(7) Is a cell slack if each 2-dimensional compactum in the cell is 1-ULC in E^n? What is the minimal dimension k such that the 1-ULC property of k-dimensional compacta implies that B is slack? (If $\dim B = m$, then $k \leqslant [(m+1)/2]$.)

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Metrizability of certain Pixley–Roy spaces

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Abstract. This paper studies metrizability of the Pixley–Roy hyperspace $\mathcal{F}[X]$ of a space X where X is a generalized ordered space of a certain type. For those generalized ordered spaces constructed from separable linearly ordered spaces, necessary and sufficient conditions for metrizability of $\mathcal{F}[X]$ are obtained. Metrization theorems for the hyperspace of other generalized ordered spaces are obtained by placing restrictions on the one-sided nature of neighborhoods. For example, it is proved that if X is any first-countable subspace of any ordinal, then $\mathcal{F}[X]$ is metrizable.

1. Introduction and definitions. In [PR] Carl Pixley and Prabir Roy presented an easily described space which could be used in place of an older and more complicated example given by Mary Ellen Rudin [R_1] in her study of completable Moore spaces. In today's terminology, Pixley and Roy associated with each space X one of its possible "hyperspaces", i.e., topological spaces whose ground-set is the power set $\mathcal{P}(X)$. It soon became apparent that Pixley and Roy had, in fact, discovered an elegant and useful general technique for constructing certain kinds of examples, and various versions of their construction have been studied in [PT], [vDTW] and [vD_1]. In this paper we give necessary and sufficient conditions for metrizability of the Pixley–Roy hyperspace of certain lines.

The lines on which our Pixley–Roy spaces are constructed are certain *generalized ordered spaces*. Begin with a linearly ordered set $(Y, <)$ and let λ denote the usual open-interval topology associated with $<$. Select three disjoint, possibly empty, subsets $A, B, C \subset Y$ and let τ be the topology on Y having the collection

$$\lambda \cup \{[x, y[\mid x \in A,\ y > x\} \cup \{]x, y] \mid x < y \in B\} \cup \{\{x\} \mid x \in C\}$$

as a base. The space (Y, τ) is then called a generalized ordered space on $(Y, <)$ and can be denoted by $GO_Y(A, B, C)$. The standard reference for the basic properties of generalized ordered spaces is [L] whose terminology and notation we usually follow.

The Pixley–Roy hyperspace of any space X is constructed as follows. Let $\mathcal{F}[X]$ be the collection of all nonempty finite subsets of X and topologize $\mathcal{F}[X]$ by using basic open neighborhoods of the form

$$[F, W] = \{S \in \mathcal{F}[X] \mid F \subset S \subset W\}$$

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where $F \in \mathscr{F}[X]$ and where W is an open set containing F. (Here we are following the notation of [vD$_1$].) For each positive integer n let

$$\mathscr{F}_n[X] = \{F \in \mathscr{F}[X] \mid \operatorname{card}(F) \leqslant n\}$$

be topologized as a subspace of $\mathscr{F}[X]$. Our first main result is

I. THEOREM. *Let $(Y, <)$ be a linearly ordered set whose usual interval topology λ is separable, and let $X = (Y, \tau)$ be a generalized ordered space constructed on Y. Then the following are equivalent:*

a) *$\mathscr{F}[X]$ is metrizable;*

b) *$\mathscr{F}_2(X)$ is metrizable;*

c) *if we define*

$$I = \{x \in Y \mid \{x\} \in \tau\},$$

$$L = \{x \in Y - I \mid \,]\leftarrow, x] \in \tau\},$$

$$R = \{x \in Y - I \mid [x, \rightarrow[\in \tau\},$$

$$E = Y - (I \cup L \cup R),$$

then

(i) *E is countable,*

(ii) *R and L are each F_σ-subsets of the space (S, τ_S) where $S = R \cup L$,*

(iii) *R can be written as $R = \bigcup \{R_n \mid n \in \omega_0\}$ in such a way that if $x \in E \cap \operatorname{cl}_\tau(R_n)$ then for some $y < x$, $]y, x[\cap R_n = \emptyset$,*

(iv) *L can be written as $L = \bigcup \{L_n \mid n \in \omega_0\}$ in such a way that if $x \in E \cap \operatorname{cl}_\tau(L_n)$ then for some $z > x$, $]x, z[\cap L_n = \emptyset$,*

(v) *the set $K_0 = \{x \in Y \mid x$ has a neighbor point (see below) x' in Y and neither x nor x' is τ-isolated$\}$ is countable.*

