\cdot f(y_1) = f(a)\cdot f(y_0)\cdot f(y_1) = f(a)\cdot f(y_0\cdot y_1) = f(a)\cdot f(b)\,.$$

If (ii) or (iv) are satisfied, the proof is similar.

COUNTEREXAMPLE. Let R be the real numbers' set and consider the multiplicative semigroup $G = R\setminus\{O\}$ and $H = R$. If $\Omega = \{(x, y)\colon xy<0\}$ and $f\colon G\to H$ is defined by

$$f(t) = \begin{cases} 0 & \text{if } t<0\,,\\ c \neq 0,1 & \text{if } t>0\,,\end{cases}$$

then f is a solution of the functional equation (*) on Ω, but, for every $(a, b) \in N$, $f(a\cdot b) \neq f(a)\cdot f(b)$.

In this example the conditions on the sets which appear in (i) and (ii) are sometimes fulfilled (for instance when $a, b>0$), but H is not a regular semigroup; on the other hand, even if G is a regular semigroup, the sets which appear in (iii) and (iv) are always empty.

Now we have to introduce some classes of subsets of G.

DEFINITION 1. Let $i>0$ be a fixed integer and $\mathfrak{F}$ a collection of subsets of G with the following properties:

(i) if $T_1 \in \mathfrak{F}$ and $T_2 \supset T_1$, then $T_2 \in \mathfrak{F}$,

(ii) if $T \in \mathfrak{F}$ and $x \in G$, then $x\cdot T \in \mathfrak{F}$ $(T\cdot x \in \mathfrak{F})$,

(iii) if $T_1, T_2, \ldots, T_i \in \mathfrak{F}$, then $T_1 \cap T_2 \cap \ldots \cap T_i \neq \emptyset$.

[2] A semigroup X is said to be *right regular* (*left regular*) if the right-hand (left-hand) cancellation law holds in it, that is:

$$\alpha\cdot\varrho = \beta\cdot\varrho \Rightarrow \alpha = \beta \quad (\varrho\cdot\alpha = \varrho\cdot\beta \Rightarrow \alpha = \beta), \quad (\alpha, \beta, \varrho \in X)\,.$$

A semigroup X is said to be *regular* if it is left and right regular.

Such a family $\mathfrak{F}$ is said a *left* (*right*) σ_i*-class*. If $\mathfrak{F}$ is a left (right) σ_i-class for every $i>0$ [3], $\mathfrak{F}$ is said a left (right) σ-class.

DEFINITION 2. Let $\mathfrak{F}$ be a left (right) σ_i-class. A set $U\subset G$ is said to be *consistent* with $\mathfrak{F}$ if there exists a left (right) σ_i-class $\tilde{\mathfrak{F}}$ with $\tilde{\mathfrak{F}} \supset \mathfrak{F}$ and $U \in \tilde{\mathfrak{F}}$.

Let $\mathfrak{F}$ be a left (right) σ_i-class and consider the family Σ_i^l (Σ_i^r) of all left (right) σ_i-classes $\mathfrak{G} \supset \mathfrak{F}$ partially ordered by inclusion. It is easy to see that every chain in Σ_i^l (Σ_i^r) has an upper bound; therefore, by Zorn's lemma, there exists a maximal left (right) σ_i-class $\mathfrak{G}_0$, with $\mathfrak{G}_0 \supset \mathfrak{F}$.

Here are some examples of σ_i-classes and σ-classes:

1) Let G be a group and $\mathfrak{G}$ a proper linearly invariant ideal in G (see [3], [6]). Then $\mathfrak{F} = \{F\colon F^c \in \mathfrak{G}\}$ is a σ-class [4].

2) Let G be a direct group (see for instance [5] or [13]) and $\mathfrak{G}$ the class of the upper (lower) bounded subsets of G. Then $\mathfrak{F} = \{F\colon F^c \in \mathfrak{G}\}$ is a σ-class.

3) Let G be a semigroup and μ a left (right) invariant probability measure. Then $\mathfrak{F} = \{F\colon \mu(F)>1-1/i\}$ is a left (right) σ_i-class.

4) Let G be a semigroup and $(G, \mathfrak{M}, \mu)$ a σ-finite measure space such that for every $x \in G$ and $M \in \mathfrak{M}$, $\mu(x\cdot M)\leqslant\mu(M)$ $(\mu(M\cdot x)\leqslant\mu(M))$. Moreover, let $V_n\uparrow G$ be a sequence of sets with finite and positive measure such that:

(i) $\mu(V_{n+1})\sim\mu(V_n)$,

(ii) for every $a \in G$, there exist $k, \nu>0$ such that $V_n\subset a\cdot V_{n+k}$ $(V_n\subset V_{n+k}\cdot a)$, for every $n\geqslant\nu$.

