Squeezing m-cells to $(m-1)$-cells in E^n

by

C. D. Bass (Pembroke, N. C.)

Abstract. An m-cell B in E^n is said to be *squeezed* to an $(m-1)$-cell if there is a map f of E^n onto E^n such that $f(B)$ is an $(m-1)$-cell, $f|(E^n-B)$ is a homeomorphism onto $E^n-f(B)$, and $f|B$ is related to a natural projection onto a spanning $(m-1)$-cell in B. It is known that every 2-cell (3-cell) in E^3 can be squeezed to a 1-cell (2-cell), and that there exist m-cells in E^n ($n \geqslant 4$, $2 \leqslant m \leqslant n-1$) which cannot be squeezed to $(m-1)$-cells. In this paper we develop a technique for squeezing m-cells in dimensions $n \geqslant 5$. We then apply this technique to derive sufficient conditions for a cell to be squeezable, and to distinguish some wildly embedded m-cells which can be squeezed in many different ways to cells of lower dimension.

1. Introduction. We are interested here in the problem of squeezing an m-cell in Euclidean space E^n to an $(m-1)$-cell in E^n. By an m-cell we mean a topological image of the set

$$B^m = \{(x_1, \ldots, x_m) \in E^m |\ x_1^2 + \ldots + x_m^2 \leqslant 1\}.$$

The boundary of this set we denote as S^{m-1}. A topological image of this set is known as an $(m-1)$-sphere. There is a canonical projection map π from B^m onto B^{m-1} defined by the equation: $\pi(x_1, \ldots, x_{m-1}, x_m) = (x_1, \ldots, x_{m-1})$. The so-called squeezing problem is to find a map of E^n onto E^n which realizes the effect of π on an m-cell embedded in E^n. More precisely, let B denote an m-cell in E^n ($m \geqslant 2$). We say that a map f of E^n onto E^n *squeezes* B *to an* $(m-1)$-*cell* B' if there exist embeddings g from B^m onto B, and h from B^{m-1} onto B' such that (i) $f|(E^n-B)$ is a homeomorphism onto E^n-B' and (ii) $f \circ g = h \circ \pi$. We say that B can be *squeezed to* B' *along* g or that g *supports a squeezing map*.

Eaton and Daverman proved in [12] and [13] that each 2-cell (3-cell) in E^3 can be squeezed to an arc (disk). It was found subsequently, that their techniques could be used to prove that each 2-cell in E^n can be squeezed to an arc. However, Daverman [6] gave examples of m-cells in E^n, $2 < m \leqslant n-1$, $n \geqslant 4$, which cannot be squeezed at all to $(m-1)$-cells. Our present efforts will yield some conditions that are sufficient for a cell to be squeezable. Yet, there remain some interesting questions regarding the incomplete classification of squeezable cells. Some of these are stated at the conclusion of this paper.

Our approach to squeezing problems emphasizes the role of the embedding g from B^m onto an m-cell B. A useful idea is to associate with each such embedding g the decomposition G of E^n whose non-degenerate elements are the particular spanning arcs $g\pi^{-1}(x)$, $x \in \operatorname{Int} B^{m-1}$, in B. The question, whether B can be squeezed to an $(m-1)$-cell along g, is equivalent to the question: is the decomposition space E^n/G homeomorphic to E^n. A principle result, Theorem 2.1, gives a sufficient condition for an m-cell to be squeezed along a particular embedding g for dimensions $2 \leqslant m \leqslant n-1$, $n \geqslant 5$. A similar result for cells of dimension n is a consequence of the "mismatch theorem" for sewings of crumpled cubes, [14, Theorem 3] and [7, Theorem 5.1]. The author is indebted to Robert Daverman for stimulating conversations and suggestions regarding the development of these results.

2. The main results. The theorems stated in this section, our main objectives, are proved in Section 5. The material in Sections 3 and 4 is confined to technical preliminaries similar to well-known techniques of [3] and [7]. Section 6 treats further consequences of these results.

Let "d" denote the usual metric on E^n. If X is a compact space, f and g are continuous functions from X into E^n; we define the distance ϱ between f and g by $\varrho(f, g) = \sup_{x \in X} d(f(x), g(x))$. A set $Y \subset E^n$ is 1-ULC, *uniformly locally simply connected*, if to each $\varepsilon > 0$ there corresponds a number $\delta > 0$ such that each map $f\colon S^1 \to Y$, for which $\operatorname{diam} f(S^1) < \delta$, extends to a map $g\colon B^2 \to Y$, for which $\operatorname{diam} g(B^2) < \varepsilon$.

THEOREM 2.1. *Let B denote an m-cell in E^n, $n \geqslant 5$, $2 \leqslant m \leqslant n-1$. Suppose that $g\colon B^m \to B$ is a homeomorphism onto B; F is an F_σ set in B such that*

1) *$(E^n - B) \cup F$ is 1-ULC, and*

2) *each arc $g\pi^{-1}(x)$, $x \in \operatorname{Int} B^{m-1}$, contains at most one point of F.*

Then B can be squeezed to an $(m-1)$-cell along g.

The use of a "local homotopy" criterion in this result is motivated in part by an example, due to Eaton, of a "dogbone" decomposition of E^n into points and tame arcs [15]. This construction can be used to define an embedding $g\colon B^2 \to E^n$ such that $g(B^2)$ is locally flat modulo a Cantor set C in $g(BdB^2)$; yet, g does not support a squeezing map. The hypothesis of Theorem 2.1 fails to apply because some of the arcs $g\pi^{-1}(x)$ contain two points of C. To perceive the limitations of this result, one can exploit a construction of Daverman [7, Example 13.2] to define an embedding $g'\colon B^2 \to E^n$ such that $g'(B^2)$ is locally flat modulo a Cantor set C in $g'(BdB^2)$. Again, unusual properties of g' and C imply that the hypothesis of Theorem 2.1 is not satisfied. However, the embedding g' does support a squeezing map. A careful comparison of these two examples might suggest how to strengthen Theorem 2.1.

THEOREM 2.2. *Let B denote an m-cell in E^n, $n \geqslant 5$, $2 \leqslant m \leqslant n-1$. Suppose that B contains a 0-dimensional F_σ set F such that $(E^n - B) \cup F$ is 1-ULC and F is locally flat relative to B and BdB. Then for each homeomorphism g of B^m onto B and each number $\varepsilon > 0$, there exists a homeomorphism $g'\colon B^m \to B$ such that $\varrho(g, g') < \varepsilon$ and B can be squeezed to an $(m-1)$-cell along g'.*

Some special types of m-cells to which the latter theorem applies are discussed in Section 5. These examples, of course, all share the property that many different embeddings g of B^m can be used to do the squeezing. Daverman's construction [8] can be used to give examples, not subject to the hypothesis of Theorem 2.2, which can be squeezed to cells of lower dimension. It remains an open question, however, whether the hypothesis of this theorem is a necessary condition for the existence of a dense set of embeddings which support squeezing maps. The author does not know if there exists an m-cell B $(m \geqslant 3)$ in E^n $(n \geqslant 4)$ with the property that there is a unique homeomorphism $g\colon B^m \to B$, along which B can be squeezed. In Section 6 we will discuss the case that every homeomorphism from B^m onto B supports a squeezing map.

3. Special subsets of an m-cell. In this section we derive a technical result related to the hypothesis that a topological m-cell B in E^n contains nicely positioned subsets which can be used to confine homotopies of simple closed curves near B.

For notational convenience we will use the unit m-cube I^m, the m-fold product of $I = \{0 \leqslant t \leqslant 1\}$, as a standard model for m-cells, in lieu of B^m. Another notation is $N_\varepsilon(X)$, the set of points within distance ε from a compact set X, known as the *ε-neighborhood* of X. The *distance between two compact sets* X and Y is defined by $d(X, Y) = \inf_{x \in X, y \in Y} d(x, y)$. An *$\varepsilon$-map* (*$\varepsilon$-homeomorphism*) of X is a map (homeomorphism) $f\colon X \to E^n$ such that $d(x, f(x)) < \varepsilon$ for each $x \in X$.