The reader must not conclude that the sets R, L, and I defined in (c) of Theorem I are the same as the sets A, B, C used to construct the topology τ from the topology λ. It happens that $R = A$, $L = B$, $I = C$ in case the initial linearly ordered set $(Y, <)$ is dense-ordered (i.e., if $a < b$ then for some $c \in Y$, $a < c < b$), but it is not true in general that $R = A$, $L = B$ and $I = C$. Consider the Alexandroff double arrow space $(Y, <)$, i.e., the lexicographically ordered set $[0, 1] \times \{0, 1\}$. Construct a generalized ordered space X on Y by isolating each point $(x, 0) \in Y$, i.e., by taking $C = \{(x, 0) \mid x \in [0, 1]\}$ and $A = B = \emptyset$. Then the sets I, R, L and E defined in part a) of Theorem I are

$$I = \{(x, 0) \mid x \in [0, 1]\},$$

$$L = \emptyset,$$

$$R = \{(x, 1) \mid x \in [0, 1]\},$$

$$E = \emptyset.$$

(We remark that Theorem I tells us that $\mathscr{F}[X]$ is metrizable.) Obviously it is the existence of neighbor points which allows this pathology to occur where, by *neighbor points* in $(Y, <)$ we mean two points $y < z$ such that $]y, z[= \emptyset$.

Let us mention several easy consequences of Theorem I in case the linearly ordered set Y is the usual set $\boldsymbol{R}$ of real numbers.

1) If S is the Sorgenfrey line $\mathrm{GO}_{\boldsymbol{R}}(\boldsymbol{R}, \emptyset, \emptyset)$ then $\mathscr{F}[S]$ is metrizable.

2) If M is the Michael line $\mathrm{GO}_{\boldsymbol{R}}(\emptyset, \emptyset, \boldsymbol{P})$, where $\boldsymbol{P}$ is the set of irrational numbers, then $\mathscr{F}[M]$ is metrizable.

3) If $\boldsymbol{R} = \mathrm{GO}_{\boldsymbol{R}}(\emptyset, \emptyset, \emptyset)$ then $\mathscr{F}(\boldsymbol{R})$ is not metrizable.

4) If X is the mixed Sorgenfrey line $X = \mathrm{GO}_{\boldsymbol{R}}(\boldsymbol{P}, \boldsymbol{Q}, \emptyset)$, where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the sets of irrational and rational numbers respectively, then $\mathscr{F}[X]$ is not metrizable.

(The first two results were announced without proof by van Douwen in [vD$_1$].)

Our paper is organized as follows. In Section 2 we introduce the Heath plane $H(X)$ associated with a generalized ordered space X and prove that $H(X)$ is homeomorphic to $\mathscr{F}_2[X]$. It is this space which enables us to conduct a detailed study of $\mathscr{F}_2[X]$ especially when X is a generalized ordered space constructed on $\boldsymbol{R}$ because in that case $H[X]$ is so easy to picture. In Section 3 we present our proof of Theorem I, above. In Section 4 we comment on the Pixley–Roy hyperspace of certain other lines—the Ostaszewski line, the Kunen line and the van Douwen line—and prove the second main result of our paper:

II. THEOREM. *Let X be a first-countable subspace of some space of ordinals. Then $\mathscr{F}[X]$ is metrizable.*

Readers are referred to a recent paper of Mary Ellen Rudin [R$_2$] in which delicate results are obtained concerning metrizability of $\mathscr{F}[X]$ where X is a subspace of a Souslin line.

2. The Heath plane of a generalized ordered space.

2.1. DEFINITION. Let $(Y, <)$ be a linearly ordered set and let $X = \mathrm{GO}_Y(A, B, C)$. Define $H(X)$ to be the set

$$H(X) = \{(x, y) \in Y^2 \mid x \leqslant y\}$$

and topologize $H(X)$ as follows:

a) each point $(x, y) \in H(X)$ with $x < y$ is isolated;

b) each point (x, x) with $x \in C$ is also isolated;

c) each point (x, x) with $x \in A$ has neighborhoods of the form $\{x\} \times [x, y[$ where $y > x$;

d) each point (x, x) with $x \in B$ has neighborhoods of the form $]y, x] \times \{x\}$ where $y < x$;

e) each point (x, x) where $x \in Y - (A \cup B \cup C)$ has neighborhoods of the form $(]y, x] \times \{x\}) \cup (\{x\} \times [x, z[)$ where $y < x < z$, except if x is an endpoint of Y, in which case only the relevant half of the above neighborhood is used.

2.2. Remark. If the generalized ordered space $X = (Y, \tau)$ in (2.1) happens to be first countable (as it will be in our paper) we can specify a countable neighborhood base at each point $(a, a) \in H(X)$ as follows.

If $[a, \to[\, \notin \tau$ choose a sequence $a''(0) < a'(1) < a'(2) < \ldots$ having supremum a and if $]\leftarrow, a] \notin \tau$ choose a sequence $a''(0) > a''(1) > a''(2) > \ldots$ having infimum a. Then:

a) if $a \in A$, let $h(n, a) = \{a\} \times [a, a''(n)[$,

b) if $a \in B$, let $h(n, a) =]a'(n), a] \times \{a\}$,

c) if $a \in C$ let $h(n, a) = \{(a, a)\}$,

d) if $a \in Y-(A \cup B \cup C)$ let $h(n, a) = (]a'(n), a] \times \{a\}) \cup (\{a\} \times [a, a''(n)[)$.