Then, the upper density $\varrho_{\{V_n\}}$ relatively to $\{V_n\}$ (see [13]) is a subadditive function and, for every $x \in G$ and $A\subset G$, $\varrho_{\{V_n\}}(x\cdot A)\leqslant\varrho_{\{V_n\}}(A)$ $(\varrho_{\{V_n\}}(A\cdot x)\leqslant\varrho_{\{V_n\}}(A))$. Therefore $\mathfrak{F} = \{F\subset G\colon \varrho_{\{V_n\}}(F^c)<1/i\}$ is a left (right) σ_i-class.

Now we can state the following corollaries of Theorem 1.

COROLLARY 1. *If H is a right (left) regular semigroup and there exists a left (right) σ_3-class $\mathfrak{F}$ such that, for every $x \in G$, $\Omega_x \in \mathfrak{F}$ $(\Omega^x \in \mathfrak{F})$, then f is a homomorphism.*

COROLLARY 2. *Let $S\subset G$ be a set such that, for every $x \in G$,*

$$S \cap x\cdot S \neq \emptyset \qquad (S \cap S\cdot x \neq \emptyset)$$

(*this condition is satisfied if, for instance, S belongs to a left (right) σ_2-class*).

a) *If H is a right (left) regular semigroup and $\Omega\supset G\times S$ $(\Omega\supset S\times G)$, then f is a homomorphism.*

b) *If G is a group, H is a right (left) regular semigroup and $\Omega\supset\{(x, y)\colon x\cdot y \in S\}$, then f is a homomorphism.*

Proof. a) is obvious. We prove b) when H is a right regular semigroup and $S \cap x\cdot S \neq \emptyset$, for every $x \in G$. In this case, for every $(a, b) \in N$, $(b^{-1}\cdot S) \cap$

[3] It is obvious that a σ_i-class is also a σ_j-class for every $j<i$.

[4] F^c denotes the complement of F.

$\cap (b^{-1}\cdot a^{-1}\cdot S) \neq \emptyset$; so there exists $y \in G$ such that $b\cdot y \in S$ and $a\cdot b\cdot y \in S$. This implies $(b, y), (a\cdot b, y), (a, b\cdot y) \in \Omega$ and so hypothesis (i) of Theorem 1 is satisfied.

COROLLARY 3. *Let G be a left (right) regular semigroup and H a right (left) regular semigroup. Suppose $S \subset G$ be a set such that, for every $x, y \in G$, $S \cap x\cdot S \cap y\cdot S \neq \emptyset$ ($S \cap S\cdot x \cap S\cdot y \neq \emptyset$) (this condition is satisfied if, for instance, S belongs to a left (right) σ_3-class).*

If $\Omega \supset \{(x, y)\colon x\cdot y \in S\}$, then f is a homomorphism.

Proof. Suppose G and H be respectively a left and a right regular semigroup. Let $s \in S \cap (a\cdot S) \cap (a\cdot b\cdot S)$; then $s = a\cdot s_1 = a\cdot b\cdot s_2 \in S$ and $s_1 = b\cdot s_2 \in S$. This implies $(b, s_2), (a\cdot b, s_2), (a, b\cdot s_2) \in \Omega$ and so hypothesis (i) of Theorem 1 is satisfied.

2. Throughout this section, unless otherwise stated, we suppose G and H are abelian (5) semigroups; so we use $+$ to denote the binary law for G and H and we assume Ω is symmetric with respect to the diagonal of $G\times G$.

We have to introduce the following notations:

for every $s, t \in G$ and $\Gamma \subset G$, $\Gamma \neq \emptyset$,

$$\varphi_{s,t} = \{u \in G\colon u = x_1+\ldots+x_k, \text{ where } x_1 \in \Omega_s \cap \Omega_t, \ldots$$
$$\ldots x_k \in \Omega_{s+x_1+\ldots+x_{k-1}} \cap \Omega_{t+x_1+\ldots+x_{k-1}}\}$$

$$\varphi^{(s,t)} = \bigcap_{a\in G} \varphi_{s+a,t+a}, \qquad \varphi^{\Gamma} = \bigcap_{t\in\Gamma}\bigcap_{s\in G} \varphi_{s,s+t}.$$

Now it is useful to give some properties of the sets $\varphi_{s,t}$, because they will appear in the main theorems.

LEMMA 1. $\bigcup_{u\in\varphi_{s,t}} (u+\varphi_{s+u,t+u}) \subset \varphi_{s,t}$. *Moreover if G is a monoid and $0 \in \varphi_{s,t}$, then $\varphi_{s,t} = \bigcup_{u\in\varphi_{s,t}} (u+\varphi_{s+u,t+u})$.*

Proof. If $v \in \varphi_{s+u,t+u}$, we have $v = x_1+\ldots+x_k$ with

$$x_1 \in \Omega_{s+u} \cap \Omega_{t+u}, \ldots, x_k \in \Omega_{s+u+x_1\ldots+x_{k-1}} \cap \Omega_{t+u+x_1+\ldots+x_{k-1}}.$$

Now, if $u \in \varphi_{s,t}$, we can write $u = y_1+\ldots+y_r$ with

$$y_1 \in \Omega_s \cap \Omega_t, \ldots, y_r \in \Omega_{s+y_1+\ldots+y_{r-1}} \cap \Omega_{t+y_1+\ldots+y_{r-1}}.$$

Therefore, is we consider the sequence $\{y_1, \ldots, y_r, x_1, \ldots, x_k\}$, we obtain $u+v \in \varphi_{s,t}$. This implies $u+\varphi_{s+u,t+u} \subset \varphi_{s,t}$. The second statement is obvious.