DEFINITION. Let g denote a homeomorphism of I^m into E^n; ψ the natural projection of I^m onto I^{m-1}; ε a positive number. An isotopy h_t of E^n *moves points ε-parallel to fibers* if and only if, for each $x \in E^n$, either $h_t(x) = x$, $0 \leqslant t \leqslant 1$, or there exists $y \in I^{m-1}$ such that $h(x \times [0, 1]) \subset N_\varepsilon(g\psi^{-1}(y))$.

LEMMA 3.1. *Let g denote a homeomorphism of I^m onto an m-cell B in E^n; $n \geqslant 5$, $2 \leqslant m \leqslant n-1$; F an F_σ set in B such that $(E^n - B) \cup F$ is 1-ULC. Let a be a point in $(0, 1)$; P a 2-dimensional polyhedron in E^n such that $P \cap B \subset g(I^{m-1} \times [0, a))$. Then, for each $\varepsilon > 0$ there exists a homotopy $f_t\colon P \to E^n$ such that*

1) *f_0 is the inclusion,*

2) *$f_t | P - N_\varepsilon(g(I^{m-1} \times [0, a]))$ is the inclusion,*

3) *$f_t(P) \cap g(I^{m-1} \times [a, 1]) = \emptyset$ for each t, $0 \leqslant t \leqslant 1$,*

4) *for each $x \in P$ either $f_t(x) = x$ for all t, or there exists $y \in I^{m-1}$ such that $f_t(x) \in N_\varepsilon(g(y \times [0, 1]))$,*

5) *$f_1(P) \cap B \subset (E^n - B) \cup F$.*

Proof. Case 1. $m \leqslant n-2$. Put

$$\eta = \min\{\varepsilon/2, d(P, g(I^{m-1} \times [a, 1]))\}.$$

Choose $\delta>0$, $0<\delta\leqslant\eta$ such that each loop in $(E^n-B)\cup F$ having diameter $<\delta$ bounds a singular disk in $(E^n-B)\cup F$ having diameter $<\eta/3$. Choose a triangulation T of P having mesh $<\delta/3$. Using the locally non-separating property of B, we obtain a $\delta/3$-map h of the 1-skeleton T^1 of T into E^n-B such that $h|T^1-N_\eta(g(I^{m-1}\times[0,a]))$ is the inclusion. Now, since $h(T^1)\cap B=\emptyset$ and $\operatorname{diam} h(\sigma_i)<\delta$ for each 2-simplex σ_i in T, each map $h|\dot\sigma_i$ extends to a map $k_i\colon \sigma_i\to E^n$ such that $k_i(\sigma_i)\subset(E^n-B)\cup F$ and $\operatorname{diam} k_i(\sigma_i)<\eta/3$. Thus, there is an η-map $k\colon P\to(E^n-B)\cup F$ defined by:

$$k(x)=\begin{cases}k_i(x), & \text{if } x\in\sigma_i \text{ for some } \sigma_i \text{ which intersects } N_\eta(g(I^{m-1}\times[0,a]))\,,\\ x, & \text{otherwise.}\end{cases}$$

The required homotopy f_t satisfying (1)-(5) is the standard "straigh-line" homotopy between the inclusion map and this map k.

Case 2. $=n-1$. Choose a point a_1 in $(0,a)$ such that $P\cap B\subset g(I^{m-1}\times[0,a_1])$. Let $r_t\colon E^n\to E^n$ denote the natural homotopy between the identity r_0 and a retraction r_1 of E^n onto B. Choose $(m-1)$-cells $B_1,\ldots,B_k$ in I^{m-1} subject to the following: $\operatorname{Int} I^{m-1}=\bigcup_{i=1}^{n}\operatorname{Int} B_i$; for each i either $B_i\cap\operatorname{Bd} I^{m-1}=\emptyset$ or $B_i\cap\operatorname{Bd} I^{m-1}$ is an $(m-2)$-cell in $\operatorname{Bd} B_i$; for each i there exists $x_i\in\operatorname{Int} B_i$ such that $g(B_i\times[0,a_1])\subset N_\varepsilon(g(x_i\times[0,a_1]))$. For each $i=1,\ldots,k$ let U_i be an open subset of E^n such that

$$r_t(U_i)=g\big((B_i-\operatorname{cl}(\operatorname{Bd} B_i\cap\operatorname{Int} I^{m-1}))\times[0,a_1)\big)$$

and

$$r_t(U_i)\subset N_\varepsilon(g(x_i\times[0,a_1]))-g(I^{m-1}\times[a,1])\,.$$

Note that any two maps from B^2 into U_i are homotopic in $N_\varepsilon(g(x_i\times[0,a_1]))$ missing $g(I^{m-1}\times[a,1])$. The homotopy f_t that we wish to define will adjust 2-simplexes of P within these neighborhoods in accordance with requirements (2) and (4).

The next step is to distinguish certain open subsets of each U_i which help to confine homotopies of simple closed curves near B in accordance with condition 5. For each $i=1,\ldots,k$ choose connected open sets V_i, V_{i1}, V_{i2}, and V_{i3} with the following properties:

$$V_i\cap B=g(\operatorname{Int} B_i\times(0,a_1));$$

$$V_{i1}\cap V_{i2}=\emptyset;\quad V_{i1}\cup V_{i2}=V_i-B;\quad g(\operatorname{Int}(B_i\times\{0\})\subset V_{i3};$$

$$g(\operatorname{Int}(B_i\cap\operatorname{Bd} I^{m-1})\times[0,a_1])\subset V_{i3};$$

The sets $V_{i1}\cap V_{i3}$, $V_{i2}\cap V_{i3}$ are arcwise connected; and each loop in $V_{ij}-B$ $(j=1,2,3)$ is contractible in $(U_i-B)\cup F$.

To see that this homotopy property can be obtained for some choice of V_{i1}, V_{i2}, we first observe that V_{i1} and V_{i2} can be chosen so that each loop in V_{i1} can be homotoped into $g(B_i\times(0,a_1))$ missing V_{i2} and vice-versa. Then, the technique used in case 1 can be applied to adjust any map $\varphi\colon B^2\to V_{ij}\cup g(B_i\times(0,a_1))$, $j=1,2$, on the 1-skeleton of a fine triangulation of B^2, thereafter replacing the images under φ of small 2-simplexes by singular disks in $V_{ij}\cup F$. To obtain V_{i3} we choose a number c in $(0,a_1)$ very close to 0; then require that $V_{i3}\cap B=g(B_i\times[0,c))$ if $B_i\cap\operatorname{Bd} I^{m-1}=\emptyset$; if $B_i\cap\operatorname{Bd} I^{m-1}\neq\emptyset$, then

$$V_{i3}\cap B\subset g(B_i\times[0,c))\cup N_c(g(B_i\cap\operatorname{Bd} I^{m-1})\times[0,a_1])\,.$$

It follows that short arcs in V_{i3} can be approximated by short arcs in $V_{i3}-B$. Thus by "pushing" the images of 1-complexes into E^n-B, we are able to adjust any map $\varphi\colon B^2\to V_{i3}$ to a map $\varphi'\colon B^2\to(U_i-B)\cup F$.