2.3. Remark on terminology. In [H], R. W. Heath introduced a class of particularly simple spaces which have since become standard in the study of Moore spaces and which are often called "Heath V-spaces". The reader will note that $H(R)$ as defined above, after 45°-rotation, is the usual Heath V-space and that if $X = \mathrm{GO}_R(P, Q, \emptyset)$ then (again after 45°-rotation) $H(X)$ is another familiar example of the Heath-type. These observations explain our choice of a name for the space $H(X)$.

2.4. Theorem. *Let $X = \mathrm{GO}_Y(A, B, C)$. Then $\mathscr{F}_2[X]$ is homeomorphic to $H(X)$.*

Proof. For $\{a\} \in \mathscr{F}_2[X]$ define $g(\{a\}) = (a, a)$ and for any two element set $S \in \mathscr{F}_2[X]$, define $g(S) = (\min(S), \max(S))$. The resulting function g is clearly a bijection and the topology induced on $H(X)$ by g is precisely the topology described in Definition (2.1). ■

2.5. Lemma. *Let $(Y, <)$ be a linearly ordered set whose usual open-interval topology λ is separable and let $(Y, \tau) = \mathrm{GO}_Y(A, B, C)$. Let $J = \{x \in Y \mid x$ has a neighbor point y and neither x nor y is τ-isolated$\}$. If $H(Y, \tau)$ is metrizable, then J is countable.*

Proof. Since (Y, λ) is separable, (Y, λ) is hereditarily separable [BL] and therefore any uncountable subset of (Y, λ) contains a λ-limit point of itself. Furthermore, since (Y, λ) is separable, it is first countable; hence so is (Y, τ). Let Q be any countable dense subset of (Y, λ).

For each $x \in Y$ let basic neighborhoods of (x, x) in $H(Y, \tau)$ be defined as in (2.2). Let $\mathscr{U} = \{h(1, x) \mid x \in Y\} \cup \{\{(x, y)\} \mid$ in $Y\}$. Since $\mathscr{U}$ is an open cover of the metrizable space $H(Y, \tau)$ there must be a locally finite open cover $\mathscr{V}$ of $H(Y, \tau)$ which refines $\mathscr{U}$. For each $x \in Y$, $h(1, x)$ is the unique member of $\mathscr{U}$ containing (x, x) so that if we choose $V(x) \in \mathscr{V}$ with $x \in V(x)$ then $V(x) \subset h(1, x)$ and hence if $x_1 \neq x_2$ belong to Y then $V(x_1) \neq V(x_2)$.

For contradiction, suppose J is uncountable. The points of J occur in pairs, namely x and its neighbor point $y(x)$. Let $J' = \{x \in J \mid x < y(x)\}$. Then J' is also uncountable. For each $x \in J'$, $x < y(x)$ and there is a point $u(x) \in Q$ having $y(x) < u(x)$ and $\{y(x)\} \times [y(x), u(x)[\, \subset V(y(x))$. Since Q is countable and J' is uncountable, there must be a point $q_0 \in Q$ such that the set $J'' = \{x \in J' \mid u(x) = q_0\}$ is uncountable. But then, with respect to the topology λ, J'' contains a limit point, say p, of itself. Since $]\leftarrow, p] =]\leftarrow, y(p)[$ is a λ-open set and since p is not τ-isolated, there must be a sequence $x(1) < x(2) < \ldots$ in J'' which τ-converges to p. But then every neighborhood of (p, p) in the space $H(Y, \tau)$ must intersect infinitely many of the sets $\{y(x(n))\} \times [y(x(n)), q_0[\, \subset V(y(x(n)))$ and therefore $\mathscr{V}$ cannot be locally finite at (p, p), which is impossible. ■

An entirely analogous argument establishes

2.6. Lemma. *Suppose $(Y, \tau) = \mathrm{GO}_Y(A, B, C)$ where the usual open-interval topology λ on Y is separable. Let $E = \{x \in Y \mid$ each τ-neighborhood of x contains points on both sides of $x\}$. If $H(Y, \tau)$ is metrizable, then E is countable.* ■

2.7. Lemma. *Let $(Y, \tau) = \mathrm{GO}_Y(A, B, C)$ where the usual open-interval topology λ on Y is separable. Suppose $H(Y, \tau)$ is metrizable. Let $I = \{x \in Y \mid \{x\} \in \tau\}$, $R = \{x \in Y-I \mid [x, \to[\, \in \tau\}$ and $L = \{x \in Y-I \mid]\leftarrow, x] \in \tau\}$. Then R and L can be written as $R = \bigcup \{R(n) \mid n \in \omega_0\}$ and $L = \bigcup \{L(n) \mid n \in \omega_0\}$ in such a way that if $x \in E = Y-(R \cup L \cup I)$ and if $x \in \mathrm{cl}_\tau(R_n)$ (respectively, $x \in \mathrm{cl}_\tau(L_n)$) then there is some point $y < x$ (respectively, some point $z > x$) such that $]y, x[\, \cap R_n = \emptyset$ (respectively, $]x, z[\, \cap L_n = \emptyset$).*