COROLLARY 4. *Let G be a topological group. If, for every $a \in G$, $0 \in \operatorname{Int}(\varphi_{s+a,t+a})$ (6), then $\varphi_{s,t}$ is open. If, for every $a \in G$, $\varphi_{s+a,t+a}$ is of the second category at 0 (7), then $\varphi_{s,t}$ is of the second category at each of its points.*

(5) If G and H are not commutative, the notations here introduced are still meaningfull and Lemmas 1,2 and Corollaries 4,5 are still true.

(6) $\operatorname{Int}(X)$ denotes the interior of the set X.

(7) A subset X of a topological space is said to be of the *second category at a point* x_0 if, for every neighbourhood U of x_0, $U \cap X$ is a set of the second category (see, for instance, [12]).

Proof. It follows immediately from Lemma 1, as the map $x \to a+x$ is a homeomorphism.

EXAMPLE 1. Let $G = (\boldsymbol{R}, +)$ and suppose $\Omega \supset \{(x, y)\colon |y| \leqslant |g(x)|\}$ where $g\colon \boldsymbol{R} \to \boldsymbol{R}$ is a nowhere vanishing continuous (or at least lower semicontinuous) function. Then, for every $s, t \in \boldsymbol{R}$, $\varphi_{s,t} = \boldsymbol{R}$. We know indeed, by Corollary 4, $\varphi_{s,t}$ is open for every s, t; therefore we have only to show the boundary of $\varphi_{s,t}$ is empty. On the contrary, if u_0 belongs to the boundary of $\varphi_{s,t}$, as $\varrho = \min(|g(s+u_0)|, |g(t+u_0)| > 0)$, there exists a neighbourhood U of u_0 such that, for every $u \in U$, $\min(|g(s+u)|, |g(t+u)|) > \varrho/2$. If we choose $u \in U \cap \varphi_{s,t}$ with $|u-u_0| < \varrho/2$, we obtain $u_0 \in (u+\varphi_{s+u,t+u}) \subset \varphi_{s,t}$, contrary to our assumption.

Remark 1. In Example 1 we supposed $g(x) \neq 0$ for every $x \in \boldsymbol{R}$. If otherwise $E = \{x\colon g(x) = 0\}$ is not empty, but is a discrete set and,

$$\inf_{x\in E} \left[\min(|D_-g(x)|, |D^-g(x)|)\right] > 1, \qquad \inf_{x\in E} \left[\min(|D_+g(x)|, |D^+g(x)|)\right] > 1$$

(here $D_-g(x)$, $D^-g(x)$, $D_+g(x)$, $D^+g(x)$ are the Dini derivates of g at x), we have still $\varphi_{s,t} = \boldsymbol{R}$, for every $s, t \notin E$.

If E is not a discrete set but, for every $s, t \notin E$ and $u \in (E-s) \cup (E-t)$ there exist $x_1, x_2 \in \varphi_{s,t}$ with $x_1 < u < x_2$ such that

$$u-x_1 < \min(|g(s+x_1)|, |g(t+x_1)|) \quad \text{and} \quad x_2-u < \min(|g(s+x_2)|, |g(t+x_2)|)$$

we have still $\varphi_{s,t} = \boldsymbol{R}$ for every $s, t \notin E$.

EXAMPLE 2. Let G and g as in Example 1. If $\Omega \supset \{(x, y)\colon 0 \leqslant y \leqslant |g(x)|\}$, then, for every $s, t \in G$, $\varphi_{t,s} = [0, +\infty)$.

LEMMA 2. $\varphi_{s,t}+\varphi^{(s,t)} \subset \varphi_{s,t}$ *and* $\varphi^{(s,t)}+\varphi^{(s,t)} \subset \varphi^{(s,t)}$.

Proof. If $\varphi^{(s,t)} = \emptyset$ the proof is trivial. Suppose $\varphi^{(s,t)} \neq \emptyset$ and let $u \in \varphi_{s,t}$, $v \in \varphi^{(s,t)}$; then we have $v \in \varphi_{s+u,t+u}$ and, from Lemma 1, $u+v \in \varphi_{s,t}$. Let now $u, v \in \varphi^{(s,t)}$. As, for every $a \in G$, $u \in \varphi_{s+a,t+a}$ and $v \in \varphi_{s+a+u,t+a+u}$, if follows $u+v \in \varphi_{s+a,t+a}$. Since a is arbitrary, we get $u+v \in \varphi^{(s,t)}$.

