shows that μ: $(h, k) \mapsto hk$: $H\times K \to G$ is an open map. μ is perfect as K is compact. So, one may quote Corollary 14 and Proposition 5 (2).

(2) By a theorem essentially due to Iwasawa [21; 1.4], $G = H_0 K_0$.

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FACHBEREICH MATHEMATIK DER UNIVERSITÄT KONSTANZ

Accepté par la Rédaction le 9. 8. 1978

Universal mappings and weakly confluent mappings

by

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Abstract. All spaces are compact Hausdorff. Let f: $X\to Y$ be continuous; f is *universal* provided f has a coincidence with every map from X into Y, and f is *weakly confluent* provided every subcontinuum of Y is the image under f of a subcontinuum of X. This paper discusses relationships between these types of maps. Some results are: (1) *Universal maps onto locally connected metric continua are weakly confluent*; (2) *If* $H^1(X)\approx 0$, *then weakly confluent maps from* X *onto the* 2-*dimensional disk* B^2 *are universal* (*hence, have fixed points if* $X\subset B^2$); (3) CE-*maps between compact metric absolute retracts are universal.* Result (1) generalizes theorems of Grispolakis and Tymchatyn and of Mazurkiewicz. In relation to (1), it is shown that universal maps between metric continua need not be weakly confluent (which answers a question of Grispolakis and Tymchatyn). Result (2) strengthens theorems of Hamilton and of the author. Examples show that (2) would be false if $H^1(X)\not\approx 0$ and that the analogue of (2) for B^3 is false. Result (3) is shown to be false for monotone maps. Various applications to fixed point theorems are given.

1. Introduction. For each $n = 1, 2, \ldots$, let B^n denote the closed unit ball in Euclidean n-space R^n, $B^n = \{v\in R^n\colon \|v\|\leqslant 1\}$, and let $S^{n-1} = \{v\in B^n\colon \|v\| = 1\}$. A mapping (= continuous function) f from a topological space Z into B^n is AH-*essential* (this terminology comes from [14, p. 156]) provided that

$$f|f^{-1}(S^{n-1})\colon f^{-1}(S^{n-1})\to S^{n-1}$$

can not be extended to a mapping defined on all of Z into S^{n-1}. AH-essential mappings are usually simply called essential mappings, and their existence is equivalent to Z having covering dimension $\geqslant n$ [25, 5.1, p. 44]. In this paper, *essential mapping* means a mapping which is not homotopic to any constant mapping; an *inessential mapping* is a mapping which is not essential. A mapping f from a topological space Z_1 to a topological space Z_2 is *universal* [12] provided that for any mapping g: $Z_1\to Z_2$ there exists a point $p\in Z_1$ such that $g(p) = f(p)$. A *continuum* is a compact connected Hausdorff space. Let X and Y be compact Hausdorff spaces. A mapping f: $X\to Y$ is *weakly confluent* [16] provided that for each continuum $B\subset Y$ there is a continuum $A\subset X$ such that $f[A] = B$. A mapping f: $X\to Y$ is *monotone* [33, p. 127] provided that $f^{-1}(y)$ is connected for each $y\in Y$. Note that, by [33, 2.2, p. 138], monotone mappings between compact Hausdorff spaces are weakly confluent.

The following result was proved in [23, Thm. I] for X compact metric and B^2, in [10, 3.5] for X compact Hausdorff and B^2, and in [9, 4.3] for X compact Hausdorff and B^n (more generally, any connected n-manifold with an appropriate generalization of the definition above of AH-essential):

(1.1) *Any* AH-*essential mapping from a compact Hausdorff space* X *onto* B^n *is weakly confluent.*

There have been several applications of (1.1). For example: By using (1.1) for the case of B^2 and by using [25, 5.1, p. 44], stated above, it follows easily that (see [10, 3.6] and, for X compact metric, see [23, Thm. II]):

(1.2) *Any compact Hausdorff space* X *of covering dimension* $\geqslant 2$ *contains a nondegenerate indecomposable continuum.*

It is known that any AH-essential mapping from a topological space Z onto B^n is universal [19]. The converse is also true since if $f\colon Z \to B^n$ is not AH-essential, so that there is an extension $F_1\colon Z \to S^{n-1}$ of $f|f^{-1}(S^{n-1})$, then $g = -F_1$ does not agree with f at any point of Z. Thus (as has also been observed in [9, 5.4]):

(1.3) *A mapping from a topological space* Z *onto* B^n *is* AH-*essential if and only if it is universal.*

For future reference let us combine (1.1) and (1.3) to obtain:

(1.4) *Any universal mapping from a compact Hausdorff space* X *onto* B^n *is weakly confluent.*

In (2.3) we generalize (1.4) by proving a theorem which implies, for example, that any universal mapping from a compact Hausdorff space onto any locally connected metric continuum is weakly confluent (see (2.6)). In (2.14) and (2.16) we give examples which show, among other things, that universal mappings between metric continua need not, in general, be weakly confluent. Thus, these examples answer a question in [9, Section 5] where it was asked if universal mappings between compact Hausdorff spaces must be weakly confluent. Other properties of these examples are discussed in connection with an application, (2.8), of (2.6) to universal images of special types of continua. We mention that in (2.11) we use universal mappings to strengthen results of Segal and Read (see (2.12) and (2.13)).