Proof. Let $\mathscr{B} = \bigcup \{\mathscr{B}(n) \mid n \in \omega_0\}$ be a σ-discrete base for $H(Y, \tau)$. Let Q be a countable dense subset of (Y, λ). For each $x \in Y$ choose a neighborhood system $\{h(n, x)\}$ for $(x, x) \in H(Y, \tau)$ as explained in (2.2). Let $R(n) = \{x \in R \mid$ for some (unique) $B(n, x) \in \mathscr{B}(n)$, $x \in B(n, x) \subset h(1, x)\}$. For each $x \in R(n)$ there is a point $u(n, x) \in Q$ such that $u(n, x) > x$ and $\{x\} \times [x, u(n, x)[\, \subset B(n, x)$. For each $q \in Q$ define $R(n, q) = \{x \in R(n) \mid u(n, x) = q\}$. For each $q \in (Q \cap E) \cup (Q \cap L)$ select a sequence $q(0) < q(1) < q(2) < \ldots$ having supremum q and define $R(n, q, k) = \{x \in R(n, q) \mid x \leqslant q(k)\}$ while for each $q \in Q-(E \cup L)$ and each k let $R(n, q, k) = R(n, q)$. Clearly $R = \bigcup \{R(n, q, k) \mid n, k \in \omega_0$ and $q \in Q\}$. Now fix $n, k \in \omega_0$ and $q \in Q$ and suppose that $x \in E \cap \mathrm{cl}_\tau(R(n, q, k))$. Suppose there is a sequence $r_1 < r_2 < \ldots$ of points of $R(n, q, k)$ having $x = \sup\{r_i \mid i \geqslant 1\}$. For each i, $r_i < u(r_i) = q$ so that $x \leqslant q$. If $x = q$ then each point y of $R(n, q, k)$ satisfies $y < q(k) < q$ so that $q = x = \sup\{r_i \mid i \geqslant 1\}$ is impossible. Hence $x < q$. But then, because $x \in E$, every basic neighborhood of (x, x) in $H(X)$ meets infinitely many of the sets $\{r_i\} \times [r_i, q[\, \subset B(n, r_i)$ and it follows from our choice of $B(n, r_i) \subset h(1, r_i)$ that the distinct points r_i correspond to distinct sets $B(n, r_i)$. But that is impossible because the collection $\mathscr{B}(n)$ is discrete in $H(Y, \tau)$ and hence the assertion of (2.7) about R is established. The corresponding assertion about the decomposition of L is analogously established. ■

2.8. Lemma. *With notation as in (2.7) and writing $S = R \cup L$, each of R and L is an F_σ-subspace of (S, τ_S).*

Proof. Write $R = \bigcup \{R(n, q, k) \mid n, k \in \omega_0, q \in Q\}$ as in the proof of (2.7). We assert that for each n, q and k, $L \cap \mathrm{cl}_\tau(R(n, q, k)) = \emptyset$. For fix $y \in L$. The proof given to show that (if $x \in E \cap \mathrm{cl}_\tau(R(n, q, k))$ then for some $x' < x$ the set $]x', x[$ is disjoint from $R(n, q, k)$) is still valid and yields a point $y' < y$ having $]y', y[\, \cap$

$\cap R(n, q, k) = \emptyset$. But then $]y', y]$ is a τ-neighborhood of y which is disjoint from $R(n, q, k)$. Therefore we see that

$$R = \bigcup \{(S \cap \mathrm{cl}_\tau(R(n, q, k))| \ n, k \in \omega_0, q \in Q\}$$

is indeed an F_σ-subspace of (S, τ_S). The assertion about L is proved similarly. ∎

2.9. Remark. After completing this paper, the authors learned that Jerry Vaughan had also observed that Heath V-spaces can be embedded in suitable Pixley–Roy spaces.

3. Proof of Theorem I. In this section we complete the proof of Theorem I of the Introduction. Obviously if $X = (Y, \tau)$ and if $\mathscr{F}[X]$ is metrizable then so is its subspace $\mathscr{F}_2[X]$. Thus (a) implies (b). That (b) implies (c) is the content of Section 2 so that it will be enough to prove that (c) implies (a). Since $\mathscr{F}[X]$ is known to be regular, we can complete the proof by exhibiting a σ-locally finite base for the topology of $\mathscr{F}[X]$.

This section is divided into three parts. In (3.1) we define a collection Φ; in (3.2) we prove that Φ is σ-locally finite; in (3.3) we prove that Φ is a base for $\mathscr{F}[X]$. It will be convenient (in the proof of (3.3)) to assume that the linearly ordered set $(Y, <)$ has no endpoints. This assumption involves no loss of generality, for if Y has a right endpoint q we simply adjoin a copy of ω_0 to the right of q and if Y has a left endpoint p we adjoin a copy of ω_0^*, the set ω_0 with the ordering reversed, before p.

