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Equivariant embeddings of finite abelian group actions in euclidean space

by

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Abstract. Let X be a finite dimensional compact metric space and let G be a finite abelian group which acts on X. This paper shows that X equivariantly embeds in a euclidean space with an orthogonal G-action. Moreover, a minimum dimension for the euclidean space is obtained.

1. Introduction. Mostow [8] first showed that every action of a compact Lie group with a finite number of non-conjugate isotropy subgroups on a finite dimensional, separable, metric space can be equivariantly embedded in a linear action of the group on some euclidean space. However, Mostow's theorem said nothing about the required dimensions of the euclidean space. Copeland and de Groot [3] went on to find dimensions for the euclidean space in the case of an action of a cyclic group of prime order. Kister and Mann [7] extended this result to actions of compact abelian Lie groups with a finite number of distinct isotropy subgroups.

In [1] the present author obtained improvements on the results of Copeland and de Groot using methods different from those employed previously. A consequence is that, if X is a compact n-dimensional metric space with a free Z_p-action, then X equivariantly embeds in R^{2n+1} with an orthogonal Z_p-action.

The present work provides improvements on the results of Kister and Mann in the case of a finite dimensional compact metric space with a finite abelian group acting on it. The methods used here are extensions of the ideas found in [1]. An important corollary of this work is contained in the theorem stated below.

THEOREM (1.1). *Let $G = R_1\oplus\ldots\oplus R_r\oplus H_1\oplus\ldots\oplus H_s$ be a finite abelian group, where the R's are cyclic groups of order $\neq 2$ and the H's are all of order 2. Suppose X is a compact n-dimensional metric space with a free G-action. Then X equivariantly embeds in an orthogonal G-action on R^N, where $N = \max\{2n+1, 2r+s\}$.*

2. Equivariant spaces and equivariant maps. Throughout the remainder of this paper let G denote a finite abelian group and let X be a compact metric space. If G acts on X, then X is called a *G-space*, frequently denoted as (X, G). An equivariant map $f\colon (X, G)\to(Y, G)$ between two G-spaces is an equivariant ε-map if $\operatorname{diam} f^{-1}y<\varepsilon$ for every $y\in fX$. If (X, G) is a compact metric G-space and (Y, G) is a separable metric G-space, then $(Y, G)^{(X,G)}$ is the subspace of the metric space Y^X

(with metric defined by $d(f,g) = \sup_{x \in X} d(fx, gx)$) consisting of all equivariant maps from (X, G) to (Y, G). In fact, $(Y, G)^{(X,G)}$ is complete if Y is complete.

If (Y, G) is a G-space, then $Gy = \{gy|\ g \in G\}$ is called the *orbit of* y, and $GS = \bigcup_{g \in G} gS$ is called the *orbit of* S, where y is an element in Y, and S is a subset of Y. A subset S of Y is called *sectional* if $S \cap Gy = \{y\}$ for each y in S, and any one-to-one function $\chi\colon (Y/G) \to Y$ is called a *section.*

(2.1), which is stated here, is used in proving (2.2) below. A special case of (2.1), namely that of $G = Z_p$, is found in Jaworowski [5, p. 235]. After the proof in [5] is examined, it is clear that it holds for G a finite abelian group.

COVERING LEMMA (2.1). *Let (X, G) be a compact metric G-space and let A be an equivariant closed subspace of X such that G acts freely outside of A. Suppose C is an equivariant open cover of $X-A$. Then there exists a countable, locally finite, equivariant, open cover B of $X-A$ which is a refinement of C and which satisfies the following*:

(i) $\lim_{d(V,A)\to 0} (\operatorname{diam} \operatorname{St} V) = 0$ *for* $V \in B$;

(ii) *If* $V \in B$, *the* $\operatorname{Cl} V \subset X-A$;

(iii) *Every neighborhood of A in X contains all but a finite number of elements of B*;

(iv) *For every $V \in B$, the elements in $\{gV|\ g \in G\}$ are mutually disjoint*;

(v) *If* $\dim(X-A) \leqslant n$, *the* $\operatorname{Ord} B \leqslant |G|(n+1)-1$;

(vi) *If ε is a given positive number, then B can be chosen such that* $\operatorname{mesh} B < \varepsilon$.