Having studied the question of when universal mappings are weakly confluent in Section 2, we investigate the question of when weakly confluent mappings are universal in Section 3. In (3.2) we show that if X is a compact Hausdorff space which is cohomologically acyclic in dimension one, then any weakly confluent mapping from X onto B^2 is universal. Thus, the converse of (1.4) is true for B^2 when $H^1(X)\approx 0$. In (3.3) we give an example of a monotone mapping from B^3 onto B^3 which is not universal. Hence, the analogue of (3.2) for B^3 is false. In (3.4) we use (3.2) to prove a fixed point theorem which strengthens results in [11, Thm. 1] and [28] — see [3.5]. Some other applications of (3.2) are given in (3.5) and, in (3.6)–(3.7), we generalize (3.2). In (3.8)–(3.12) we give results about when near-homeomorphisms and CE-maps (all of which are weakly confluent) are universal. Among other results, we show in (3.10) that any CE-map from one compact metric absolute retract onto another is universal. By the example in (3.3) mentioned above, this is not true for monotone mappings (comp., (3.11)).

For future use, note the following definitions. Let Y be a metric continuum. An *ε-map* $f_\varepsilon\colon Y \to Z$ is a mapping such that $f_\varepsilon^{-1}(z)$ has diameter less than ε for each $z \in Z$. The continuum Y is *arc-like* provided that for each $\varepsilon > 0$ there is an ε-map from Y onto the unit interval $[0, 1]$. This is equivalent to the definition of snake-like in [15, p. 224]. The continuum Y is *circle-like* provided that for each $\varepsilon > 0$ there is an ε-map from Y onto S^1.

A space is said to be *nondegenerate* provided that it contains more than one point.

For any compact Hausdorff space Y, the symbol $C(Y)$ will denote the space of all nonempty subcontinua of Y with the Vietoris topology — see [24] and [29]. For future use let us note that if X and Y are compact Hausdorff spaces, then a mapping $f\colon X \to Y$ is weakly confluent if and only if the mapping $\hat{f}\colon C(X) \to C(Y)$, defined by

$$\hat{f}(A) = \{f(a)\colon a \in A\} \quad \text{for each } A \in C(X)$$

is surjective (i.e., $\hat{f}[C(X)] = C(Y)$).

Other definitions and notation will be given as they are needed or may be found in appropriate references at the end of the paper.

2. When universal mappings are weakly confluent.

The first main results in this section are (2.3) and its consequences in (2.5), (2.6), and (2.8). We will use the following notation:

(2.1) Notation. Let X and Y be spaces. We write $B \in E(X, Y)$ to mean that B is a closed connected subset of Y such that any mapping from a closed subset of X into B can be extended to a mapping of all of X into B. Such is the case, for example, when B is an absolute retract.

(2.2) Lemma. *Let* X *and* Y *be compact Hausdorff spaces and let* $B \in E(X, Y)$. *If* $f\colon X \to Y$ *is a universal mapping, then there exists a component* K *of* $f^{-1}(B)$ *such that* $f|K\colon K \to B$ *is universal; hence* $f[K] = B$.

Proof. Let $A = f^{-1}(B)$ and let $f_A = f|A\colon A \to B$. We first prove that f_A is universal. To do this, let $g\colon A \to B$ be a mapping. Since $B \in E(X, Y)$, g can be extended to a mapping $\bar{g}\colon X \to B$. Since $f\colon X \to Y$ is universal, there exists a point $p \in X$ such that $\bar{g}(p) = f(p)$. Since $\bar{g}(p) \in B$ and $f(p) = \bar{g}(p)$, $f(p) \in B$ and, thus, $p \in A$. Hence, $\bar{g}(p) = g(p)$ and $f(p) = f_A(p)$. Thus, $g(p) = f_A(p)$. Therefore, we have proved that f_A is universal. To complete the proof of (2.2), let $\mathscr{F} = \{A_\lambda\colon \lambda \in \Lambda\}$ denote the family of all the components of A and, for each $\lambda \in \Lambda$, let $f_\lambda = f|A_\lambda\colon A_\lambda \to B$. Suppose that f_λ is not universal for any $\lambda \in \Lambda$. Then, for each $\lambda \in \Lambda$, there is a mapping $h_\lambda\colon A_\lambda \to B$ such that $f_\lambda(x) \neq h_\lambda(x)$ for each $x \in A_\lambda$. Since $B \in E(X, Y)$, h_λ can be extended to a mapping $\bar{h}_\lambda\colon X \to B$ for each $\lambda \in \Lambda$. For each $\lambda \in \Lambda$, since $\bar{h}_\lambda(x) \neq f(x)$ for each $x \in A_\lambda$, we see that there is a closed open subset U_λ of A such that $U_\lambda \supset A_\lambda$

and $\bar{h}_\lambda(x) \neq f(x)$ for each $x \in U_\lambda$ (see [15, Thm. 2, p. 169]). Since A is compact, there are finitely many such sets U_λ, denoted by $U_{\lambda(1)}, U_{\lambda(2)}, \ldots, U_{\lambda(n)}$, such that $A = \bigcup_{i=1}^{n} U_{\lambda(i)}$. Let

$$k_1 = \bar{h}_{\lambda(1)}|U_{\lambda(1)}$$

and, for each $j = 2, 3, \ldots, n$, let

$$k_j = \bar{h}_{\lambda(j)}|(U_{\lambda(j)} - \bigcup_{i=1}^{j-1} U_{\lambda(i)}) .$$