3.1. Definition of the collection Φ. Let λ be the usual open-interval topology on Y and let Q be a countable dense subset of (Y, λ). Let J be a nonempty finite subset of Q, say $J = \{q_i| \ 0 \leqslant i \leqslant k\}$, where $q_0 < q_1 < \ldots < q_k$. We will say that an element $F \in \mathscr{F}[X]$ *interlaces* J provided $F \cap J = \emptyset$ and for each $i \in \{1, 2, \ldots, k\}$ the set $F \cap]q_{i-1}, q_i[$ is a non-empty order-convex subset of Y.

Next let $K_1 = \{x \in X| \ x$ has a neighbor point x' in Y and at least one of x and x' is λ-isolated$\}$. Because (Y, λ) is separable, K_1 is countable. Define $\hat{K} = K_0 \cup K_1 \cup E$ where K_0 and E are as described in part (c) of the theorem. Then $\hat{K}$ is countable.

Because R and L are each F_σ-subsets of (S, τ_S) there are τ-closed sets $C_0 \subset C_1 \subset C_2 \subset \ldots$ and $D_0 \subset D_1 \subset D_2 \subset \ldots$ such that

$$L = \bigcup \{S \cap C_n| \ n \in \omega_0\} \text{ and } R = \bigcup \{S \cap D_n| \ n \in \omega_0\},$$

and where

$$R \cap C_n = \emptyset = L \cap D_n \quad \text{for each } n \in \omega_0.$$

Now for each finite set $J \subset Q$ and each finite set $K \subset \hat{K}$ and each $n \in \omega_0$, define

$$\mathscr{F}(J, K, n) = \{F \in \mathscr{F}[X]| \ F \text{ interlaces } J, \ F \cap \hat{K} = K,$$
$$F \cap R \subset (\bigcup_{i=0}^{n} R_i) \cap (\bigcup_{i=0}^{n} D_i), \ F \cap L \subset (\bigcup_{i=0}^{n} L_i) \cap (\bigcup_{i=0}^{n} C_i)\}.$$

For each $x \in X$, select a τ-neighborhood base $\{g(n, x)| \ n \in \omega_0\}$ at x such that $g(n, x) \supset g(n+1, x)$ and such that

a) if $x \in I$ then $g(n, x) = \{x\}$ for each n;

b) if $x \in R$ then $g(n, x) \subset [x, \rightarrow[$ for each n;

c) if $x \in L$ then $g(n, x) \subset]\leftarrow, x]$ for each n;

d) each $g(n, x)$ is order-convex.

For any finite set $F \in \mathscr{F}[X]$ define $G(n, F) = \bigcup \{g(n, x)| \ x \in F\}$ and for each J, K and n as above and each $l \in \omega_0$ define

$$\Phi(J, K, n, l) = \{[F, G(l, F)]| \ F \in \mathscr{F}(J, K, n) \text{ and the collection}$$
$$\{g(l, x)| \ x \in F\} \text{ is pairwise disjoint and } G(l, F) \cap J = \emptyset\}.$$

Finally define

$$\Phi = \bigcup \{\Phi(J, K, n, l)| \ J \subset Q, K \subset \hat{K} \text{ are finite}; \ n, l \in \omega_0\}.$$

3.2. Lemma. *Each collection* $\Phi(J, K, n, l)$ *is locally finite in* $\mathscr{F}[X]$.

Proof. Fix $T \in \mathscr{F}[X]$ and write $\psi = \Phi(J, K, n, l)$. If $T \cap J \neq \emptyset$ then $[T, X]$ is a neighborhood of T meeting no member of ψ. If $K \not\subset T$ choose $p \in K - T$; then $[T, X - \{p\}]$ is a neighborhood of T meeting no member of ψ. Hence assume that both $T \cap J = \emptyset$ and $K \subset T$.

We will define a τ-open set W containing T such that the open set $[T, W]$ meets at most $4^{\mathrm{card}(T)}$ members of ψ. We begin by letting

$$T' = \{x \in X| \ x \text{ is a neighbor point of some } t(x) \in T \text{ and neither } x \text{ nor } t(x) \text{ is } \tau\text{-isolated}\}.$$

Let $\tilde{T} = T \cup T'$. Then $\mathrm{card}(\tilde{T}) \leqslant 2 \, \mathrm{card}(T)$. Because the set T is finite there is an integer m so large that:

a) the collection $\{g(m, t)| \ t \in T\}$ is pairwise disjoint and $J \cap (\bigcup \{g(m, t)| \ t \in T\}) = \emptyset$;

b) if $t \in R \cap T$ then $g(m, t) \cap (\bigcup \{C(i)| \ 0 \leqslant i \leqslant n\}) = \emptyset$ and if $t \in L \cap T$ then $g(m, t) \cap (\bigcup \{D(i)| \ 0 \leqslant i \leqslant n\}) = \emptyset$;

c) if $t \in E \cap T$ and if $t \notin \mathrm{cl}_\tau(R(i))$ for some $i \in \{0, 1, \ldots, n\}$ then $g(m, t) \cap R(i) = \emptyset$ and if $t \notin \mathrm{cl}_\tau(L(j))$ for some $j \in \{0, 1, 2, \ldots, n\}$ then $g(m, t) \cap L(j) = \emptyset$;

d) if $t \in E \cap T$ and if $t \in \mathrm{cl}_\tau(R(i))$ for some $i \in \{0, 1, 2, \ldots, n\}$ then $g(m, t) \cap]\leftarrow, t[\cap R(i) = \emptyset$ and if $t \in \mathrm{cl}_\tau(L(j))$ for some $j \in \{0, 1, \ldots, n\}$ then $g(m, t) \cap]t, \rightarrow[\cap L(j) = \emptyset$.