Now, choose a triangulation T of P with sufficiently small mesh that each 2-simplex of T which intersects B lies in one of the open sets $V_i\cup V_{i3}$. We suppose further that the vertices of T are in E^n-B; that no 1-simplex of T intersects $\operatorname{Bd} B$; and that each 1-simplex pierces B at each point of intersection. Let τ_j be a 1-simplex of T such that $\tau_j\cap B\neq\emptyset$. For each 2-simplex σ_q which intersects B choose an integer $i(q)$ such that $\sigma_q\subset V_{i(q)}\cup V_{i(q),3}$. Choose disjoint segments $A_1,\ldots,A_s$ in τ_j such that $\tau_j\cap B\subset\bigcup_{p=1}^{s}\operatorname{Int} A_p$; each A_p contains just one point of $\tau_j\cap B$ and $\bigcup_{p=1}^{s}A_p\subset\bigcap_{\sigma_q\in\operatorname{st}(\tau_j,T)}V_{i(q)}$. Applying the method of [20, p. 66], we replace each A_p by an arc A_p' such that $A_p'\cap B=\emptyset$ and A_p is homotopic to A_p' in $\bigcap_q V_{i(q)}\cup V_{i(q,3)}$ keeping endpoints fixed. A_p' runs near a fiber $g(x\times[0,a])$ for some $x\in I^{m-1}$. By repeating this procedure for all the other 1-simplexes of T, we obtain a homotopy $\theta_t\colon P\to E^n$ such that θ_0 is the identity; θ_t is the identity on each simplex σ which does not intersect B; $\theta_t(P)\cap g(I^{m-1}\times[a,1])=\emptyset$ for each t, $0\leqslant t\leqslant1$; $\theta_1(T^1)\subset E^n-B$; and for each 2-simplex σ_q intersecting B, $\theta_t(\dot\sigma_q)\subset V_{i(q)}\cup V_{i(q),3}$.

For each 2-simplex σ_q in P, intersecting B, $\theta_1(\dot\sigma_q)$ is a loop in

$$V_{i(q),1}\cup V_{i(q),2}\cup(V_{i(q),3}-B)\,.$$

Subdivide $\dot\sigma_q$ into arcs $\alpha_0,\ldots,\alpha_t=\alpha_0$ such that, for each j, $\theta_1(\alpha_j)$ is in one of the open sets $V_{i(q),e}$, $e=1,2$, or 3, but any two consecutive arcs α_j, α_{j+1} are in distinct neighborhoods. For each α_j such that $\theta_1(\alpha_j)\subset V_{i(q),1}$, choose a companion arc β_j spanning $\dot\sigma_q$ between the endpoints of α_j. We require these β_j's to be disjoint. Since the endpoints of each α_j are mapped by θ_1 into $V_{i(q),1}\cap V_{i(q),3}$, there exists an arc β_j' in this open set joining the endpoints of $\theta_1(\alpha_j)$. We use a similar procedure to obtain arcs β_j and β_j' for each α_j such that $\theta_1(\alpha_j)\subset V_{i(q),2}$. Subsequently, we extend $\theta_1|\dot\sigma_j'$ in the obvious way to the finite graph $\dot\sigma_q\cup(\bigcup_j\beta_j)$. Each of the boundaries of the components of $\dot\sigma_q-\bigcup_j\beta_j$ are mapped by this extension k_q into one of the sets $V_{i(q),e}$, $e=1,2,3$. Thus by the local homotopy properties of these neighborhoods, we extend k_q to a map $k_q'\colon\sigma_q\to(U_{i(q)}-B)\cup F$. Carrying out a similar procedure for all the 2-simplexes which intersect B leads to a map $k\colon P\to(E^n-B)\cup F$. Since $\theta_1(\sigma_q)$ is homotopic to $k(\sigma_q)$, for each σ_q in $N_\varepsilon(x_{i(q)}\times[0,a_1])$, missing

$g(I^{m-1}\times[a,1])$ and keeping $\theta_1(\dot\sigma_q)$ fixed, the map k is realized by a homotopy ω_t having the properties: $\omega_0=\theta_1$; $\omega_t=\theta_1$ on each simplex of P in E^n-B; $\omega_t(P)\cap g(I^{m-1}\times[a,1])=\emptyset$ for each t; $\omega_1(P)\subset(E^n-B)\cup F$; and for each 2-simplex σ_q, either $\omega_t|\theta_1(\sigma_q)$ is the inclusion, or $\sigma_q\cap B\neq\emptyset$ and $\omega_t(\theta_1(\sigma_q))\subset N_\varepsilon(x_{i(q)}\times[0,a_1])$. The required homotopy f_t may now be identified as the composition of θ_t and ω_t. This completes the proof.

4. Engulfing. In this section we establish the engulfing technique needed for the proof of Theorem 2.1. Throughout this section we assume $n\geqslant 5$, $2\leqslant m\leqslant n-1$, whenever m denotes the dimension of a cell embedded in E^n; unless we specifically state otherwise.

PROPOSITION 4.1. *Let g denote a homeomorphism of I^m onto an m-cell B in E^n; F an F_σ set in B such that $(E^n-B)\cup F$ is 1-ULC and each arc $g(x\times[0,1])$, $x\in I^{m-1}$, contains at most one point of F. Let a, b be numbers such that $0<a<b<1$; P_1 and P_2 disjoint 2-dimensional polyhedra in E^n such that $P_1\cap B\subset g(I^{m-1}\times[0,a])$ and $P_2\cap B\subset g(I^{m-1}\times(b,1])$. Then for each $\varepsilon>0$ there exist an isotopy h_t of E^n and two disjoint closed subsets X_1 and X_2 of I^{m-1} such that*

1) *h_0 is the identity,*

2) *$h_t|g(I^{m-1}\times[a,b])\cup\big(E^n-N_\varepsilon(g(I^{m-1}\times[0,a]\cup[b,1]))\big)$ is the identity for all t, $0\leqslant t\leqslant 1$,*

3) *h_t moves points ε-parallel to the fibers $g(x\times[0,1])$, $x\in I^{m-1}$, and*

4) *$h_i(P_i)\cap B\subset g(X_i\times[0,1])$, $i=1,2$.*

Proof. The proof of this lemma uses two applications of Lemma 3.1 to provide homotopies needed for engulfing. More specifically, we obtain a homotopy $f_t\colon P_1\cup P_2\to E^n$ such that f_0 is the inclusion; f_t is the inclusion on

$$P_1\cup P_2-N_\varepsilon(g(I^{m-1}\times[0,a]\cup[b,1]))$$

for each t; $f_t(P_1\cup P_2)\cap g(I^{m-1}\times[a,b)]=\emptyset$ for each t; f_t moves points ε-parallel to the fibers $g(x\times[0,1])$; and $f_1(P_1\cup P_2)\subset(E^n-B)\cup F$. Since each arc $g(x\times[0,1])$ contains at most one point of F, the projections $\psi g^{-1}(f_1(P_1)\cap B)$ and $\psi g^{-1}(f_1(P_2)\cap B)$ are disjoint closed subsets of I^{m-1}. We take X_1 and X_2, respectively, to be the disjoint closures of two open sets U_1 and U_2 in I^{m-1} such that $\psi g^{-1}(f_1(P_1)\cap B)\subset U_1$ and $\psi g^{-1}(f_1(P_2)\cap B)\subset U_2$. Choose an open set V containing $f_1(P_1\cup P_2)$ such that $V\cap B\subset g((X_1\cup X_2)\times[0,1])$.

The remainder of the proof amounts to an application of Bing's radial engulfing technique [17, p. 185–193]. An isotopy H_t is obtained, such that $H_1(V)$ contains $P_1\cup P_2$ and points are moved ε-parallel to the fibers $g(x\times[0,1])$. The required isotopy h_t is then obtained by reversing H_t. We leave further details to the reader.

The next result is essentially the same as Proposition 4.1 of [3]. The proof, which we omit, uses radial engulfing.