Define $k\colon A \to B$ by

$$k(x) = \begin{cases} k_1(x), & \text{if } x \in U_{\lambda(1)}, \\ k_j(x), & \text{if } x \in (U_{\lambda(j)} - \bigcup_{i=1}^{j-1} U_{\lambda(i)}) \text{ for some } j > 1 . \end{cases}$$

Clearly, k is continuous and $k(x) \neq f(x)$ for all $x \in A$. This contradicts the fact that $f|A = f_A$ is universal. Thus, $f_{\lambda'}\colon A_{\lambda'} \to B$ is universal for some $\lambda' \in \Lambda$. Since universal mappings are surjective [12, Prop. 1, p. 433], $f[A_{\lambda'}] = B$. We have proved (2.2).

Aside from its use in proving the following theorem, (2.2) can be used to see very quickly that certain mappings are not universal — see (2.15).

(2.3) THEOREM. *Let X and Y be compact Hausdorff spaces such that $E(X, Y)$ is a dense subset of $C(Y)$. If $f\colon X \to Y$ is a universal mapping, then f is weakly confluent.*

Proof. By (2.2), each $B \in E(X, Y)$ must be the image of a subcontinuum K of X, i.e., $E(X, Y) \subset \hat{f}[C(X)]$ where $\hat{f}\colon C(X) \to C(Y)$ is as defined at the end of Section 1. Since X is compact Hausdorff, we have by [15, Thm. 1, p. 45] and [15, Thm. 14, p. 139] that $C(X)$ is compact. Hence, since $\hat{f}$ is continuous, $\hat{f}[C(X)]$ is compact. Thus, since $E(X, Y) \subset \hat{f}[C(X)]$ and $E(X, Y)$ is dense in $C(Y)$, we have that $C(Y) = \hat{f}[C(X)]$. Therefore, f is weakly confluent and we have proved (2.3).

In (2.5) and (2.6) we give two consequences of (2.3). The proof of (2.5) uses the following well-known fact:

(2.4) LEMMA. *Let Z be an arcwise connected metric continuum. For each $\varepsilon > 0$ there exists an absolute retract $Z_\varepsilon \subset Z$ such that each point of Z is within ε of some point of Z_ε.*

Proof. Let F be an ε-dense finite subset of Z. Let $A_1 \subset Z$ be an arc from one point of F to another point of F. If $F \not\subset A_1$, then let $A_2 \subset Z$ be an arc irreducible from some point of $F - A_1$ to A_1. If $F \not\subset (A_1 \cup A_2)$, then let $A_3 \subset Z$ be an arc irreducible from some point of $F - (A_1 \cup A_2)$ to $A_1 \cup A_2$. Continue this process finitely many times until $F \subset \bigcup_{i=1}^{n} A_i$. Then $Z_\varepsilon = \bigcup_{i=1}^{n} A_i$ has the desired properties.

(2.5) THEOREM. *Let Y be a metric continuum such that the family*

$$\mathscr{B} = \{Z \in C(Y)\colon Z \text{ is arcwise connected}\}$$

is a dense subset of $C(Y)$. If X is any compact Hausdorff space and $f\colon X \to Y$ is a universal mapping, then f is weakly confluent.

Proof. Let $\mathscr{A} = \{A \in C(Y)\colon A$ is an absolute retract$\}$. Since, by hypothesis, $\mathscr{B}$ is a dense subset of $C(Y)$, it follows from (2.4) that $\mathscr{A}$ is a dense subset of $C(Y)$. Hence, since $\mathscr{A} \subset E(X, Y)$, we have that $E(X, Y)$ is a dense subset of $C(Y)$. Therefore, by (2.3), f is weakly confluent.

The following result is of prime interest—it is discussed in Section 1 and in (2.7).

(2.6) COROLLARY. *Let Y be a locally connected metric continuum. If X is any compact Hausdorff space and $f\colon X \to Y$ is a universal mapping, then f is weakly confluent.*

Proof. The corollary is a special case of (2.5) since, as follows easily from [15, Thm. 1, p. 260], the family $\mathscr{L}$ of all locally connected subcontinua of Y is a dense subset of $C(Y)$ and since each member of $\mathscr{L}$ is arcwise connected (by [15, Thm. 2, p. 253] and [15, Thm. 1, p. 254]).

(2.7) Remarks. By (1.3) we see that (2.6) contains (1.1) as a special case. Our proofs are different than the proofs in the papers cited above (1.1)—these papers use some homotopy theory and facts about covering spaces. Thus, our proof of (2.6) yields a different proof of (1.2) than those in [10] and [23]. We hasten to point out that the full generality of [9, 4.3] for connected n-manifolds is not implied by (2.6) since AH-essential mappings and universal mappings are not (in general) related when their range space is not B^n.