(Because of (iii) and (iv) in condition (c) of Theorem I, we can arrange a).) Now let $W = G(m, T) = \bigcup \{g(m, t)| \ t \in T\}$.

To complete the proof of (3.2) suppose $[F, G(l, F)] \in \psi$ and $[F, G(l, F)] \cap [T, W] \neq \emptyset$. Since $[F, G(l, F)] \cap [T, W] \neq \emptyset$ we have $T \cup F \subset G(l, F) \cap W$. We will show that $F \subset \tilde{T}$. The major step in establishing $F \subset \tilde{T}$ is the following

SUBLEMMA. *Suppose* $z \in F$ *and* $t \in T$ *satisfy* $t \in g(l, z)$ *and* $z \in g(m, t)$. *Then* $z \in T$.

Proof. If $t \in I$ then $z \in g(m, t) = \{t\}$ and there is nothing to prove, so assume $t \notin I$. If $z \in \hat{K}$ then $z \in \hat{K} \cap F = K \subset T$ and again there is nothing to prove, so assume $z \notin \hat{K}$. In particular, $z \notin E$ so $z \in R \cup I \cup L$. If $z \in I$ then $t \in g(l, z) = \{z\}$ which forces $z = t \in T$, so assume $z \notin I$; hence $z \in R \cup L$. Consider the case where $z \in R$ (the other case being analogous). Then $t \in g(l, z) \subset [z, \rightarrow[$ so that $z \leqslant t$. For contradiction, suppose $z < t$. If $t \in R$ then $z \in g(m, t) \subset [t, \rightarrow[$; since $z < t$, $t \notin R$. We already know that $t \notin I$. Consider the case where $t \in L$. Then, by choice of m, $g(m, t) \cap \cap (\bigcup \{D_i|\ 0 \leqslant i \leqslant n\}) = \emptyset$ and that is impossible because $z \in g(m, t) \cap \cap (\bigcup \{D_i|\ 0 \leqslant i \leqslant n\})$. Hence $t \in E$. Choose $i \in \{0, 1, \ldots, n\}$ such that $z \in R_i$. Because $z \in R_i \cap g(m, t)$, it follows from our choice of m that $t \in \mathrm{cl}_\tau(R_i)$. But then $g(m, t) \cap]\leftarrow, t[\cap R_i = \emptyset$ contrary to the fact that $z \in g(m, t) \cap R_i$ and $z < t$. Therefore $z = t \in T$ and the sublemma is proved.

We now return to the proof of (3.2). For contradiction, suppose some x satisfies

(*) $x \in F$ and yet $x \notin \hat{T}$.

Because $F \cap \hat{K} = K \subset T \subset \hat{T}$ we must have

(**) $x \notin \hat{K}$.

Now choose the unique $t \in T$ having $x \in g(m, t)$. Choose the unique i having $x \in F \cap]q_{i-1}, q_i[$. Because the convex set $g(m, t)$ meets $]q_{i-1}, q_i[$ and contains neither q_{i-1} nor q_i, $t \in g(m, t) \subset]q_{i-1}, q_i[$. Hence the unique $y \in F$ having $t \in g(l, y)$ must have $y \cap]q_{i-1}, q_i[$. If $y = x$ the sublemma could be applied (with $z = x$) to conclude that $x \in T$, contrary to (*). Therefore $\mathrm{card}(F \cap]q_{i-1}, q_i[) > 1$. If card $(F \cap]q_{i-1}, q_i[) \geqslant 3$ then, because F interlaces J, we would have $x \in F \cap]q_{i-1}, q_i[\subset K_1 \subset \hat{K}$, contrary to (**). Therefore

(***) $\mathrm{card}(F \cap]q_{i-1}, q_i[) = 2$ and $F \cap]q_{i-1}, q_i[= \{x, y\}$ where x and y are neighbor points.