PROPOSITION 4.2. *Let g denote a homeomorphism of I^m onto an m-cell B in E^n; "a" a point of $(0,1)$; V an open set containing $g(I^{m-1}\times[a,1])$; $\varepsilon>0$. Then there exist $\delta>0$ and a neighborhood W of $g(I^{m-1}\times[a,1])$ with the following properties: If P is an $(n-3)$-polyhedron in $N_\delta(g(I^{m-1}\times[0,a]))$, then there exists an isotopy H_t of E^n such that*

1) *H_0 is the identity;*

2) *H_t is the identity on W and also on $E^n-N_\varepsilon(g(I^{m-1}\times[0,a]))$ for all t, $0\leqslant t\leqslant 1$;*

3) *$P\subset H_1(V)$; and*

4) *H_t moves points ε-parallel to the fibers $g(x\times[0,1])$, $x\in I^{m-1}$.*

We are now prepared to establish the main tool for squeezing an m-cell. The lemma below shows that large portions of two disjoint families of arcs lying in an m-cell can be engulfed using an open set which initially contains only one endpoint of each arc. Elements of the two families which fail to be engulfed are "mismatched." The engulfing thereby defines a shrinking effect on the cell.

LEMMA 4.3. *Let g be a homeomorphism of I^m onto an m-cell B in E^n; F an F_σ set in B such that $(E^n-B)\cup F$ is 1-ULC and each $g(x\times[0,1])$, $x\in I^{m-1}$, contains at most one point of F. Let a and b be points in $(0,1)$, $a<b$; $\varepsilon>0$; V an open set containing $g(I^{m-1}\times[a,b])$. Then there exist open sets U_1 and U_2 in I^{m-1} such that $I^{m-1}=U_1\cup U_2$ and an isotopy h_t of E^n such that*

1) *h_t is the identity on $g(I^{m-1}\times[a,b])$ for all t;*

2) *h_t moves points ε-parallel to the arcs $g(x\times[0,1])$, $x\in I^{m-1}$; and*

3) *$h_1g((U_1\times[0,a])\cup(U_2\times[b,1]))\subset V$.*

Proof. The isotopy h_t is obtained in typical fashion by stacking three isotopies H_t, φ_t, and k_t, each of which moves points close to the fibers $g(x\times[0,1])$, $x\in I^{m-1}$. In order to control the tracks of these isotopies we initially choose a number δ such that $0<\delta<\varepsilon/4$, and to every pair of points $x,y\in I^{m-1}$, such that $d(g(x+[0,1]),g(y\times[0,1]))<3\delta$, there corresponds a point $z\in I^{m-1}$ such that $g(x\times[0,1])\cup g(y\times[0,1])\subset N_{\varepsilon/2}(g(z\times[0,1]))$. Apply Proposition 4.2 twice to obtain a positive number η and a neighborhood W of $g(I^{m-1}[a,b])$ such that: for every pair of $(n-3)$-dimensional polyhedra P_1 and P_2; $P_1\subset N_\eta(g(I^{m-1}\times[0,a]))$, $P_2\subset N_\eta(g(I^{m-1}\times[b,1]))$; there exists an isotopy H_t of E^n such that

4) H_0 is the identity,

5) H_t is the identity on W and outside $N_\eta(g(I^{m-1}\times[0,a]\cup[b,1]))$,

6) $P_1\cup P_2\subset H_1(V)$, and

7) H_t moves points δ-parallel to the fibers $g(x\times[0,1])$, $x\in I^{m-1}$.

Choose an open set W' such that $g(I^{m-1}\times[a,b])\subset W'$ and $\mathrm{cl}W'\subset W$. Put

$$\lambda=\min\{\delta,d(E^n-W,\mathrm{cl}W'),d(E^n-W',g(I^{m-1}\times[a,b]))\}.$$

Choose PL n-manifolds M_1 and M_2 such that $g(I^{m-1}+[0,a])\subset\mathrm{Int}M_1$, $M_1\subset N_\eta(g(I^{m-1}\times[0,a]))$, $g(I^{m-1}\times[b,1])\subset\mathrm{Int}M_2$, and $M_2\subset N_\eta(g(I^{m-1}\times[b,1]))$. Triangulate M_1 and M_2 with triangulations T_1 and T_2, respectively, having mesh less than λ. Let K_1 and K_2 be the subcomplexes of T_1^2 and T_2^2, respectively, (the

2-skeleta of T_1 and T_2), obtained by taking all simplexes which do not intersect cl W'. For each $i = 1, 2$ let L_i denote the dual of K_i in T (the polyhedron consisting of all simplexes in the first barycentric subdivision of T_i which do not intersect K_i). Note that, for each i, $L_i - W$ lies in a subcomplex Q_i of L_i such that $\dim Q_i \leqslant n-3$; and L_i contains a neighborhood of $g(I^{m-1} \times [a, b])$.

Apply Proposition 4.1, with K_1, K_2, and

$$\min\{\delta, d(E^n - M_1, g(I^{m-1} \times [0, a])), d(E^n - M_2, g(I^{m-1} \times [b, 1]))\};$$

corresponding, respectively, to P_1, P_2, and ε; to obtain an isotopy k_t of E^n and two disjoint closed sets X_1 and X_2 in I^{m-1} satisfying conditions (1)-(4) of the conclusion of that proposition. Note that k_t is the identity outside $M_1 \cup M_2$. Choose open sets U_1 and U_2 in I^{m-1} such that $I^{m-1} = U_1 \cup U_2$, $k_1^{-1} g(\mathrm{cl}\, U_1 \times [0, a]) \subset M_1 - K_1$, and $k_1^{-1} g(\mathrm{cl}\, U_2 \times [b, 1] \subset M_2 - K_2$ ($X_1 \subset U_2$ and $X_2 \subset U_1$.)

Let H_t be an engulfing isotopy satisfying (4)-(7) above with P_1 and P_2 identified as Q_1 and Q_2, respectively. Since $H_t|W$ is the identity for all t, $H_1(V)$ contains $L_1 \cup L_2$ as well as $Q_1 \cup Q_2$.

By "stretching" $H_1(V)$ linearly across the join structure between each L_i and K_i we obtain an isotopy φ_t of E^n such that φ_t is fixed outside a neighborhood of $M_1 \cup M_2$ and on $g(I^{m-1} \times [a, b])$, $\varrho(\varphi_t, \text{identity}) < \lambda$ for all t, φ_0 is the identity, $k_1^{-1}(g(\mathrm{cl}\, U_1 \times [0, a])) \subset \varphi_1(H_1(V))$, and $k_1^{-1}(g(\mathrm{cl}\, U_2 \times [b, 1])) \subset \varphi_1(H_1(V))$. Observe that each of H_t, φ_t, and k_t reduce to the identity on $g(I^{m-1} \times [a, b])$; H_t and k_t move points δ-parallel to the fibers $g(x \times [0, 1])$, $x \in I^{m-1}$; and each φ_t is a δ-homeomorphism of E^n. Thus we define h_t by:

$$h_t(x) = \begin{cases} k_{1-3t} \circ k_1^{-1}(x) & \text{if } 0 \leqslant t \leqslant 1/3, \\ \varphi_{2-3t} \circ \varphi_1^{-1} \circ k_1^{-1}(x) & \text{if } 1/3 \leqslant t \leqslant 2/3, \\ H_{3-3t} \circ H_1^{-1} \circ \varphi_1^{-1} \circ k_1^{-1}(x) & \text{if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

It follows from the choice of δ that h_t moves points ε-parallel to the fibers. This completes the proof.

5. Shrinking the vertical arcs of an m-cell. The shrinking process which is needed to prove Theorem 2.1 is based upon Lemma 4.3 and the following technical result due to Cannon and Daverman [5, Lemma 2.2]. We include a proof here for completeness.