POLYHEDRAL REPLACEMENT LEMMA (2.2). *Let (X, G) be a compact metric G-space and let A be an equivariant, closed subspace such that G acts freely outside A. For a given positive number ε, there exists a compact Hausdorff G-space (Z, G) such that*:

(i) *Z contains A as an equivariant, closed subspace*;

(ii) *There exists a countable, locally finite, simplicial complex K with $|K| = Z-A$ and a free simplicial action of G on $|K|$*;

(iii) *There is an equivariant ε-map $f\colon (X, G) \to (Z, G)$ such that $f|_A = 1_A$, $f(X-A) \subset |K|$, and $f^{-1}\{\operatorname{St}(V)|\ V \in K^0\}$ forms a locally finite, equivariant, open cover of $X-A$ of mesh less than ε; and*

(iv) *If* $\dim(X-A) \leqslant n$, *then* $\dim K \leqslant n$.

(2.2) is a modification of Lemma (2.3) in [5]. The proof given in [5] is valid in the present case of G a finite abelian group. The essential idea is to replace $X-A$ by the equivariant polyhedron generated by the nerve of the equivariant cover found in (2.1).

3. Equivariant general position. Given a set S in R^N, $L(S)$ denotes the affine span of S in R^N. Let C be a convex body in R^N and let S be a subset of C. Then $L_C(S) = L(S) \cap C$ and $L_C(S)$ is the affine span of S in C. Several geometric notions found in [1], including those of equivariant general position and of equivariant (q, T)-position, are summarized in the following definition.

DEFINITION (3.1). Suppose (R^N, G) is a G-space and (C, G) is an equivariant subspace of (R^N, G) where C is a convex set in R^N. Let Q and T be equivariant subsets of C and let q be a positive integer. Q is said to be in equivariant (q, T)-*general position in* C if every sectional subset S of Q satisfies the following conditions:

(1) S is in general position in C;

(2) For every subset S_1 of S containing less than $q+1$ elements, $L_C(S_1) \cap T = \emptyset$.

EQUIVARIANT GENERAL POSITION LEMMA (3.2). *Suppose R^N has an orthogonal G-action and the convex body C is an equivariant subspace of R^N. Let $Q = \{q_1, q_2, \ldots\}$ be a countable set; let $\varphi\colon Q \to C$ be a function; and let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers. If T is an equivariant closed subset of C and $\dim(L_C(T)) = k < N$, then there exists a function $\psi\colon Q \to C$ satisfying the following*:

(1) *$G(\psi Q)$ is in equivariant $(N-k, T)$-general position in C; and*

(2) $d(\varphi q_i, \psi q_i) < \varepsilon_i$.

Proof. Let $B(\varphi q_i, \varepsilon_i) = \{y \in C|\ d(\varphi q_i, y) < \varepsilon_i\}$. Pick ψq_1 to be any point in $B(\varphi q_1, \varepsilon_1) \cap (C-T)$. Then $E_1 = G(\psi q_1)$ satisfies conditions (1) and (2).

Assume $E_j = G(\{\psi q_1, \ldots, \psi q_j\})$ is defined and satisfies conditions (1) and (2). Let

$$P_1 = \bigcup \{L_C(S)|\ S \subset E_j;\ \#(S) \leqslant N;\ S \text{ is sectional}\}.$$

It follows that $P_1 = GP_1$ is a closed, nowhere dense set in C. Therefore, $C-GP_1$ is an equivariant, open, dense subset of C. If c is any element in $C-GP_1$, then $E_j \cup Gc$ is in equivariant general position.

Let

$$P = \bigcup \{L_C(S \cup T)|\ S \subset E_j;\ \#(S) \leqslant N-k-1;\ S \text{ is sectional}\}.$$

As was the case above for P_1, it follows that $P = GP$ is a closed, nowhere dense set in C. Therefore, $C-GP$ is an equivariant, open, dense subset of C and $(C-GP) \cap T = \emptyset$. Furthermore, $0 = (C-GP_1) \cap (C-GP)$ is an open, dense subset of C which does not intersect T.

Let $D = B(\varphi q_{j+1}, \varepsilon_{j+1}) \cap (C-T)$. D is open in C; thus GD is open in C. And, since the G-action is orthogonal, $GD \cap T = \emptyset$. Hence, $U = O \cap GD$ is open and equivariant. Let y be in D such that $Gy \subset U$. Define $\psi q_{j+1} = y$ and $E_{j+1} = E_j \cup Gy$. E_{j+1}, so constructed, satisfies (1) and (2).

Thus, using induction, a function ψ that satisfies the conclusion of the lemma can be constructed.

Lemma (3.2) is used in the next section to embed equivariant polyhedra.