Remark 2. If $a \in \varphi^{(s,t)}$, then, for every $n \in N$, $na \in \varphi^{(s,t)}$ and $\varphi_{s,t}+na \subset \varphi_{s,t}$, moreover, if G is a monoid and $0 \in \varphi^{(s,t)}$, then $\varphi_{s,t}+\varphi^{(s,t)} = \varphi_{s,t}$ and $\varphi^{(s,t)}+ +\varphi^{(s,t)} = \varphi^{(s,t)}$.

Remark 3. Let G be a monoid and H a regular monoid. If Ω contains $(0, t)$ or $(t, 0)$ for some $t \in G$, then $f(0) = 0$ and so $\tilde{\Omega} \supset \{(0, t), t \in G\} \cup \{(t, 0), t \in G\}$. In this case we can always assume, without loss of generality, that Ω itself contains the set $\{(0, t), t \in G\} \cup \{(t, 0), t \in G\}$. So we have $0 \in \varphi^{(s,t)}$.

COROLLARY 5. *Let $\emptyset \neq J \subset G$ and denote by T the subsemigroup generated by J. Then, if $\varphi^{(s,t)} \supset J$, we have $\varphi^{(s,t)} \supset T$ and $\varphi_{s,t} \supset \varphi_{s,t}+T$.*

Proof. $T = \bigcup_{n=1}^{\infty} \left(\sum_1^n J\right)$. As, for every $n \geqslant 1$, $\sum_1^n J = J+\ldots+J \subset \varphi^{(s,t)}+\ldots+\varphi^{(s,t)} \subset \varphi^{(s,t)}$, we have $T \subset \varphi^{(s,t)}$. Hence, by Lemma 2, $\varphi_{s,t}+T \subset \varphi_{s,t}$.

EXAMPLE 3. Let $R = (R, +, \cdot, Q)$ be the vector space of the real numbers over the field Q of the rational numbers and S be a Hamel basis of R over Q whose elements are supposed positive. If $G = R$ and $\Omega \supset G \times L$ where $L = \{s/n,\ n \in N,\ s \in S\}$, then for every $s, t \in G$, $\varphi_{s,t}$ and $\varphi^{(s,t)}$ contain

$$S^+ = \{x:\ x = c_1 s_1 + \ldots + c_n s_n,\ 0 \leqslant c_i \in Q,\ s_i \in S,\ n = 1, 2, \ldots\}.$$

Now we state the main result.

THEOREM 2. *Let G and H be abelian semigroups and H be also regular. Consider the functional equation* (*) *and let $(a, b) \in N$. If there exist elements $\alpha, \beta, \gamma, \delta$ belonging to G defined and connected by the following relations*:

(i) $\alpha \in \varphi_{a,a+\beta+\delta}$, $\beta \in \varphi_{a+b,a+\alpha}$, $\gamma \in \varphi_{b,a+b+\beta}$,

(ii) $\delta \in \Omega_{a+\alpha+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})]$,

then $(a, b) \in \tilde{\Omega}$. If for every $(a, b) \in N$ there exist $\alpha, \beta, \gamma, \delta$ satisfying (i) *and* (ii), *then f is a homomorphism.*

Proof. As $\alpha \in \varphi_{a,a+\beta+\delta}$, $\beta \in \varphi_{a+b,a+\alpha}$ and $\gamma \in \varphi_{b,a+b+\beta}$, there exist $x_1, \ldots, x_n \in G$ with the following properties:

$x_1 + \ldots + x_s = \alpha$ with

$$x_1 \in \Omega_a \cap \Omega_{a+\beta+\delta},\ \ldots,\ x_s \in \Omega_{a+x_1+\ldots+x_{s-1}} \cap \Omega_{a+\beta+\delta+x_1\ldots+x_{s-1}},$$

$x_{s+1} + \ldots + x_r = \beta$ with

$$x_{s+1} \in \Omega_{a+b} \cap \Omega_{a+\alpha},\ \ldots,\ x_r \in \Omega_{a+b+x_{s+1}+\ldots+x_{r-1}} \cap \Omega_{a+\alpha+x_{s+1}+\ldots+x_{r-1}},$$

$x_{r+1} + \ldots + x_n = \gamma$ with

$$x_{r+1} \in \Omega_b \cap \Omega_{a+b+\beta},\ \ldots,\ x_n \in \Omega_{b+x_{r+1}+\ldots+x_{n-1}} \cap \Omega_{a+b+\beta+x_{r+1}+\ldots+x_{n-1}}.$$