By using (2.6) we can determine the universal images of certain continua (in the following result, h.l.c. means hereditarily locally connected [15, p. 268] and a *dendrite* is a locally connected metric continuum which contains no simple closed curve [15, p. 300]):

(2.8) THEOREM. *If f is a universal mapping from X onto a nondegenerate continuum Y where X is* (1) *an arc or a circle,* (2) *a graph,* (3) *an h.l.c. metric continuum, then Y is* (1′) *an arc,* (2′) *an acyclic graph,* (3′) *a dendrite, respectively.*

Proof. Since X satisfies (1), (2), or (3) and since f is continuous, Y is a locally connected metric continuum [15, Thm. 2, p. 256]. Hence, by (2.6), f is weakly confluent. Thus, by [6; II.1, II.3, and II.6], we see that if X satisfies (1), (2), or (3), then Y satisfies (1), (2), or (3) respectively. Since the range of any universal mapping has the fixed point property [12, Prop. 2, p. 433],

(*) Y has the fixed point property.

Now, assume that X satisfies (1). Then, since Y satisfies (1) and (*), Y satisfies (1′). Next, assume that X satisfies (2). Then, since Y satisfies (2) and (*), Y satisfies (2′). Finally, assume that X satisfies (3). Then, since Y satisfies (3), Y is one-dimensional by (1.2). Suppose that Y contains a simple closed curve S. Then, since Y is one-dimensional, it follows from [15, Thm. 2, p. 354] that S is a retract of Y. Thus, Y does not have the fixed point property. This contradicts (*). Hence, Y does not contain a simple closed curve. Therefore, since Y is a locally connected metric continuum, Y satisfies (3′). This completes the proof of (2.8).

(2.9) Remark. As regards parts (2) and (3) of (2.8), more information about the behavior of f on the branchpoints of X can be obtained from the weak confluence of f by using [6, II.1 and II.2]. For example, if X is a graph then each branchpoint of the graph Y is the image under f of a branchpoint of X. However, the order of a branchpoint in $Y = f(X)$ may be larger than the order of any branchpoint in X—this is seen by letting f be the "natural" monotone mapping from X onto Y where X and Y are drawn in Figure 1 below (f is the mapping which sends the

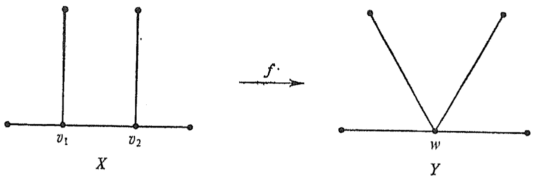

Fig. 1

arc $\overline{v_1v_2}$ to the point w; it follows easily that f is universal—see [12, Props. 7 and 8]).

We note the following corollary to (2.8):

(2.10). Corollary. *If Y is a locally connected metric continuum such that every mapping from any continuum onto Y is universal, then Y is an arc (or a point).*

Proof. Since Y is a continuous image of $[0, 1]$ by [15, Thm. 2, p. 256], (2.10) follows from part (1) of (2.8).

Recall the definition of arc-like near the end of Section 1. In [12, Thm. 3] it was shown that every mapping from a connected space onto an arc-like continuum is universal. We have the following related result which simultaneously strengthens theorems in [31] and [30]—see (2.12) and (2.13) below.

(2.11) Theorem. *Let Y be an arc-like continuum. Then, for any continuum X and any mapping f from X onto Y, $\hat{f}\colon C(X) \to C(Y)$ is universal.*

Proof. Let $\varepsilon > 0$, Since Y is arc-like, there is an ε map f_ε from Y onto $[0, 1]$. Let $g = f_\varepsilon \circ f$. In the proof of [14, 4.1] it was shown that for any mapping k from a continuum onto $[0, 1]$, $\hat{k}$ is AH-essential. Thus, by (1.3), $\hat{g}\colon C(X) \to C([0, 1])$ is universal. Since $g = f_\varepsilon \circ f$, clearly $\hat{g} = \hat{f}_\varepsilon \circ \hat{f}$. By [14, 2.5], $\hat{f}_\varepsilon$ is an ε-map. Hence, we have shown that, for each $\varepsilon > 0$, there exists an ε-map $\hat{f}_\varepsilon$ from $C(Y)$ into $C([0, 1])$ such that $\hat{f}_\varepsilon \circ \hat{f}$ is universal. Therefore, by [12, Lemma 1, p. 436], $\hat{f}$ is universal.

(2.12) Corollary (Segal [31, Thm. 3]). *If Y is an arc-like continuum, then $C(Y)$ has the fixed point property.*

Proof. Let $\hat{\imath}$ be the identity map from Y onto Y. By (2.11), $\hat{\imath}\colon C(Y) \to C(Y)$ is universal. Therefore, since $\hat{\imath}$ is the identity map for $C(Y)$, $C(Y)$ has the fixed point property by [12, Prop. 3, p. 433].

(2.13) Corollary (Read [30, Thm. 4]). *If Y is an arc-like continuum, then every mapping from any continuum onto Y is weakly confluent.*

Proof. Let f be a mapping from a continuum X onto Y. By (2.11), $\hat{f}\colon C(X) \to C(Y)$ is universal. Clearly, then, $\hat{f}[C(X)] = C(Y)$ [12, Prop. 1, p. 433]. Therefore, f is weakly confluent.