LEMMA 5.1. *Let g be a homeomorphism from I^m into E^n; ε a positive number; and $\mathscr{P} = \{0 = t_0 < t_1 < \ldots < t_k = 1\}$, $k \geqslant 2$, a partition of $[0, 1]$ such that* $\operatorname{diam} g(x \times [t_{i-1}, t_i]) < \varepsilon$ *for each $i = 1, \ldots, k$. Suppose further that, for each $x \in I^{m-1}$, either $g(x \times [0, t_2])$ or $g(x \times [t_{k-2}, 1])$ has diameter less than ε. Then there exists a homeomorphism $g'\colon I^m \to E^n$ such that*

1) $g'(x \times [0, 1]) = g(x \times [0, 1])$ *for each $x \in I^{m-1}$;*

2) $\operatorname{diam} g'(x \times [t_{i-1}, t_i]) < \varepsilon$ *for each $i = 1, \ldots, k-2$, $x \in I^{m-1}$;*

3) $\operatorname{diam} g'(x \times [t_{k-2}, 1]) < \varepsilon$ *for each $x \in I^{m-1}$.*

Proof. We use Urysohn's Lemma to define a sequence of functions $f_i\colon I^{m-1} \to [0, 1]$, $i = 0, 1, \ldots, k-1$. Each of these is to determine the image of $I^{m-1} \times \{t_i\}$, for some i, under the homeomorphism g'. Then we obtain g' by extending linearly between these levels.

Choose positive numbers $a_1, a_2, \ldots, a_{k-2}$ such that $t_i < a_i < t_{i+1}$ and $\operatorname{diam} g(x \times [a_i, t_{i+2}]) < \varepsilon$ for each i. Put $X_0 = \{x \in I^{m-1} |\ \operatorname{diam} g(x \times [0, t_2] \geqslant \varepsilon)\}$; $X_1 = \{x \in I^{m-1} |\ \operatorname{diam} g(x \times [t_{k-2}, 1]) \geqslant \varepsilon\}$. Since these are disjoint closed sets, there exist open sets $U_0, U_1, \ldots, U_{k-2}$ in I^{m-1} such that $X_0 \subset \mathrm{cl}\, U_0$, $X_1 \subset I^{m-1} - U_{k-2}$, and $\mathrm{cl}\, U_{i-1} \subset U_i$ for each $i = 1, \ldots, k-2$. Applying Urysohn's Lemma to each pair $(\mathrm{cl}\, U_{i-1}, I^{m-1} - U_i)$, we obtain maps $f_i\colon I^{m-1} \to [t_i, a_i]$ such that $f_i(\mathrm{cl}\, U_{i-1}) = t_i$ and $f_i(I^{m-1} - U_i) = a_i$ for each i. Let f_0 and f_{k-1} denote the constant maps mapping I^{m-1} onto "0" and "1" respectively. It follows that $\operatorname{diam} g(x \times [f_{i-1}(x), f_i(x)]) < \varepsilon$ for each $i = 1, \ldots, k-1$ and $x \in I^{m-1}$. Define a map θ on $I^{m-1} \times \{t_0, \ldots, t_{k-2}, 1\}$ by the rule $\theta(x, t_i) = (x, f_i(x))$, $x \in I^{m-1}$, $i = 0, \ldots, k-2$; $\theta(x, 1) = (x, 1)$. We obtain a natural extension of θ to a function $\theta'\colon I^m \to I^m$ by extending linearly on each fiber $\{x\} \times [t_{i-1}, t_i]$. The required embedding g' is the composition $g \circ \theta'$.

The proof of Theorem 2.1. Let B be an m-cell in E^n, $g\colon B^m \to B$, and F an F_σ set in B satisfying the hypothesis of Theorem 2.1. By Bing's Shrinking Criterion [16, p. 92] it suffices to prove that, for every $\varepsilon > 0$, there is a homeomorphism $h\colon E^n \to E^n$ such that $\operatorname{diam} hg\pi^{-1}(x) < \varepsilon$ for each $x \in B^{m-1}$; for each $x \in B^{m-1}$ there is some $y \in B^{m-1}$ such that $g\pi^{-1}(x) \cup hg\pi^{-1}(x) \subset N_\varepsilon(g\pi^{-1}(y))$; and for each $z \in E^n$, either $h(z) = z$ or there is some $y \in B^{m-1}$ such that $\{z\} \cup \{h(z)\} \subset N_\varepsilon(g\pi^{-1}(y))$. Actually, it is sufficient to establish just the first and third of the conditions. The shrinking homeomorphism h can be realized by an isotopy h_t which moves points ε-parallel to the fibers $g\pi^{-1}(x)$.

Let $\theta\colon I^m \to B^m$ be a one-to-one map, δ a number such that $0 < \delta < \varepsilon/3$; for each $x \in I^{m-1}$, $\theta(x \times [0, 1]) = g\pi^{-1}(y)$ for some $y \in B^{m-1}$; and $\operatorname{diam} g\pi^{-1}(z) < \varepsilon/3$ for each z such that $d(g(\pi^{-1}(z)), B - g\theta(I^m) < \delta$ (Fig. 1). Put $g_0 = g \circ \theta$. Choose a partition $\mathscr{P} = [0 = t_0 < t_1 < \ldots < t_k = 1]$ such that $\operatorname{diam} \theta_0(\{x\} \times [t_{i-1}, t_i]) < \delta$ for each $x \in I^{m-1}$; $i = 1, \ldots, k$. The remainder of the proof uses Lemmas 4.3 and 5.1 inductively to prove: for each integer $i = 0, 1, \ldots, k-2$ and each number $\lambda > 0$, there exist embeddings $g_0, \ldots, g_i$ of I^m onto $g_0(I^m)$ and homeomorphisms $h_0, \ldots, h_i$ of E^n onto E^n such that

(i.1) $g_i(\{x\} \times [0, 1]) = h_i g_0(x \times [0, 1])$ for each $x \in I^{m-1}$;

(i.2) $\operatorname{diam} g_i(\{x\} \times [t_{j-1}, t_j]) < \delta$, for each $j = 1, \ldots, k-i-1$;

(i.3) $\operatorname{diam} g_i(\{x\} \times [t_{k-i-1}, 1]) < \delta$, $x \in I^{m-1}$; and

(i.4) for each $x \in E^n$, either $h_i(x) = x$ or $\{x, h_i(x)\} \subset N_\lambda(g_0(\{y\} \times [0, 1]))$ for some $y \in I^{m-1}$.

Clearly, the embedding g_0 specified already and the choice $h_0 = $ identity suffice for the case $i = 0$. Suppose, then, that g_i and h_i, satisfying these properties, exist for some arbitrary $i < k-2$. Choose γ, $0 < \gamma < \lambda/2$ such that for any two points x, y of I^{m-1}, $d(g_0(\{x\} \times [0, 1]), g_0(\{y\} \times [0, 1])) < \gamma$ implies that each of the arcs

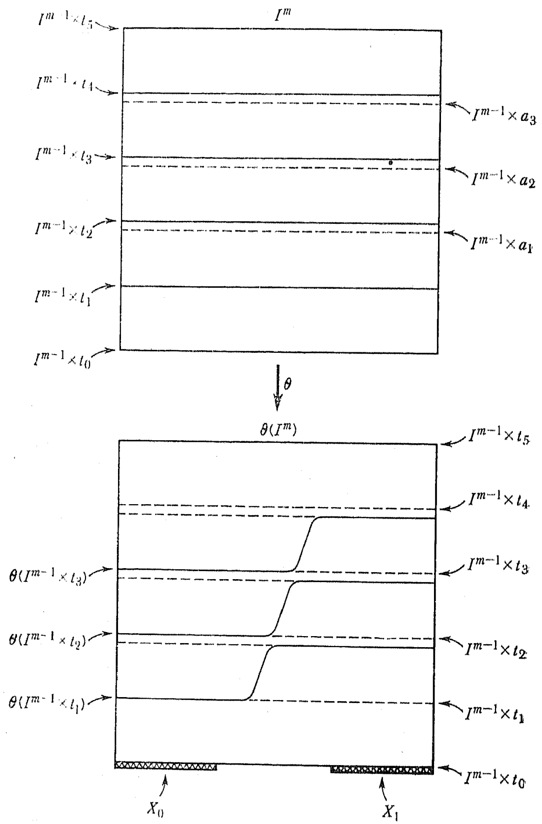

Fig. 1

$g_0(\{x\}\times[0,1])$, $g_0(\{y\}\times[0,1])$ lies in the $\lambda/2$-neighborhood of $g_0(z\times[0,1])$ for some $z\in I^{m-1}$. By the induction hypothesis g_i and h_i can be chosen so that (i.1)-(i.3) are satisfied and h_i moves points γ-parallel to the fibers $g_0(y\times[0,1])$, $y\in I^{m-1}$. An application of Lemma 4.3 provides a homeomorphism $\omega_i\colon E^n\to E^n$ (realized by an isotopy of E^n) and open sets U_1 and U_2 in I^{m-1}, $I^{m-1}=U_1\cup U_2$, such that