By (ii), $\delta = b+\gamma+x_0$ with $x_0 \in \Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma}$ and $x_0+b+\gamma \in \Omega_{a+\alpha+\beta}$. Because of the properties of $x_0, x_1, \ldots, x_n$, Ω contains therefore the following points:

$$(a, x_1), (a+x_1, x_2), \ldots, (a+x_1+\ldots+x_{r-1}, x_r),$$
$$(b, x_{r+1}), (b+x_{r+1}, x_{r+2}), \ldots, (b+x_{r+1}+\ldots+x_{n-1}, x_n),\ (b+x_{r+1}+\ldots+x_n, x_0),$$
$$(a+b, x_{s+1}), (a+b+x_{s+1}, x_{s+2}), \ldots, (a+b+x_{s+1}+\ldots+x_{n-1}, x_n),$$
$$(a+b+x_{s+1}+\ldots+x_n, x_0), (a+b+x_{s+1}+\ldots+x_n+x_0, x_1), \ldots$$
$$\ldots, (a+b+x_{s+1}+\ldots+x_n+x_0+\ldots+x_{s-1}, x_s),$$
$$(a+x_1+\ldots+x_r,\ b+x_{r+1}+\ldots+x_n+x_0).$$

By the equation (*) we can easily deduce

$$f(a+b) + \sum_{i=0}^{n} f(x_i) = f(a+b+\sum_{i=0}^{n} x_i) = f(a)+f(b) + \sum_{i=0}^{n} f(x_i);$$

now the regularity of H implies $f(a+b) = f(a)+f(b)$.

REMARK 4. We can always suppose, without loss of generality, that G and H are monoids and f maps the zero element of G in the zero element of H.

As a matter of fact we can consider the monoids $\hat{G} = G \cup \{O\}$ and $\hat{H} = H \cup \{O'\}$ obtained by adding elements O and O' which play the role of unit for G and H respectively. Now, if $\hat{f}$: $\hat{G} \to \hat{H}$ is the function defined by:

$$\hat{f}(x) = \begin{cases} O' & \text{if } x = O, \\ f(x) & \text{if } x \in G \end{cases}$$

it is easy to see thai $\hat{f}$ satisfies the functional equation (*) on

$$\hat{\Omega} = \Omega \cup (\{O\} \times \hat{G}) \cup (\hat{G} \times \{O\}) \quad \text{and} \quad \hat{N} = (\hat{G} \times \hat{G}) \setminus \hat{\Omega} = N.$$

Therefore f is a homomorphism iff $\hat{f}$ is a homomorphism. It follows that we can always choose α, β and γ equal to O because, for every $s, t \in G$, $\hat{\varphi}_{s,t} = \varphi_{s,t} \cup \{O\}$.

Remark 5. The hypotheses (i) and (ii) of Theorem 2 show that $\alpha, \beta, \gamma, \delta$ are mutually connected, but note that δ is defined in a different way than the others. The following simple counterexample shows that it is not possible to weaken the hypotheses of Theorem 2 by asking that there exist elements $\alpha, \beta, \gamma, \delta \in G$ which satisfy the following conditions:

(i′) $\alpha \in \varphi_{a,a+\beta+\delta}$, $\beta \in \varphi_{a+b,a+\alpha}$, $\gamma \in \varphi_{b,a+b+\beta}$,

(ii′) $\Omega_{a+\alpha+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})] \neq \emptyset$.

Let $G = \{0, 1, 2\}$ be the additive group of integers mod(3), $H = (R, +)$ and f: $G \to R$ be defined by:

$$f(0) = 0, \quad f(1) = 1, \quad f(2) = -1.$$

In this case f satisfies the functional equation (*) on $\Omega = G^2 \setminus N$ where $N = \{(1, 1), (2, 2)\}$; but clearly f is not a homomorphism because $-1 = f(1+1) \neq f(1)+f(1) = 2$ and $1 = f(2+2) \neq f(2)+f(2) = -2$. Nevertheless, if we choose $\alpha = 2$, $\beta = 0$, $\gamma = 0$, $\delta = 2$ when $(a, b) = (1, 1)$ and $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1$ when $(a, b) = (2, 2)$, the above mentioned conditions (i′) and (ii′) are satisfied.

Remark 6. Here we give another simple counterexample which shows that, if the set

$$E = \Omega_{a+\alpha+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})]$$

is empty for any choice of elements α, β, γ, then the statement of the Theorem 2 is not necessarily true.

Let $N = 1, 2, \ldots, n, \ldots$, $G = (N, \cdot)$, $H = (N, +)$ and

$$\Omega = \{(m, n) \in N \times N:\ (m, n) = 1\}\ (^8).$$

A function f: $N \to N$ which satisfies the functional equation $f(m \cdot n) = f(m)+f(n)$ on the restricted domain Ω or on $N \times N$ is called respectively *additive* or *completely additive*.

(8) $(m, n) = 1$ means that 1 is the highest common divisor of m and n.

Here it is easy to see that $E = \emptyset$ for any choice of elements α, β, γ; and it is indeed well known that an additive function is not necessarily a completely additive function.

Remark 7. A carefull reading of the proof of Theorem 2 shows that the hypotheses about commutativity of G and H are not necessary if (i) and (ii) can be satisfied choosing α and β equal to the unit element of G (see Remark 4); furthermore it is sufficient that H is right regular.