4. Polyhedral replacement embedding lemma.

LEMMA (4.1). *Let K be a countable, locally finite, n-dimensional, simplicial complex with a free simplicial G-action and $K^0 = G(\{v_i\}_{i=1}^{\infty})$. Suppose R^{N+k}, $N > n$, has an orthogonal G-action and the convex body C is an equivariant subspace of R^{N+k}. Let*

$Q = \{q_1, q_2, \ldots\}$ *be a countable set of points in* C *and let* $\varepsilon_1, \varepsilon_2, \ldots$ *be a sequence of positive numbers. Let* T *be an equivariant subset of* C *and* $\dim(L_C(T)) = k \geqslant n$. *If* G *is free on* $C-T$, *then there exists an equivariant embedding* $h\colon (|K|, G) \to (C-T, G)$ *such that* $d(hv_i, q_i) < \varepsilon_i$.

Proof. Let $\psi\colon Q \to C$ be a function that satisfies Lemma (3.2). Let $G(\psi Q) = G(\{\psi q_i = r_i\})$. Define $h\colon K^0 \to C$ by $hgv_i = gr_i$, $g \in G$. Extend h linearly to all the simplexes of K. Note that $d(hv_i, q_i) = d(r_i, q_i) = d(\psi q_i, q_i) < \varepsilon_i$. The details which show that h is an equivariant embedding can be found in the proof of (3.1) in [2].

The following notation pertains to (4.2) below. Let X be a compact metric G-space of dimension $\leqslant n$, where G acts freely outside of a closed equivariant subspace A. Suppose R^{N+k}, $N > n$, is an orthogonal G-space and suppose the convex body C is an equivariant subspace of R^{N+k}. Furthermore, let G be free outside the equivariant closed subspace T of C where $\dim(L_C(T)) = k \geqslant n$.

If $w\colon (A, G) \to (T, G)$ is a fixed equivariant embedding and $j\colon (X, G) \to (C, G)$ is an equivariant map such that $j|_A = w$, then, corresponding to a given positive number η, the uniform continuity of j implies that there exists a positive number δ such that, if $d(x, x') < \delta$, then $d(jx, jx') < \frac{1}{6}\eta$. In addition, corresponding to δ, let (K, G), $(Z = |K| \cup A, G)$, and $f\colon (X, G) \to (Z, G)$ be as in (2.2).

Finally, denote by $K^0 = G(\{v_i\}_{i=1}^{\infty})$ and by $GB = G\{V_i\}$ the locally finite, equivariant, open cover of $X-A$, where K is generated by the nerve of B.

POLYHEDRAL REPRESENTATION EMBEDDING LEMMA (4.2). *There exists* $h\colon (Z, G) \to (C, G)$ *such that*:

(i) *h is an equivariant embedding*;

(i) $h|_A = w$;

(iii) $h|_{Z-A}$ *is a simplicial homeomorphism*; *and*

(iv) $d(gh(v_i), j(f^{-1}(\mathrm{St}gv_i))) < \frac{1}{2}\eta$, *for each* $g \in G$.

Proof. Define $D = \{V_i \in B \mid d(V_i, A) < \frac{1}{2}\delta\}$ and let $D' = B - D$.

For each $V_i \in D'$, choose $x_i \in V_i$. Then choose $p_i \in (C-T) \cap B(j(x_i), \varepsilon_i)$ where $\varepsilon_i = \frac{1}{4}\eta$ and where $B(j(x_i), \varepsilon_i)$ is the open ball of radius ε_i in C around $j(x_i)$. Similarly, $gp_i \in (C-T) \cap B(gj(x_i), \varepsilon_i)$ for each $g \in G$.

For each $V_i \in D$ there exists $a_i \in A$ such that $d(V_i, A) = d(\overline{V}_i, A) = d(\overline{V}_i, a_i)$, and there exists $x_i \in V_i$ such that $d(x_i, a_i) < \delta$. Choose $p_i \in (C-T) \cap B(w(a_i), \varepsilon_i)$, where $\varepsilon_i = \min\{\frac{1}{2}d(\overline{V}_i, a_i), \frac{1}{6}\eta\}$. Similarly, $gp_i \in (C-T) \cap B(gw(a_i), \varepsilon_i)$ for each $g \in G$.