(i.5) $\operatorname{diam}\omega_i g_i(\{x\}\times[t_{i-1},t_i])<\delta$, $1\leqslant i\leqslant k-i$ $(t_{k-i}=1)$, $x\in I^{m-1}$;

(i.6) $\operatorname{diam}\omega_i g_i(\{x\}\times[0,t_2])\leqslant\delta$, for each $x\in U_1$;

(i.7) $\operatorname{diam}\omega_i g_i(\{x\}\times[t_{k-i-2},1])<\delta$, for each $x\in U_2$; and

(i.8) for each $y\in E^n$, either $\omega_i(y)=y$ or $\{y,\omega_i(y)\}\subset N_\gamma(g_0(z\times[0,1]))$ for some $z\in I^{m-1}$.

The numbers a and b in this application are t_2 and t_{k-i-2}, respectively, γ corresponds to ε. Put $h_{i+1}=\omega_i\circ h_i$. Lemma 5.1 provides the necessary embedding g_{i+1} so that properties $(i+1.1)-(i+1.4)$ are satisfied by the pair h_{i+1},g_{i+1}. For this application, $h_{i+1}g_0$ corresponds to g in Lemma 5.1; δ to ε; $k-i$ to k; and g_{i+1} to g'.

The homeomorphism h_{k-2} is the homeomorphism satisfying Bing's Shrinking Criterion, since $(k-2.4)$ applies with $\lambda=\delta$ and, for each $x\in I^{m-1}$,

$$\begin{aligned}h_{k-2}g_0(\{x\}\times[0,1]) &= g_{k-2}(\{x\}\times[0,1])\\ &= g_{k-2}(\{x\}\times[0,t_1])\cup g_{k-2}(\{x\}\times[t_1,1])\end{aligned}$$

implies that $\operatorname{diam}h_{k-2}g_0(\{x\}\times[0,1])<\varepsilon$. This completes the proof.

The proof of Theorem 2.2. Let g denote an embedding of B^m into E^n, $B=g(B^m)$; ε a positive number; F a 0-dimensional F_σ set in B such that the hypothesis of Theorem 2.2 is satisfied. By continuity there exists $\delta>0$ such that $d(g(x),g(y))<\varepsilon$ whenever $d(x,y)<\delta$, $x,y\in B^m$. By Theorem 2.1 it suffices to show that there is a homeomorphism θ of B^m onto B^m such that $\varrho(\theta,\text{identity})<\delta$ and $\theta g^{-1}(F)$ intersects each segment $\pi^{-1}(x)$, $x\in B^{m-1}$, in at most one point. Then, of course, the required embedding g' is $g\circ\theta^{-1}$. In order to obtain θ we describe below a sequence of homeomorphisms $\omega_1,\omega_2,\ldots$ of B^m onto B^m and a sequence $\delta_1,\delta_2,\ldots$ of positive numbers such that $\varrho(\omega_i,\text{identity})<\delta_i$ for each i and $\sum_{i=1}^{\infty}\delta_i<\delta$. Each ω_i produces a slight simplication of the intersections $g^{-1}(F)\cap\pi^{-1}(x)$, $x\in B^{m-1}$. Then we recursively define homeomorphisms $\theta_0,\theta_1,\theta_2,\ldots$ by $\theta_0=$ identity and $\theta_i=\omega_i\circ\theta_{i-1}$ for each $i>1$. Clearly, the limit θ will be continuous and surjective (for δ sufficiently small). Additional care in specifying the sequence $\{\delta_i\}$ insures that θ is injective as well.

Put $\delta_1=\delta/2$. Let $C_1,C_2,\ldots$ be compact 0-dimensional sets in B^m such that $g^{-1}(F)=\bigcup_{i=1}^{\infty}C_i$; $C_i\subset C_{i+1}$ for each i; and each C_i is locally flat relative to B^m and $\operatorname{Bd}B^m$. Thus, there exists a finite collection $\{B(1,1),B(1,2),\ldots,B(1,k(1))\}$ of disjoint locally flat m-cells in B^m such that

(1.1) $\operatorname{diam}B(1,i)<\delta_1$ for each $i=1,\ldots,k(1)$;

(1.2) for each i, either $B(1,i)\cap\operatorname{Bd}B^m=\emptyset$ or $B(1,i)\cap\operatorname{Bd}B^m$ is a locally flat $(m-1)$-cell in $\operatorname{Bd}B^m$.

(1.3) $C_1\subset\bigcup_{i=1}^{k(1)}\operatorname{Int}B(1,i)\cup\operatorname{Int}(B(1,i)\cap\operatorname{Bd}B^m)$.

Choose a collection $\{D(1,1),D(1,2),\ldots,D(1,k(1))\}$ of disjoint flat $(m-1)$-cells in $\operatorname{Int}B^{m-1}$ such that, for each i,

(1.4) If $B(1,i)\cap\operatorname{Bd}B^m\neq\emptyset$, then one component of $\pi^{-1}(D(1,i))\cap\operatorname{Bd}B^m$ lies in $\operatorname{Int}(B(1,i)\cap\operatorname{Bd}B^m)$;

(1.5) if $B(1,i)\cap \operatorname{Bd}B^m=\emptyset$, then $\pi^{-1}(D(1,i))\cap \operatorname{Int}B(1,i)=\emptyset$.

See Figure 2 for an illustration of this arrangement. Using the local flatness of each $B(1,i)$ and condition 1.2 we obtain a homeomorphism ω_1 of B^m such that

(1.6) ω_1 is the identity outside $\bigcup_{i=1}^{k(1)} B(1,i)$;

(1.7) $\omega_1(C_1)\cap(B(1,i))\subset\pi^{-1}(\operatorname{Int}D(1,i))$ for each i;

(1.8) $\omega_1(\operatorname{Bd}B^m)=\operatorname{Bd}B^m$.

To define $\omega_1|B(1,i)$, for $B(1,i)\subset\operatorname{Int}B^m$, we shrink a subcell of $B(1,i)$ radially toward an interior point. If $B(1,i)$ intersects $\operatorname{Bd}B^m$ we shrink toward a point of $\operatorname{Int}(B(1,i)\cap\operatorname{Bd}B^m)$. Using the uniform continuity of ω_1^{-1}, choose $\lambda_1>0$ such

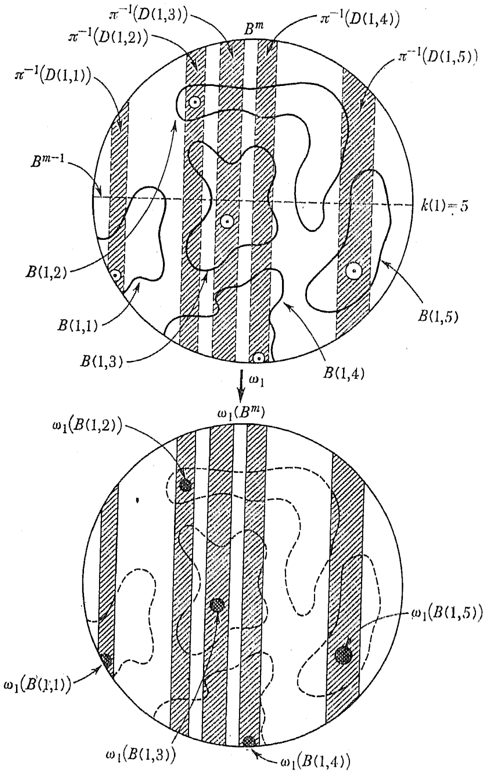

Fig. 2

that $d(\omega_1(x),\omega_1(y))\geqslant\lambda_1$ whenever $d(x,y)\geqslant 1$. For notational convenience put $\theta_1=\omega_1$, $\delta_0=3\delta=\theta\delta_1$, and $C_0=\emptyset$.