Consider the case where $x < y$; the other case is analogous. If $\{x, y\} \cap T \neq \emptyset$ then $x \in \hat{T}$ contrary to (*); hence $\{x, y\} \cap T = \emptyset$. But then either $q_{i-1} < t < x < y < q_i$ or else $q_{i-1} < x < y < t < q_i$. If $t < x < y$ then $g(l, y) \subset [y, \rightarrow[$ so that $t \in g(l, y)$ would be impossible. Hence $x < y < t$. But then because $x \in g(m, t)$, convexity of $g(m, t)$ forces $y \in [x, t] \subset g(m, t)$ and so we have $y \in g(m, t)$ and $t \in g(l, y)$. Now the sublemma may be applied (with $z = y$) to conlcude that $y \in T$. But then in the light of (***), $x \in \hat{T}$, contrary to (*). Therefore, assumption (*) is untenable and the proof of (3.2) is complete. ■

3.3. LEMMA. *The collection* $\Phi = \bigcup \{\Phi(J, K, n, l)|\ J \subset Q$ *and* $K \subset \hat{K}$ *are both finite and* $n, l \in \omega_0\}$ *is a base for* $\mathscr{F}[X]$.

Proof. Suppose $[F, W]$ is a basic open set in $\mathscr{F}[X]$. Define an equivalence relation $\sim$ on F by the rule that $x \sim y$ if the set $\{z \in F|\ z$ lies between x and $y\}$ is

an order-convex subset of Y. Let $F_1, F_2, \ldots, F_k$ be the distinct equivalence classes of $\sim$ listed in such a way that if $a \in F_i$ and $b \in F_{i+1}$ then $a < b$. Let $x_i' = \min(F_i)$ and $x_i'' = \max(F_i)$. Then $x_i' \leqslant x_i'' < x_{i+1}'$. Furthermore $]x_i'', x_{i+1}'[\neq \emptyset$ because the equivalence classes F_i and F_{i+1} are distinct. Therefore for $1 \leqslant i < k$ we may choose $q_i \in Q \cap]x_i'', x_{i+1}'[$. Since Y has no endpoints we may also choose $q_0, q_k \in Q$ with $q_0 < x_1'$ and $x_k'' < q_k$. Then F interlaces the set $J = \{q_i|\ 0 \leqslant i \leqslant k\}$. Let $K = \hat{K} \cap F$. Since F is finite, there is an $n \in \omega_0$ having

$$F \cap R \subset (\bigcup \{D_i|\ 0 \leqslant i \leqslant n\}) \cap (\bigcup \{R_i|\ 0 \leqslant i \leqslant n\})$$

and

$$F \cap L \subset (\bigcup C_i|\ 0 \leqslant i \leqslant n\}) \cap (\bigcup \{L_i|\ 0 \leqslant i \leqslant n\}).$$

Then $F \in \mathscr{F}(J, K, n)$. Next observe that since $\{g(j, x)|\ j \in \omega_0\}$ is a neighborhood base at x for each $x \in F$, it is possible to choose l so large that the collection $\{g(l, x)|\ x \in F\}$ is disjoint and $G(l, F) \cap J = \emptyset$. But then $[F, G(l, F)]$ belongs to $\Phi(J, K, n, l) \subset \Phi$ and $F \in [F, G(l, F)] \subset [F, W]$. ■

4. Certain other lines.

In $[\mathrm{vD}_1]$, van Douwen announced that $F[[0, \omega_1]]$ is metrizable. This fact is a corollary of our first theorem as well as of Theorem II of Section 1.

4.1. THEOREM. *Suppose X is a first-countable T_1 space in which each point has a countable neighborhood. Then $\mathscr{F}[X]$ is metrizable.*

Proof. Since X is first-countable, $\mathscr{F}[X]$ is a regular, metacompact, first-countable space. Fix $F \in \mathscr{F}[X]$ and for each $x \in X$ let $B(x)$ be a countable open set containing x. Let $V = \bigcup \{B(x)|\ x \in F\}$. The set $[F, V]$ is a countable, open subspace of $\mathscr{F}[X]$; since $\mathscr{F}[X]$ is first-countable, $[F, V]$ is metrizable. Thus we have proved that $\mathscr{F}[X]$ is locally separable metrizable and the conclusion now follows from our next lemma.

4.2. LEMMA. *If Y is regular, metacompact, and locally separable metrizable, then Y is metrizable.*

Proof. Let $\mathscr{U}$ be a point-finite open cover of Y by separable, metrizable subsets. Then for each $U_0 \in \mathscr{U}$, the collection $\mathscr{S}\mathrm{t}(U_0, \mathscr{U}) = \{U \in \mathscr{U}|\ U \cap U_0 \neq \emptyset\}$ is countable. If $n \geqslant 1$ define $\mathscr{S}\mathrm{t}^{n+1}(U_0, \mathscr{U}) = \{U \in \mathscr{U}|$ for some $U_1 \in \mathscr{S}\mathrm{t}^n(U_0, \mathscr{U}), U \cap U_1 \neq \emptyset\}$. Each $\mathscr{S}\mathrm{t}^n(U_0, \mathscr{U})$ is countable; hence so is the collection $\mathscr{C}(U_0) = \bigcup \{\mathscr{S}\mathrm{t}^n(U_0, \mathscr{U}\}: n \geqslant 1\}$. Let $C(U_0) = \bigcup \mathscr{C}(U_0)$. It is clear that if $C(U_0) \neq C(U_1)$ for $U_0, U_1 \in \mathscr{U}$, then $C(U_0) \cap C(U_1) = \emptyset$. Thus $\{C(U)|\ U \in \mathscr{U}\}$ is a partition of Y into open and closed subspaces, so it is sufficient to show that each subspace $C(U)$ is metrizable. But that is immediate since $C(U)$ is regular and, being a countable union of open, second countable subspaces, is also second countable. ■