By the Equivariant General Position Lemma (3.2), there exists a countable, equivariant set $G(\{q_i \mid i = 1, 2, \ldots\})$ in C with the property that $d(gp_i, gq_i) < \varepsilon_i$ for each $g \in G$. Furthermore, the following inequalities hold:

(1) For $V_i \in D'$, $d(j(x_i), q_i) \leqslant d(j(x_i), p_i) + d(p_i, q_i) < \varepsilon_i + \varepsilon_i = \frac{1}{4}\eta + \frac{1}{4}\eta = \frac{1}{2}\eta$.

(2) For $V_i \in D$, $d(j(x_i), q_i) \leqslant d(j(x_i), w(a_i)) + d(w(a_i), p_i) + d(p_i, q_i) < \frac{1}{6}\eta + \frac{1}{6}\eta + \frac{1}{6}\eta = \frac{1}{2}\eta$.

Finally, define $h\colon (Z, G) \to (C, G)$ as follows:

$$h|_A = w.$$

For each $g \in G$, let $h(gv_i) = gh(q_i)$ for the vertices $v_i \in K^0$. Then extend h linearly from K^0 to all of $|K| = Z - A$.

By definition, h is equivariant, $h|_A = w$, and $h|_{Z-A}$ is simplicial. (1) and (2) above imply that condition (iv) of the lemma is satisfied. On A, h is clearly one-to-one. Lemma (4.1) tells us that on $|K|$ h is an embedding into the complement of T in C. The details which show that h is continuous can be found in the proof of (3.2) in [2].

5. Function space lemmas. Results relating to some function spaces will be discussed here. For this discussion, (X, G), (C, G), A, T, and w are the same as in Section 4. Then, one defines

$M = (C, G)^{(X,G)}$;

$M' = \{\varphi \in M \mid \varphi|_A = w\}$;

$M'(\varepsilon) = \{\varphi \in M' \mid \varphi \text{ is an } \varepsilon\text{-map}\}$; and

$E = \{\varphi \in M' \mid \varphi \text{ is an embedding}\}$.

(5.1) *M' is complete.*

Since M is complete and M' is closed in M, statement (5.1) follows.

Note that the map $f\colon (X, G) \to (Z, G)$ used in the introduction to (4.2) above was an equivariant δ-map. Consequently, given an $\varepsilon > 0$, δ could have been chosen less than ε from the beginning.

(5.2) $M'(\varepsilon) \neq \emptyset$ *and, hence,* $M' \neq \emptyset$.

Statement (5.2) is obtained by letting $k = h \circ f$, where h satisfies (4.2). Since f is an equivariant δ-map, and hence an ε-map from (X, G) to (Z, G), then $k \in M'(\varepsilon)$.

Suppose S_1 is the set of ε-maps in C^X and suppose S_2 is the set of embeddings in C^X. In [4, pp. 57–59] it is shown that S_1 is open and dense in C^X and that $S_2 = \bigcap_\varepsilon S_1$. By adjusting the proof in [4] using (2.2), one proves the following.

(5.3) *For every positive number* ε, $M'(\varepsilon)$ *is dense in* M'.

Furthermore, $M'(\varepsilon) = M' \cap S_1$ and this implies that

$$\bigcap_\varepsilon M'(\varepsilon) = \bigcap_\varepsilon (M' \cap S_1) = M' \cap \big(\bigcap_\varepsilon S_1\big) = M' \cap S_2 = E.$$

The next two statements follow from the above observation.

(5.4) *For every positive number* ε, $M'(\varepsilon)$ *is open in* M'.

(5.5) $h \in \bigcap M'(\varepsilon)$ *if and only if* $h\colon (X, G) \to (C, G)$ *with* $h|_A = w$ *is an equivariant embedding. In particular,* $\bigcap_\varepsilon M'(\varepsilon) = E$.

Finally, using (5.5), (5.4), (5.3), and (5.1), the following result is established.

(5.6) *E is a dense G_δ set in M', and, hence,* $E \neq \emptyset$.

The following theorem is a corollary of (5.6).

THEOREM (5.7). *Let (X, G) be a compact n-dimensional metric G-space and let A be a closed equivariant subspace such that A equivariantly embeds in R^k, via w, with an orthogonal G-action. Suppose $G = R_1 \oplus \ldots \oplus R_r \oplus H_1 \oplus \ldots \oplus H_s$ acts freely on $X-A$. Then (X, G) equivariantly embeds in an orthogonal G-action on $R^N \times R^M$ via an equivariant embedding which extends w and where*

(i) $N+M = (n+1)+\max\{n, k\}$, *if* $2r+s \leqslant n+1$;

(ii) $N+M = (2r+s)+\max\{2n+1-2r-s, k\}$, *if* $n+1 < 2r+s \leqslant 2n+1$;

(iii) $N+M = (2r+s)+k$, *if* $2n+1 < 2r+s$.