In (2.14) and (2.16) we give two examples related to (2.3)-(2.8). Recall that a metric continuum X is *Suslinian* [17] provided that X does not contain uncountably many mutually disjoint nondegenerate subcontinua.

In (2.8) we showed that the universal image of an arc is an arc (or a point). The following example shows that a universal image of the "simplest" non-locally connected arc-like continuum need not be arc-like and, in fact, need not even be acyclic. The example also shows that universal mappings between Suslinian atriodic continua in the plane need not be weakly confluent—thus, the example answers the question in [9, Section 5] mentioned here in Section 1.

(2.14) Example. Let X and Y be the continua in the plane R^2 drawn in Figure 2 below. The continuum $X = I \cup W \cup C$ and the continuum $Y = I' \cup W' \cup P$ where

$$I = \{(0, y) \in R^2\colon\ -1 \leqslant y \leqslant +1\},$$
$$W = \{(x, \sin[1/x]) \in R^2\colon\ 0 < x \leqslant +1\},$$
$$C = \{(+1, y) \in R^2\colon\ -1 \leqslant y \leqslant \sin[1]\},$$
$$I' = I,$$
$$W' = W,$$
$$P = \{(0, y) \in R^2\colon\ -3 \leqslant y \leqslant -1\} \cup \{(x, -3) \in R^2\colon\ 0 \leqslant x \leqslant +1\} \cup \cup \{(+1, y) \in R^2\colon\ -3 \leqslant y \leqslant \sin[1]\}.$$

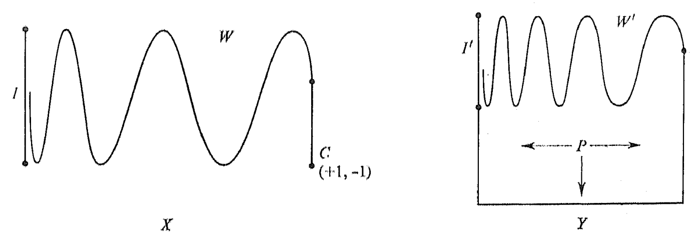

Fig. 2

Let h be a homeomorphism from C onto $I' \cup P$ such that $h(+1, -1) = (0, +1)$. Define f from X onto Y by

$$f(x, y) = \begin{cases} h(x, y), & \text{if } (x, y) \in C, \\ (x, y), & \text{if } (x, y) \in I \cup W \end{cases}$$

and note that f is a mapping since $h(+1, \sin[1]) = (+1, \sin[1])$. To see that f is universal, let $g\colon X \to Y$ be a mapping. For the purpose of proof, assume that $g(x, y) \neq f(x, y)$ for each $(x, y) \in C \cup W$. Then, by going along $C \cup W$ from $(+1, -1)$ towards I, we see that $g(x, y)$ is "ahead" of $f(x, y)$ for each $(x, y) \in C \cup W$, i.e., $f(x, y)$ is in the unique arc in Y from $(0, +1)$ to $g(x, y)$ for each $(x, y) \in C \cup W$. Thus, by an easy sequence argument, it follows that $g[I] \subset I'$. Hence, since $I = I'$, $g(x_0, y_0) = (x_0, y_0)$ for some $(x_0, y_0) \in I$. Therefore, $g(x_0, y_0) = f(x_0, y_0)$. This completes the proof that f is universal. We see that f is not weakly confluent since, for the continuum

$$M' = W' \cup I' \cup \{(0, y) \in R^2\colon\ -3 \leqslant y \leqslant -1\},$$

there is no subcontinuum of X which maps by f onto M'. The other properties attributed to the example in the paragraph above are obvious.

(2.15) Remark. Let X and Y be as in (2.14). The "most natural" mapping from X onto Y is the quotient map which identifies the two points $(+1, -1)$ and $(0, -1)$ in X. However, as follows easily from the second part of the conclusion to (2.2), this map is not universal.

The next example shows that a universal image of an arc-like continuum need not be arc-like even though the image is acyclic and one-dimensional. As in the case of the previous example, it shows that universal mappings need not be weakly confluent.

(2.16) Example. Let X and Y be the continua in the plane R^2 drawn in Figure 3 below. The continuum $X = W_1 \cup A \cup W_2$ and the continuum $Y = W_1' \cup A' \cup W_2'$ where

$$W_1 = \{(x, \sin[1/x]+3) \in R^2\colon\ 0 < x \leqslant +1\},$$
$$A = \{(0, y) \in R^2\colon\ -4 \leqslant y \leqslant +4\},$$
$$W_2 = \{(x, \sin[1/x]-3) \in R^2\colon\ -1 \leqslant x < 0\},$$
$$W_1' = W_1,$$
$$A' = \{(0, y) \in R^2\colon\ 0 \leqslant y \leqslant +4\},$$
$$W_2' = \{(x, -\sin[1/x]+3) \in R^2\colon\ -1 \leqslant x < 0\}.$$

Define a mapping f from X onto Y by

$$f(x, y) = \begin{cases} (x, y), & \text{if } (x, y) \in X \text{ and } y \geqslant 0, \\ (x, -y), & \text{if } (x, y) \in X \text{ and } y \leqslant 0. \end{cases}$$