Now, we proceed inductively to define $\delta_2,\delta_3,\ldots;\ \lambda_2,\lambda_3,\ldots;\ \omega_2,\omega_3,\ldots;$ and $\theta_2,\theta_3,\ldots$ in a manner similar to the procedure above. Suppose that for each $j=1,\ldots,r-1$ there exist positive numbers δ_j,λ_j; homeomorphisms ω_j,θ_j; and finite collections $\{B(j,i)\}$, $\{D(j,i)\}$ of disjoint locally flat m-cells and $(m-1)$-cells, respectively, such that

(j.0) $\delta_j=\min\{\delta_{j-1}/2,\lambda_{j-1}/6\}$;

(j.1) $\operatorname{diam}B(j,i)<\delta_j$ for each $i=1,\ldots,k(j)$;

(j.2) for each i, either $B(j,i)\cap\operatorname{Bd}B^m=\emptyset$ or $B(j,i)\cap\operatorname{Bd}B^m$ is an $(m-1)$-cell in $\operatorname{Bd}B(j,i)$;

(j.3) $\theta_{j-1}(C_j)\subset\bigcup_{i=1}^{k(j)}\operatorname{Int}B(j,i)\cup\operatorname{Int}(B(j,i)\cap\operatorname{Bd}B^m)$;

(j.4) if $B(j,i)\neq\emptyset$, then one component of $\pi^{-1}(D(j,i))\cap\operatorname{Bd}B^m$ lies in $\operatorname{Int}(B(j,i)\cap\operatorname{Bd}B^m)$;

(j.5) if $B(j,i)\cap\operatorname{Bd}B^m=\emptyset$, then $\pi^{-1}(D(j,i))\cap\operatorname{Int}B(j,i)\neq\emptyset$;

(j.6) ω_j is the identity outside $\bigcup_{i=1}^{k(j)}B(j,i)$;

(j.7) $\omega_j\theta_{j-1}(C_j)\cap B(j,i)\subset\pi^{-1}(\operatorname{Int}D(j,i))$ for each i;

(j.8) $\omega_j(\operatorname{Bd}B^m)=\operatorname{Bd}B^m$;

(j.9) for some integer $l(j)\leqslant k(j)$, $j\geqslant 2$,

$$\theta_{j-1}(C_{j-1})\subset\bigcup_{i=1}^{l(j)}\operatorname{Int}B(j,1)\cup\operatorname{Int}(B(j,i)\cap\operatorname{Bd}B^m)\subset\bigcup_{i=1}^{k(j-1)}\omega_{j-1}(B(j-1),i);$$

(j.10) $\theta_j=\omega_j\circ\theta_{j-1}$; and

(j.11) $d(\theta_j(x),\theta_j(y))\geqslant\lambda_j$ whenever $d(x,y)\geqslant 1/j$.

Put $j=r$; $\delta_r=\min\{\delta_{r-1}/2,\lambda_{r-1}/6\}$. In the manner described above for the case $j=1$, we choose disjoint m-cells $B(r,1),\ldots,B(r,k(r))$ in B^m as well as $(m-1)$-cells $D(r,1),\ldots,D(r,k(r))$ in B^{m-1} with sufficient care that (r.1)-(r.5) and (r.9) are satisfied. Then ω_r and θ_r are defined using radial shrinking within the cells $B(r,i)$ so that (j.6)-(j.8) and (j.10) are satisfied. Finally, λ_r satisfying (r.11) exists by uniform continuity of θ_r^{-1}.

As indicated above, we define $\theta=\lim_{r\to\infty}\theta_r$. It follows from the specification of $\{\delta_r\}$ (1.0, 2.0, etc.) that θ is continuous and that $\varrho(\theta,\text{identity})<\delta$. Conditions (r.8) for $r=1,2,\ldots$ assure that θ is surjective. All that remains to be shown is that θ is injective and that each set $\pi^{-1}(x)$, $x\in B^{m-1}$, contains at most one point of $\theta(F)$.

Let x and y be distinct points of B^m. Let N be a positive integer such that $d(x,y)\geqslant 1/N$. Then $d(\theta_N(x),\theta_N(y))\geqslant\lambda_N$ by (N.11). Moreover, the numbers $\delta_{N+1},\delta_{N+2},\ldots$ have been chosen (conditions r.0) to assure that $d(\theta_N(x),\theta(x))<\lambda_N/3$ and $d(\theta_N(y),\theta(y))<\lambda_N/3$. Hence, we have $d(\theta(x),\theta(y))\geqslant\lambda_N-2\lambda_N/3>0$. I.e., θ is injective.

Now suppose that x and y are distinct points of F. It suffices to show that two such points are mapped by θ into two distinct segments $\pi^{-1}(z)$ and $\pi^{-1}(w)$, respectively. Let M be a positive integer such that x and y are both in C_M. There is also a positive integer M' such that $d(\theta_{r-1}(x), \theta_{r-1}(y)) \geqslant 2\delta_r$ if $r \geqslant M'$, as shown above. Put $R = \max\{M, M'\}$. By Properties R.1 and R.3 in the program for defining θ_R, there exist disjoint m-cells $B(R, s)$ and $B(R, t)$ such that $\theta_{R-1}(x) \in B(R, s)$ and $\theta_{R-1}(y) \in B(R, t)$. By (R.7) there exist disjoint $(m-1)$-cells $D(R, s)$ and $D(R, t)$ in B^{m-1} such that $\theta_R(x) \in \pi^{-1}(\operatorname{Int} D(R, s))$ and $\theta_R(y) \in \pi^{-1}(\operatorname{Int} D(R, t))$. It now follows from $(R+1.9)$, $(R+2.9)$, etc., and from $(R+1.6)$, $(R+2.6)$, etc. that $\theta(x) \in \pi^{-1}(\operatorname{Int} D(R, s))$; whereas $\theta(y) \in \pi^{-1}(\operatorname{Int} D(R, t))$. This completes the proof.

We mention now some applications of Theorem 2.2. The first, an unpublished result of Daverman and Eaton, is an immediate consequence of Theorem 2.2 and [18, Lemma 2].

COROLLARY 5.2. *Each 2-cell in E^n, $n \geqslant 5$, can be squeezed to an arc.*

For $n = 4$ this result is due to Sher [19].

The following theorem, proved by techniques of [18, Theorem 4], [10, Theorem 3.3], and [9, Theorem 4], provides a useful vehicle for further applications of Theorem 2.2.