Remark. In (5.7) above, let $R^N \times R^M = (R^{2r} \times R^{s+e}) \times (R^{M-k} \times R^k)$, where e is some nonnegative integer, and let the cyclic group R_j have order $|R_j|$. For each $j = 1, \ldots, r$, suppose α_j: $R^2 \to R^2$ is the rotation about the origin through the angle $2\pi/|R_j|$. Then the required orthogonal G-action on $R^N \times R^M$ is the product of the actions generated by the α_j $(j = 1, \ldots, r)$, the action on R^{s+e} generated by the antipodal involution, the trivial action on R^{M-k}, and the given G-action on R^k. Note that this action is free on $R^N - \{0\}$.

Proof. Consider
$$R^N = R^2 \times \ldots \times R^2 \times R^i,$$
where
$$i = \begin{cases} s, & \text{if} \quad n+1 \leqslant 2r+s, \\ n+1-2r, & \text{if} \quad 2r+s < n+1. \end{cases}$$
There is a unique copy of R^2 corresponding to each R_j in the decomposition of G. Recall that each R_j is cyclic of order $\neq 2$. For each $j = 1, \ldots, r$, let α_j: $R^2 \to R^2$ be the rotation about the origin in R^2 through the angle $2\pi/|R_j|$. Define β: $R^i \to R^i$ by $\beta x = -x$. Then $\gamma = (\alpha_1, \ldots, \alpha_r, \beta)$ generates an orthogonal G-action on R^N that is free on $R^N - \{0\}$. This G-action, together with the given orthogonal G-action on R^M, provides an orthogonal G-action on $R^N \times R^M$ which, using (5.6), implies the conclusion of the theorem.

Remark. An immediate corollary of (5.7) is Theorem (1.1), since $k = 0$ in the case of a free G-action.

6. G-actions and isotropy subgroups. Let X be a G-space and $x \in X$. Then $G_x = \{g \in G |\ gx = x\}$ denotes the isotropy subgroup of x. If H is a subgroup of G, then $X^H = \{x |\ H \subset G_x\}$ denotes the fixed point set of H. In addition, let $X_H = \{x |\ H = G_x\}$. Furthermore, suppose X is a compact n-dimensional metric space and G is a finite abelian group. Order the isotropy subgroups of the G-action on X as follows:
$$H_1, H_2, \ldots, H_m \quad \text{where} \quad H_i \underset{\text{subgroup}}{\subset} H_j$$
only if $j \leqslant i$. An ordering of this type will be called canonical [6]. Given a canonical ordering, define
$$X_i = \{x \in X |\ G_x = H_j \text{ for some } j \leqslant i\}.$$

Then $X_1 \subset X_2 \subset \ldots \subset X_m = X$ is a sequence of closed G-subspaces of X and $X_i - X_{i-1} = X_{H_i}$. Note that G/H_i acts freely on X_{H_i} and that the G-action and the G/H_i-action coincide on X_{H_i}.

DEFINITION (6.1). Let G be a finite abelian group. G is said to have a *cyclic decomposition of type* (r, s) if G is isomorphic to a direct sum
$$R_1 \oplus \ldots \oplus R_r \oplus H_1 \oplus \ldots \oplus H_s,$$
where the R's are cyclic groups of order $\neq 2$ and the H's are all of order 2.

THEOREM (6.2). *Let X be a compact n-dimensional metric space and let G be a finite abelian group acting on X with $H_1, \ldots, H_t$ distinct nontrivial isotropy subgroups. Suppose G has a cyclic decomposition of type (r, s) and G/H_i has a cyclic decomposition of type (r_i, s_i), for $i = 1, \ldots, t$, and suppose X^G embeds in R^k. Then X equivariantly embeds in an orthogonal G-action on R^N, where*
$$N \leqslant \max\{n+1, 2r+s\} + \sum_{i=1}^{t} \max\{n+1, 2r_i+s_i\} + \max\{k, n\}.$$

Proof. We can assume that $G = H_0, H_1, \ldots, H_t, H_{t+1} = (e)$ forms a canonical ordering of the isotropy subgroups. The proof proceeds, by induction, on the number of H_i's. By (5.7), $X_1 = X^G \cup X_{H_1}$ equivariantly embeds in an orthogonal G-action on R^{N_1}, where
$$N_1 = \begin{cases} (n+1)+\max\{n, k\}, & \text{if } 2r_1+s_1 \leqslant n+1, \\ (2r_1+s_1)+\max\{2n+1-2r_1-s_1, k\}, & \text{if } n+1 < 2r_1+s_1 \leqslant 2n+1, \\ (2r_1+s_1)+k, & \text{if } 2n+1 < 2r_1+s_1, \end{cases}$$
$$N_1 \leqslant \max\{n+1, 2r_1+s_1\} + \max\{k, n\}.$$
The result is true for $i = 1$.