It is clear that f is not weakly confluent since for the continuum

$$M' = \{(x, y) \in Y\colon\ y \geqslant +2\},$$
$$f^{-1}(M') = W_1 \cup W_2 \cup \{(0, y) \in X\colon\ 2 \leqslant |y| \leqslant 4\}$$

and, thus, there is no subcontinuum of X which maps by f onto M'. Now we show that f is universal. We will use the following fact: (#) *Any mapping from an arc onto an arc is universal* (comp., [12, Prop. 8, p. 434]). Now, to see that f is universal, let $g\colon X \to Y$ be a mapping. Note that $f[A] = A'$, and hence, by (#), $f|A\colon A \to A'$

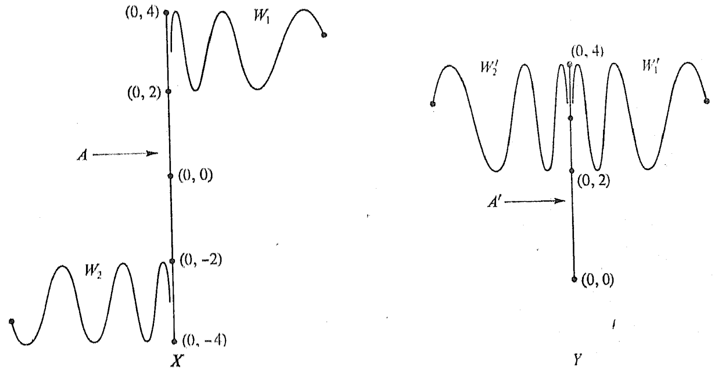

Fig. 3

is universal. Thus: If $g[A] \subset A'$, then there exists a point $a \in A$ such that $g(a) = f(a)$. Assume that $g[A] \not\subset A'$. Then, since $g[A]$ is arcwise connected, we see that $g[A] \subset W_i'$ for some $i = 1$ or 2. Hence, it follows easily that $g[X] \subset W_i'$. Let $B' = g[X]$. Note that B' is an arc (or a point) in W_i'. Hence, letting $B = f^{-1}(B')$, it follows from the definition of f that B is an arc (or a point) in W_i and that $f[B] = B'$. Thus, by (#), $f|B\colon B \to B'$ is universal. Hence, since $g[B] \subset B'$, there is a point $b \in B$ such that $g(b) = f(b)$. This completes the proof that f is universal.

(2.17) Remark. A mapping is said to be *pseudo-confluent* [18] provided that each irreducible subcontinuum of its range is the image of a subcontinuum of its domain. Clearly, weakly confluent mappings are pseudo-confluent. The mappings f in (2.14) and (2.16) are not pseudo-confluent since the continua M' in these examples are irreducible. Thus, universal mappings between metric continua need not even be pseudo-confluent.

3. When weakly confluent mappings are universal and some fixed point theorems. Let X be a compact Hausdorff space. Then, $H^1(X)$ denotes the one-dimensional cohomology group of X based on open covers and with integer coefficients. Thus, $H^1(X) \approx 0$ if and only if every mapping from X into S^1 is inessential [4, 8.1, p. 226].

If f is a universal mapping from a compact Hausdorff space X onto B^2, then f is weakly confluent by (1.4). The result in (3.2) shows that the converse is true provided that $H^1(X)\approx 0$. The converse is not true in general, as is noted in [9, Section 5], since the mapping f from the annulus $A = \{v \in R^2\colon 2^{-1} \leqslant ||v|| \leqslant 1\}$ onto B^2 given by $f(v) = (2||v||-1)v$, for each $v \in A$, is monotone, thus weakly confluent, but is not universal. The following lemma will be used in the proof of (3.2).

(3.1) Lemma. *Let X be a compact Hausdorff space. If f is a weakly confluent mapping from X onto B^2, then*

$$f|f^{-1}(S^1)\colon f^{-1}(S^1) \to S^1$$

is essential.

Proof. Let $p\colon B^2-\{(0,0)\} \to S^1$ denote radial projection, i.e., $p(v) = v/||v||$ for each $v \in (B^2-\{(0,0)\})$. For each $n = 1, 2, \ldots$, let A_n denote the annulus in B^2 given by

$$A_n = \{v \in B^2\colon 1-2^{-n} \leqslant ||v|| \leqslant 1\}$$

and let $X_n = f^{-1}(A_n)$. Let L denote the limit of a convergent subnet of $\{X_n\}_{n=1}^{\infty}$ [15, Thm. 1, p. 45]. Then, since $\{A_n\}_{n=1}^{\infty}$ converges to S^1 and since $f[X_n] = A_n$ for each $n = 1, 2, \ldots$, we have that $f[L] = S^1$. Suppose that $f|L\colon L \to S^1$ is inessential. Then, since $p\circ f|L = f|L$, $p\circ f|L\colon L \to S^1$ is inessential. Hence, by [10, 2.1, p. 345], there exists an open subset U of X such that $U \supset L$, $U \cap f^{-1}(0,0) = \emptyset$, and $p\circ f|U\colon U \to S^1$ is inessential. Since L is the limit of a subnet of $\{X_n\}_{n=1}^{\infty}$, there exists an index j such that $X_j \subset U$. Thus, since $p\circ f|U\colon U \to S^1$ is inessential, $p\circ f|X_j\colon X_j \to S^1$ is inessential. Since f is weakly confluent and since $X_j = f^{-1}(A_j)$, $f|X_j\colon X_j \to A_j$ is weakly confluent. A general result in [8, 3.5] implies that a weakly confluent mapping from a compact Hausdorff space (e.g., X_j) onto an annulus (e.g., A_j) induces a monomorphism from $H^1(A_j)$ into $H^1(X_j)$. Hence, since $f|X_j\colon X_j \to A_j$ is weakly confluent and since $p\circ f|X_j\colon X_j \to S^1$ is inessential, $p|A_j\colon A_j \to S^1$ is inessential (see [33, 8.3, p. 231]). However, clearly $p|A_j\colon A_j \to S^1$ is essential. Thus, we have a contradiction. Hence, $f|L\colon L \to S^1$ is essential. Therefore (since $L \subset f^{-1}(S^1)$), we have proved (3.1).