Therefore we can state the following proposition (using the multiplicative notation):

Let G and H be semigroups and H be also regular. Consider the functional equation (*); *if for every $(a, b) \in N$ there exists $\gamma \in \varphi_{b,a\cdot b}$* [9] *such that*

$$\Omega_a \cap [b\cdot\gamma\cdot(\Omega_{b\cdot\gamma} \cap \Omega_{a\cdot b\cdot\gamma})] \neq \emptyset,$$

then f is a homomorphism [10].

This proposition generalizes some results of [3] (for instance Theorem 1) in the case of semigroups (indeed if G is a group, the subgroup generated by Y is equal to the subsemigroup generated by $Y \cup Y^{-1}$, and considering the footnote [1] and Corollary 5 we can show that $\varphi_{s,t} = G$ for every $s, t \in G$; so $\gamma = b^{-1}$ satisfies our hypothesis).

Example 4. Let G and Ω as in Example 3, H an abelian regular semigroup and $f\colon G \to H$ a solution of (*) on Ω. Then f is a homomorphism. In fact if we apply Theorem 2 and remember that for every $s, t \in G$ $\varphi_{s,t} \supset S^+$, we have

$$\tilde{\Omega} \supset (R\times(-S^+)) \cup (R\times L),$$

and so f is a solution of (*) on $\tilde{\Omega}$. Now we apply again Theorem 2 and we can see that (*) is satisfied for every $(a, b) \in R^2$, that is f is a homomorphism.

Corollary 6. *Let G be an abelian group and H an abelian regular monoid. If, for every $(a, b) \in N$, $-a \in \varphi_{a,0} \cap \varphi_{a+b,0}$, then f is a homomorphism; this condition is satisfied if, for instance,*

$$-p_1(N) \subset \bigcap_{\alpha \in p_1(N)\cup N^*} \varphi_{\alpha,0}$$

where $p_1(N) = \{t \in G\colon (t, z) \in N$ for some $z \in G\}$ and $N^* = \{t = a+b, (a, b) \in N\}$.

Proof. If $(a, b) \in N$, we have also $(b, a) \in N$ and therefore, from our hypothesis, $-b \in \varphi_{b,0} \subset \varphi_{b,b}$. Moreover, as $\varphi_{a,0} \neq \emptyset$, we have $\Omega_0 \neq \emptyset$. From the hypotheses on G and H we can deduce $f(0) = 0$ and so we may assume $\Omega_0 = \Omega^0 = G$ (see Remark 3). If we choose $\delta = 0$, $\alpha = \beta = -a$, $\gamma = -b$, it is easy to verify that (i) and (ii) of Theorem 2 are satisfied.

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[9] See footnote [5].

[10] A similar theorem can be stated if H is a left regular semigroup.

Corollary 7. *Let G be an abelian group and H an abelian regular monoid. If, for every $(a, b) \in N$, $-a \in \varphi_{a+b,a}$ and $-b \in \varphi_{b,b}$, then f is a homomorphism. In particular f is a homomorphism if $\varphi_{s,t} = G$ for every $s, t \in G$.*

Proof. $\Omega_0 \neq \emptyset$; if, on the contrary, $\Omega_0 = \emptyset$ we should have $(0, 0) \in N$ and so $0 \in \varphi_{0,0}$, contrary to our assumption. Now we have, from Remark 3, $\Omega_0 = \Omega^0 = G$. Therefore the hypotheses of Theorem 2 are satisfied if we assume $\alpha = 0$, $\beta = -a$, $\gamma = -b$, $\delta = 0$.

The following theorems give us some particular conditions under which f is a homomorphism.

Theorem 3. *Let G and H be abelian regular semigroups and suppose there exist two σ_3-classes $\mathfrak{F}$ and $\mathfrak{G}$ such that:*

(i) *the set $S = \{x \in G\colon \Omega_x \in \mathfrak{F}\}$ belongs to $\mathfrak{G}$,*

(ii) *for every $s, t \in G$, $\varphi_{s,t}$ is consistent with $\mathfrak{G}$.*

Then f is a homomorphism.

Proof. We suppose, without loss of generality, G and H are monoids (see Remark 3) and so we can choose $\alpha = 0$ for every $(a, b) \in N$. Now, if $(a, b) \in N$, the set $(a+\varphi_{a+b,a}) \cap S$ is not empty and so there exists $\beta \in \varphi_{a+b,a}$ such that $a+\beta \in S$. Analogously $(a+b+\beta+\varphi_{b,a+b+\beta}) \cap S \cap (a+\beta+S) \neq \emptyset$ and so there exists $\gamma \in \varphi_{b,a+b+\beta}$ with $b+\gamma \in S$ and $a+b+\beta+\gamma \in S$. Hence

$$E = \Omega_{a+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})] \neq \emptyset$$

and so we can choose an element $\delta \in E$. Therefore all the hypotheses of Theorem 2 are satisfied.