4.3. COROLLARY. *If X is any of the following spaces, then $\mathscr{F}[X]$ is metrizable:*

a) $[0, \omega_1[$;

b) $[0, \omega_1[^n$ *where n is finite;*

c) *the Kunen line* [JKR];

d) *the Ostaszewski line* [O];

e) *the van Douwen line* [vD_2] *or the van Douwen–Wicke line* [vDW].

Proof. Each of the listed spaces is locally countable and first countable. ■

Let us now turn to a proof of Theorem II of Section 1. Indeed, we prove a more general, but more technical, result. We say that a generalized ordered space (X, τ) is of *Sorgenfrey type* either if $[x, \to[\, \in \tau$ for every $x \in X$ or if $]\leftarrow, x] \in \tau$ for each $x \in X$. Obviously, the familiar Sorgenfrey line and any space of ordinals are of Sorgenfrey type. We will prove:

4.4. Theorem. *Let X be a first-countable generalized ordered space of Sorgenfrey type. Then $\mathcal{F}[X]$ is metrizable.*

Proof. We consider the case where $]\leftarrow, x] \in \tau$ for each $x \in X$. For each $x \in X$ choose a sequence $\langle a(x, n)\rangle$ of points of X such that $a(x, n) = x$ for every n if $\{x\} \in \tau$ and such that $a(x, n) < a(x, n+1)$ has $\sup\{a(x, n)|\ n \in \omega_0\} = x$ otherwise. For each x and n, let $g(x, n) =]a(x, n), x]$ if $\{x\} \notin \tau$ and $g(x, n) = \{x\}$ if $\{x\} \in \tau$. If $F \in \mathcal{F}[X]$, we let $n(F)$ be the least integer k such that the collection $\{g(x, k)|\ x \in F\}$ is pairwise disjoint. For each $k \in \omega_0$, let $G(k, F) = \bigcup \{g(x, k)|\ x \in F\}$.

For each $k \in \omega_0$, define a collection $\Phi(k)$ by

$$\Phi(k) = \{[F, G(k, F)]|\ F \in \mathcal{F}[X],\ n(F) \leqslant k\}.$$

Clearly $\bigcup \{\Phi(k)|\ k \in \omega_0\}$ is a base for $\mathcal{F}[X]$, so that, $\mathcal{F}[X]$ being regular, it is enough to show that each $\Phi(k)$ is locally finite in $\mathcal{F}[X]$. To that end, fix $k \in \omega_0$ and fix $T \in \mathcal{F}[X]$. Index T as $T = \{t_i|\ 1 \leqslant i \leqslant m\}$ where $t_1 > t_2 > \ldots > t_m$. Let $i_1 = 1$. Let i_{j+1} be the first integer with $t_{i_{j+1}} \notin g(t_{i_j}, k)$. Since T is finite, this induction terminates after, say, J steps. Let $T' = \{t_{i_j}|\ 1 \leqslant j \leqslant J\}$. Let $W = G(k, T')$. Then W is open and $T \subset W$ so that $[T', W]$ is a neighborhood of T in $\mathcal{F}[X]$. Suppose $[T', W]$ meets a set $[F, G(k, F)] \in \Phi(k)$. Let $x_1 = \max(F)$. Then $x_1 = t_{i_1}$ so that $g(x_1, k) = g(t_{i_1}, k)$. Because $k \geqslant n(F)$, the point $x_2 = \max(F - \{x_1\})$ has

$$x_2 = \max(F - g(x_1, k))$$

and, for the same reason that $x_1 = t_{i_1}$, $x_2 = t_{i_2}$. This induction continues, showing that $F = T'$, and hence that T does have a neighborhood which meets only a finite number of members of $\Phi(k)$, as required. ■

4.5. Remark. The technique of (4.4) can be modified slightly to show that if X is a first countable space of Sorgenfrey type, then $\mathcal{F}[X \times X]$ is metrizable, thereby generalizing 4.3(b).

4.6. Question. Is it true that for each ordinal α, the Pixley–Roy space $\mathcal{F}[[0, \alpha)]$ is paracompact? If so, then Theorem II of the Introduction is an immediate corollary since for each first-countable subspace X of $[0, \alpha)$, $\mathcal{F}[X]$ is a closed subspace of $\mathcal{F}[[0, \alpha)]$ so that $\mathcal{F}[X]$ would be a paracompact Moore space.

Added in proof. Question 4.6 was answered affirmatively in the paper *Ultraparacompactness in certain Pixley–Roy spaces* by Bennett, Fleissner and Lutzer. That paper will appear in Fundamenta Mathematicae.

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