(3.2) Theorem. *Let X be a compact Hausdorff space such that $H^1(X)\approx 0$. If f is a weakly confluent mapping from X onto B^2, then f is universal.*

Proof. Let $f_1 = f|f^{-1}(S^1)\colon f^{-1}(S^1) \to S^1$. We will show that f is AH-essential. Suppose that f is not AH-essential. Then (by definition), f_1 can be extended to a mapping $F_1\colon X \to S^1$. Since $H^1(X)\approx 0$, F_1 is inessential [4, 8.1, p. 226]. Hence, since F_1 is an extension of f_1, we have that f_1 is inessential. However, since f is weakly confluent, this contradicts (3.1). Thus, f is AH-essential. Therefore, by (1.3), f is universal.

Applications and generalizations of (3.2) are in (3.4)–(3.7).

In the paragraph above (3.1) we observed that (3.2) would be false if the condition that $H^1(X)\approx 0$ were dropped. The following example shows that the analogue of (3.2) for mappings onto B^3 is false. In fact, we give an example of a monotone mapping from B^3 onto B^3 which is not universal (comp., [3.10]–(3.11)).

(3.3) Example. Let g be a monotone mapping from B^2 onto S^2 (for example, the quotient map of B^2 which shrinks S^1 to a point). Let X denote the cone over B^2 with vertex v and let Y denote the cone over S^2 with vertex w. Note that X and Y are homeomorphic to B^3. Let $f\colon X \to Y$ be the cone map of g, i.e., $f(v) = w$ and $f(z, t) = (g(z), t)$ if $(z, t) \in (X-\{v\})$. Since g is a monotone mapping from B^2 onto S^2, it follows easily that f is a monotone mapping from X onto Y. Since $f^{-1}(S^2) = g^{-1}(S^2) = B^2$,

$$f|f^{-1}(S^2)\colon f^{-1}(S^2) \to S^2$$

is inessential. Therefore, by [12, Prop. 10, p. 434], f is not universal. This completes (3.3).

In [27] it was shown that if X is any compact uncountable proper subset of B^2, then there is a fixed point free mapping from X onto B^2. However, as a consequence of (3.2), we have the following fixed point theorem:

(3.4) Theorem. *Let X be a compact subset of B^2 such that $H^1(X)\approx 0$. If f is a weakly confluent mapping from X onto B^2, then f has a fixed point.*

Proof. By (3.2), f is universal. Let $g\colon X \to B^2$ be the inclusion map (i.e., $g(x) = x$ for each $x \in X$). Then, since f is universal, there exists a point $p \in X$ such that $g(p) = f(p)$. Therefore, p is a fixed point for f.

(3.5) Remarks. Let us note that the condition in (3.4) that $H^1(X)\approx 0$ is equivalent to the condition that X does not separate the plane R^2 [5, 2.1, p. 357]; if X is locally connected, then these conditions are equivalent to X being unicoherent [5, p. 364]. In [11, Thm. 1], (3.4) was proved for the case when X is a locally connected unicoherent subcontinuum of B^2 and f is an open mapping. In [28], (3.2) and (3.4) were proved for confluent mappings. Thus, since open mappings are confluent [3, VI, p. 214] and confluent mappings are weakly confluent, (3.2) and (3.4) are stronger results than those in [28] and [11, Thm. 1]. In this connection, recall that the converse of (3.2) is also true [by (1.4)]. Next, let us note that some other fixed point theorems can be deduced using (3.2). For example (given an open cover $\mathcal{U}$ of a space X, a mapping $f_{\mathcal{U}}\colon X \to Y$ is a *$\mathcal{U}$-map* provided that $\{f^{-1}(y)\colon y \in Y\}$ refines the cover $\mathcal{U}$: (1) *If X is a compact Hausdorff space such that there is a weakly confluent $\mathcal{U}$-map $f_{\mathcal{U}}$ from X onto B^2 for each open cover $\mathcal{U}$ of X, then X has the fixed point property.* The proof of (1) goes as follows. Since there is a $\mathcal{U}$-map from X onto B^2 for each open cover $\mathcal{U}$ of X, $H^1(X)\approx 0$ (as follows from [22] and [7, p. 261]). Hence, by (3.2), $f_{\mathcal{U}}$ is universal for each $\mathcal{U}$. Therefore, by letting f in [12, Lemma 1, p. 436] be the identity mapping from X onto X, we see that (1) now follows from [12, Lemma 1, p. 436]. Let us also note the following result: (2) *If X is an inverse limit, over a directed set Λ, of $\{X_\lambda, f_{\lambda\mu}, \Lambda\}$ where each $X_\lambda = B^2$ and each $f_{\lambda\mu}\colon X_\lambda \to X_\mu$ is weakly confluent, then X has the fixed point property.* Since each $f_{\lambda\mu}$ is weakly confluent, it follows easily that each projection $\pi_\lambda\colon X \to X_\lambda = B^2$ is weakly confluent. Also: For any given open cover $\mathcal{U}$ of X, π_λ is a $\mathcal{U}$-map for some $\lambda \in \Lambda$. Therefore, (2) follows from (1).