Remark 7. Theorem 3 is still true if, instead of (i) and (ii), we suppose that, for every $s, t, u, v \in G$,

$$(u+\varphi_{s,t}) \cap S \cap (v+S) \neq \emptyset.$$

Theorem 4. *Let G be an abelian topological group and H an abelian regular monoid. If for every x belonging to a neighbourhood U of a point x_0, $\mathrm{Int}(\Omega_x) \neq \emptyset$ and, for every $s, t \in G$, $\varphi_{s,t}$ is a dense subset of G, then f is a homomorphism.*

Proof. As $\varphi_{0,0} \neq \emptyset$, we have $\Omega^0 = G$ (see Remark 3) and therefore, for every $s,t \in G$, $0 \in \varphi_{s,t}$. Let $(a, b) \in N$. In order to apply Theorem 2 we choose $\alpha = 0$ and we have only to show there exist $\beta, \gamma \in G$ such that $\beta \in \varphi_{a+b,a}$, $\gamma \in \varphi_{b,a+b+\beta}$ and

$$\Omega_{a+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})] \neq \emptyset;$$

As $0 \in \Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma}$, the last condition is satisfied if $b+\gamma \in \Omega_{a+\beta}$. Let $\beta \in (-a+U) \cap \varphi_{a+b,a}$ (this set is not empty because $\varphi_{a+b,a}$ is dense in G and $\mathrm{Int}(-a+U) \neq \emptyset$). Now, by the hypotheses on Ω_x and $\varphi_{s,t}$,

$$W = [-b+\mathrm{Int}(\Omega_{a+\beta})] \cap \varphi_{b,a+b+\beta} \neq \emptyset;$$

so, if we choose $\gamma \in W$, all the above mentioned conditions are satisfied and then f is a homomorphism.

Remark 8. H needs to be a monoid only to assure $\Omega^0 = G$. The same remark is true for Theorems 5 and 6.

EXAMPLE 5. Let $G = (R, +)$. If Ω has at least an interior point (for instance $(0, 0)$) and there exists a continuous function $g: R \to (0, +\infty)$ such that $\Omega \supset \{(x, y): y \in Q, |y| \leqslant g(x)\}$ (Q is the rational field), then, for every $s, t \in G$, $\varphi_{s,t}$ is dense in R and we can apply Theorem 4. In this case indeed it is easy to see that $\varphi_{s,t} \supset Q$.

THEOREM 5. *Let $(G, \mathfrak{M}, \mu)$ be an abelian group and a measure space with μ invariant under translation and H be an abelian regular monoid. If there exists a set $T \subset G$ with $\mu_*(T) > 0$* [11] *such that, for every $x \in T$, $\mu_*(\Omega_x) > 0$ and moreover, for every $s, t \in G$, $\mu_*(\varphi_{s,t}^c) = 0$, then f is a homomorphism.*

Proof. The first part of the proof is the same as in Theorem 4. Therefore it is sufficient to show that β and γ can be chosen so that

$$\beta \in \varphi_{a+b,a}, \quad \gamma \in \varphi_{b,a+b+\beta}, \quad b+\gamma \in \Omega_{a+\beta}.$$

This is possible if $X = (-a+T) \cap \varphi_{a+b,a} \neq \emptyset$ and there exists $\beta \in X$ such that $Y = (-b+\Omega_{a+\beta}) \cap \varphi_{b,a+b+\beta} \neq \emptyset$.

First we prove $X \neq \emptyset$. If, on the contrary, $X = \emptyset$, denoting $-a+T = A$ and $\varphi_{a+b,a} = B$, we should have

$$A = (A \cap B) \cup (A \cap B^c) = A \cap B^c \subset B^c$$

and so $\mu_*(A) \leqslant \mu_*(B^c) = 0$ contrary to our assumption $\mu_*(A) > 0$.

If now we choose $\beta \in X$, we have $a+\beta \in T$ and so $\mu_*(-b+\Omega_{a+\beta}) = \mu_*(\Omega_{a+b}) > 0$. Now the proof $Y \neq \emptyset$ is similar to the previous one about X.

Remark 9. Theorem 5 is not true if we suppose T satisfies the weaker hypothesis $\mu^*(T) > 0$. We can indeed consider the following counter-example.

Let $G = H = R$, regarded as a vector space over the field Q of rational numbers, and m the Lebesgue measure on R. If S is a Hamel basis of R over Q and s_0 a fixed element of S, then S_0 will denote the linear subspace of R over Q spanned by $S \setminus \{s_0\}$ (S_0 is a saturated non-measurable set (see [11])). Consider the function $f: R \to R$ so defined:

$$f(x) = 0 \text{ if } x \in S_0, \quad f(x) = 1 \text{ if } x \in S_0^c.$$

Clearly f is not additive, but it satisfies the functional equation (*) on the restricted domain $\Omega = (S_0 \times R) \cup (S_0^c \times S_0)$.