Assume the result is true for $i = l$; i.e., X_l equivariantly embeds in R^{N_l} with an orthogonal G-action and $N_l \leqslant \sum_{i=1}^{l} \max\{n+1, 2r_i+s_i\} + \max\{k, n\}$. Note that $X_{l+1} = X_{H_{l+1}} \cup X_l$. Again, using (5.7), X_{l+1} equivariantly embeds in $R^{N_{l+1}}$, where
$$\begin{aligned} N_{l+1} &\leqslant \max\{n+1, 2r_{l+1}+s_{l+1}\} + \max\{N_l, n\} \\ &= \max\{n+1, 2r_{l+1}+s_{l+1}\} + N_l \\ &\leqslant \max\{n+1, 2r_{l+1}+s_{l+1}\} + \sum_{i=1}^{l} \max\{n+1, 2r_i+s_i\} + \max\{k, n\} \\ &= \sum_{i=1}^{l+1} \max\{n+1, 2r_i+s_i\} + \max\{k, n\}. \end{aligned}$$
Hence, the result is true for $i = l+1$. Note that for $i = t+1$, $2r_{t+1}+s_{t+1} = 2r+s$. Therefore, by induction, the theorem is proved.

An immediate corollary of (6.2) is the following.

THEOREM (6.3). *Let X be a compact n-dimensional metric space and let $G = R_1 \oplus \ldots \oplus R_r \oplus H_1 \oplus \ldots \oplus H_s$ be a finite abelian group acting on X with t distinct nontrivial isotropy subgroups. Suppose X^G embeds in R^k. Then X equivariantly embeds in an orthogonal G-action on R^N, where*

$$N \leqslant (t+1)(\max\{n+1, 2r+s\}) + \max\{k, n\}.$$

Remark. It is clear that, upon comparison with Theorem 2 in [7], (6.2) above does significantly lower the known dimension of the euclidean space in which a compact finite dimensional metric space can be equivariantly embedded. The following example is given for comparison.

EXAMPLE. The Kister and Mann result, Theorem 2 in [7], implies that, given the conditions of Theorem (6.2) above, the dimension of the euclidean space is

$$\begin{cases}(t+1)\{(n+1)(r+s)\}+2n+1 & \text{for } n \text{ odd},\\ (t+1)\{(n+2)r+s(n+1)\}+2n+1 & \text{for } n \text{ even}.\end{cases}$$

Let $Z_6 = Z_3 \oplus Z_2$ act on X, a compact n-dimensional metric space where n is odd. Then $r = s = 1$.

Case 1. Isotropy subgroups: (e), Z_2, Z_3, Z_6. Then $t = 2$. The following comparison for the computation of the dimension of the euclidean space is given:

(i) $K-M$ gives $(3)(n+1)(2)+2n+1 = 8n+7$.

(ii) Theorem (6.2) gives $\begin{cases}3+2+2+3 = 10 \text{ if } n = 1,\\ (n+1)+(n+1)+(n+1)+2n+1 = 5n+4 \text{ if } n>1\end{cases}$

and where k is the "worst" possible, namely $2n+1$.

Case 2. Free action. The $t = 0$ and $k = 0$.

(i) $K-M$ gives $(n+1)(2)+2n+1 = 4n+3$.

(ii) Theorem (6.2) gives $2n+1$.

CONCLUSION. Most of the results in this paper hold in the case of an arbitrary finite group. The abelian property of the group is essentially used only to obtain the cyclic decomposition of the group which, in turn, affects the calculation of the minimum dimension of the euclidean space having an orthogonal group action. In fact, if, in the case of an arbitrary finite group, one can determine the minimum dimension of a euclidean space having an orthogonal semi-free group action, then one can obtain equivariant embedding dimension results for an arbitrary finite group action analogous to all those contained in this work.

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