THEOREM 5.3. *Let B denote an m-cell in E^n, $3 \leqslant m \leqslant n-1$, $n \geqslant 5$. The following assertions are equivalent.*

(1) *B contains a 0-dimensional F_σ set F such that F is locally flat relative to B and $(E^n - B) \cup F$ is 1-ULC.*

(2) *For each 2-dimensional polyhedron $P \subset B$, $\varepsilon > 0$ ($\dim P \cap \operatorname{Bd} B \leqslant 1$ if $m = 3$), there exists an ε-homeomorphism h of B onto B such that $h(P)$ is locally flat relative to E^n.*

COROLLARY 5.4. *Suppose that an m-cell B in E^n is locally flat modulo a polyhedron P that is flat relative to E^n. Then B can be squeezed to an $(m-1)$-cell.*

Proof. The result is established by proving that each 2-complex K in such a cell B is locally flat in E^n ($\dim K \cap \operatorname{Bd} B \leqslant 1$, if $\dim B = 3$). Assume $\dim P \leqslant m-1$. Let $\varepsilon > 0$ be given. Let $f\colon S^1 \to E^n$ be a loop such that $f(S^1) \cap K = \emptyset$ and $\operatorname{diam} f(S^1) < \varepsilon$. We may suppose also that $f(S^1) \cap P = \emptyset$. Using the fact that P is flat, we extend f to a map $g\colon B^2 \to E^n$ such that $g(B^2)$ intersects P in at most a finite set of points in $P - (P \cap K)$ and $\operatorname{diam} g(B^2) < \varepsilon$. Thus, $g(B^2) \cap K$ is contained by a subpolyhedron K' of K such that $K' \subset B - P$. Since K' is locally flat $g(B^2)$ can be pushed into $E^n - (K)$ by an arbitrarily small move. This shows that $E^n - K$ is 1-ULC. Thus, K is flat by the Bryant–Seebeck taming result [4].

COROLLARY 5.5. *Suppose that B, an m-cell in E^n, $2 \leqslant m \leqslant n-1$, $n \geqslant 5$, contains a polyhedron P such that B is locally flat modulo P, P is locally flat relative to B, $\dim P \leqslant m-1$, and $\dim(P \cap \operatorname{Bd} B) \leqslant m-2$. Then B can be squeezed to an $(m-1)$-cell.*

Proof. Using standard techniques, a 2-complex K in E^n can be adjusted via an isotopy h_t, close to the identity, so that $h_1(K) \cap P$ is 0-dimensional. Using such intersections for a suitably chosen sequence of 2-complexes $K_1, K_2, \ldots$, we obtain the required F_σ set F in B so that $(E^n - B) \cup F$ is 1-ULC.

To put these results into proper perspective it is helpful to consider the special class of cells which are locally flat modulo a Cantor set. If such a Cantor set is locally flat in E^n or in the cell B which contains it, then B satisfies the hypothesis of one of the corollaries above. The examples, given by Daverman, of cells which cannot be squeezed are locally flat modulo Cantor sets which are wildly embedded in both ways, in E^n and also in the cells.

6. Slack cells. In view of the results discussed above, which reveal that a wildly embedded m-cell might be squeezed to an $(m-1)$-cell in many different ways, we are led to consider the possibility that a cell can be squeezed, using any embedding whatsoever, to an $(m-1)$-cell. More specifically, we say that a cell B is *slack* if every homeomorphism g from B^m onto B supports a squeezing map. In a forthcoming paper, the author will prove, using Theorem 2.1, that an m-cell is slack provided that each compactum of dimension $\leqslant [(m+1)/2]$ in that cell has 1-ULC complement, $2 \leqslant m \leqslant n-1$, $n \geqslant 5$. (At present it is unknown whether this result can be improved by lessening the dimension.) From this result follows the existence of slack cells which are wild at every point. For $m \leqslant n-2$, $n \geqslant 5$ such examples are obtained by specifying an arc A in E^{n-m+1}, which fails to be locally flat at each point. The product $A \times I^{m-1}$, naturally embedded in $E^{n-m+1} \times E^{m-1}$ is an m-cell in which each $(m-1)$-dimensional compactum has 1-ULC complement in E^n. For $m = n-1$ define B as $D \times I^{n-3}$, where D is a 2-cell embedded in the wild sphere described by Bing in [1]. A result of Daverman [11, Theorem 9] implies that $(n-2)$-dimensional compacta in this example have locally nice complements, a sufficient condition for B to be slack. Other examples of slack cells can be obtained, which are not factored and contain wildly embedded polyhedra of low dimension.

We conclude by stating a few open questions.

(1) Can an m-cell be squeezed to an $(m-1)$-cell if it is cellular?

(2) Can a cell be squeezed if it is locally flat at each interior point?

(3) Is there a k-cell D, $k > 1$, such that no $(k+1)$-cell can be squeezed to D?

(4) Is there a converse of Theorem 2.2? I.e., if each embedding $g\colon B^m \to B$ can be approximated by embeddings g' which support a squeezing map, then does B contain a locally flat 0-dimensional F_σ set F such that $(E^n - B) \cup F$ is 1-ULC?

(5) Let $g\colon B^m \to B$ be an embedding. Suppose that each arc A in B^{m-1} can be approximated by arcs A' such that $g\pi^{-1}(A')$ is locally flat in E^n. Is this a sufficient condition for B to be squeezable to an $(m-1)$-cell?

(6) Is every factored cell slack? (A cell B in E^n is *factored* if it is ambiently homeomorphic to $D \times I^k \subset E^{n-k} \times E^k$ for some lower dimensional cell D in E^{n-k}.)

(7) Is a cell slack if each 2-dimensional compactum in the cell is 1-ULC in E^n? What is the minimal dimension k such that the 1-ULC property of k-dimensional compacta implies that B is slack? (If $\dim B = m$, then $k \leqslant [(m+1)/2]$.)

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PEMBROKE STATE UNIVERSITY
Pembroke, North Carolina

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Metrizability of certain Pixley–Roy spaces

by

H. R. Bennett, W. G. Fleissner, and **D. J. Lutzer** * (Lubbock, Tex.)

Abstract. This paper studies metrizability of the Pixley–Roy hyperspace $\mathcal{F}[X]$ of a space X where X is a generalized ordered space of a certain type. For those generalized ordered spaces constructed from separable linearly ordered spaces, necessary and sufficient conditions for metrizability of $\mathcal{F}[X]$ are obtained. Metrization theorems for the hyperspace of other generalized ordered spaces are obtained by placing restrictions on the one-sided nature of neighborhoods. For example, it is proved that if X is any first-countable subspace of any ordinal, then $\mathcal{F}[X]$ is metrizable.

1. Introduction and definitions. In [PR] Carl Pixley and Prabir Roy presented an easily described space which could be used in place of an older and more complicated example given by Mary Ellen Rudin [R_1] in her study of completable Moore spaces. In today's terminology, Pixley and Roy associated with each space X one of its possible "hyperspaces", i.e., topological spaces whose ground-set is the power set $\mathcal{P}(X)$. It soon became apparent that Pixley and Roy had, in fact, discovered an elegant and useful general technique for constructing certain kinds of examples, and various versions of their construction have been studied in [PT], [vDTW] and [vD_1]. In this paper we give necessary and sufficient conditions for metrizability of the Pixley–Roy hyperspace of certain lines.

The lines on which our Pixley–Roy spaces are constructed are certain *generalized ordered spaces*. Begin with a linearly ordered set $(Y, <)$ and let λ denote the usual open-interval topology associated with $<$. Select three disjoint, possibly empty, subsets $A, B, C \subset Y$ and let τ be the topology on Y having the collection

$$\lambda \cup \{[x, y[\mid x \in A,\ y > x\} \cup \{]x, y] \mid x < y \in B\} \cup \{\{x\} \mid x \in C\}$$

as a base. The space (Y, τ) is then called a generalized ordered space on $(Y, <)$ and can be denoted by $GO_Y(A, B, C)$. The standard reference for the basic properties of generalized ordered spaces is [L] whose terminology and notation we usually follow.

The Pixley–Roy hyperspace of any space X is constructed as follows. Let $\mathcal{F}[X]$ be the collection of all nonempty finite subsets of X and topologize $\mathcal{F}[X]$ by using basic open neighborhoods of the form

$$[F, W] = \{S \in \mathcal{F}[X] \mid F \subset S \subset W\}$$

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