For every $s, t \in R$, $\varphi_{s,t} = S_0$ or $\varphi_{s,t*} = R$, and therefore $m_*(\varphi_{s,t}^c) = 0$. In this case, if we take $T = S_0$ we have $m^*(S_0) > 0$ and $m_*(\Omega_x) = m(R) > 0$ for every $x \in T$.

The proof of the following theorem is analogous to the previous one.

THEOREM 6. *Let G be a second category abelian topological group and H an abelian regular monoid. If there exists a second category set $T \subset G$ such that, for every $x \in T$, Ω_x is a second category set and moreover, for every $s, t \in G$, $\varphi_{s,t}^c$ is a first category set, then f is a homomorphism.*

THEOREM 7. *Let G be an abelian group, H an abelian regular monoid and $\Gamma \subset G$. If:*

a) $\varphi^\Gamma = G$,

b) *for every $(a, b) \in N$, there exist $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that*

$$(a+\gamma_1-\gamma_3, b+\gamma_3-\gamma_2) \in \Omega,$$

then f is a homomorphism.

Proof. a) implies $\Omega_0 \neq \emptyset$ and so we may assume $\Omega^0 = G$ (see Remark 3). Now we prove that there exist $\alpha, \beta, \gamma \in \varphi^\Gamma$ which satisfy the following conditions:

$$b+\beta+\gamma \in \Gamma, \quad b-\alpha \in \Gamma, \quad a+\beta \in \Gamma, \quad b+\gamma \in \Omega_{a+\alpha+\beta}.$$

This follows from b) if we choose $\gamma_1, \gamma_2, \gamma_3$ such that $a+\gamma_1-\gamma_3 \in \Omega_{b+\gamma_3-\gamma_2}$ and we denote $\alpha = b-\gamma_2$, $\beta = \gamma_3-a$ and $\gamma = a-b+\gamma_1-\gamma_3$.

If we assume $\delta = b+\gamma$ it is easy to see the hypotheses of Theorem 2 are satisfied. Indeed:

$$\alpha \in \varphi^\Gamma \subset \varphi_{a,a+\gamma_1} = \varphi_{a,a+b+\beta+\gamma} = \varphi_{a,a+\beta+\delta},$$
$$\beta \in \varphi^\Gamma \subset \varphi_{a+b-\gamma_2,(a+b-\gamma_2)+\gamma_2} = \varphi_{a+b,a+b-\gamma_2} = \varphi_{a+b,a+\alpha},$$
$$\gamma \in \varphi^\Gamma \subset \varphi_{b,b+\gamma_3} = \varphi_{b,a+b+\beta}$$

and

$$\delta \in \Omega_{a+\alpha+\beta} \cap [b+\gamma+(\Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma})]$$

(because $b+\gamma \in \Omega_{a+\alpha+\beta}$ and $0 \in \Omega_{b+\gamma} \cap \Omega_{a+b+\beta+\gamma}$). The proof is so complete.

Remark 9. If G is also a topological group and Γ is open, the hypotheses a) and b) can be replaced by:

a′) for every $t \in \Gamma$, $\varphi^{(0,t)}$ is dense in G,

b′) for every $(a, b) \in N$, there exist $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ such that

$$(a+\gamma_1-\gamma_3, b+\gamma_3-\gamma_2) \in \mathrm{Int}(\Omega).$$

Remark 10. As, in the case of groups, $\varphi^\Gamma = \varphi^{\Gamma \cup (-\Gamma)}$, we can suppose Γ symmetric. Furthermore, in order to satisfy condition b), Γ has to contain at least an element different from 0. Now, if $\Gamma_1 = \{\gamma_0\}$, $\Gamma_2 = \{0, \gamma_0\}$, $\Gamma_3 = \{0, \gamma_0, -\gamma_0\}$ we have $\varphi^{\Gamma_3} = \varphi^{\Gamma_2} = \varphi^{\Gamma_1} = \bigcap_{s \in G} \varphi_{s,s+\gamma_0}$ because, for every $s, t \in G$, $\varphi_{s,s} \subset \varphi_{s,t}$.

EXAMPLE 6. Let $G = (R, +)$ and

$$\Omega \supset \{(x, y): x+(8n-1) < y < x+(8n+1), n = 0, \pm 1, \pm 2, \ldots\}.$$

If we assume $\Gamma = \{0, 1, -1\}$ the hypotheses of Theorem 7 are satisfied and so f is a homomorphism.

Nevertheless there are some pairs (s, t) for which $\varphi_{s,t} = \{0\}$ and so Theorems 3, 4, 5, 6 cannot be applied.

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[11] If $A \subset G$, $\mu_*(A)$ and $\mu^*(A)$ are respectively the inner (or interior) and the outer (or exterior) measure induced by μ, that is (see [7]): $\mu_*(A) = \sup\{\mu(F): A \supset F \in \mathfrak{m}\}$, $\mu^*(A) = \inf\{\mu(F): A \subset F \in \mathfrak{m}\}$.

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INSTITUTO DI MATEMATICA
UNIVERSITÀ DI MILANO
Milano

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