Let us note the following generalization of (3.2):

(3.6) THEOREM. *Let Y be a compact Hausdorff space such that $H^1(Y) \approx 0$. If X satisfies the hypothesis of* (1) *or* (2) *in* (3.5), *then every weakly confluent mapping from Y onto X is universal.*

Proof. Using the fact that the composition of weakly confluent mappings is weakly confluent, (3.6) follows easily from (3.2) and [12, Lemma 1, p. 436].

(3.7) Remark. We note some special cases of (3.6). Let Z be an arclike or a circle-like continuum. Then, $X = C(Z)$ satisfies the hypothesis of (1) in (3.5) as can be seen by using the proofs in [14, 4.1 and 4.2] and then using (1.1). Thus, for Y as in (3.6), every weakly confluent mapping from Y onto $X = C(Z)$ is universal. The same is true when X is the cone over Z as can be seen by applying [12, Prop. 10], and then (1.4), to the cone map $\bar{f}_\varepsilon$ of any ε-map f_ε from Z onto $[0, 1]$ or, if Z is not arclike, onto S^1 (in which case, $f_\varepsilon\colon Z \to S^1$ is chosen to be essential by [14, 3.2]). The reader might wish to use [14, 1.3] to see easily that $\bar{f}_\varepsilon$ satisfies [12, Prop. 10] when Z is arc-like.

In (3.8)–(3.11) we will give some more results about when weakly confluent mappings are universal. However, unlike previous results, we will be considering some special types of weakly confluent mappings. A mapping f from a compact metric space X onto a compact metric space Y is a CE-*map* provided that for each $y \in Y$, $f^{-1}(y)$ has the shape of a point (see [2] or [32]). Note that CE-maps are weakly confluent since they are monotone. A mapping f from the compact metric space X onto Y is a *near-homeomorphism* provided that f is a uniform limit of homeomorphisms from X onto Y. Since a uniform limit of weakly confluent mappings is weakly confluent ([21] or [26, 2.5]), clearly near-homeomorphisms are weakly confluent. The following easy-to-prove but useful result shows when near-homeomorphisms are universal.

(3.8) LEMMA. *Let X be a metric continuum. Then,* (1) *every near-homeomorphism from X onto X is universal if and only if* (2) *X has the fixed point property.*

Proof. Let id denote the identity mapping from X onto X. If (1) holds, then id is universal and, hence [12, Prop. 3], (2) holds. Conversely, assume that (2) holds. Then [12, Prop. 3], id is universal. Let h be a homeomorphism from X onto X. Since $h \circ h^{-1} = \mathrm{id}$ and since id is universal, h is universal by [12, Prop. 4]. Thus, we have proved that every homeomorphism from X onto X is universal. Therefore, since the uniform limit of universal mappings is universal [12, Prop. 6], (1) holds.

Now we use (3.8) to prove the following theorem (if $f_i\colon X_i \to Y_i$ is a map for $i = 1$ and 2, then $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$):

(3.9) THEOREM. *Let X and Y be any compact metric absolute neighborhood retracts such that $X \times Q$ or $Y \times Q$, where Q is the Hilbert cube, has the fixed point property. If f is a* CE-*map from X onto Y, then f is universal.*

Proof. Let id denote the identity map from Q onto Q. Since f is a CE-map from X onto Y, clearly $f \times \mathrm{id}$ is a CE-map from $X \times Q$ onto $Y \times Q$. Thus, since $X \times Q$ and $Y \times Q$ are Q-manifolds [2, 44.1, p. 106] and since CE-maps between Q-manifolds are near-homeomorphisms [2, 43.2, p. 105], $f \times \mathrm{id}$ is a near-homeomorphism. We now know that $X \times Q$ and $Y \times Q$ are homeomorphic metric continua with the fixed point property. Hence, since $f \times \mathrm{id}$ is a near-homeomorphism from $X \times Q$ onto $Y \times Q$, it follows easily from (3.8) that $f \times \mathrm{id}$ is universal. Now, to see that f is universal, let g be a mapping from X into Y. Since $f \times \mathrm{id}$ is universal, there exists a point $(x, q) \in X \times Q$ such that

$$(g \times \mathrm{id})(x, q) = (f \times \mathrm{id})(x, q) .$$

Therefore, $g(x) = f(x)$ and we have proved (3.9).

(3.10) COROLLARY. *Let X and Y be any compact metric absolute retracts. If f is a* CE-*map from X onto Y, then f is universal.*

(3.11) Remarks. In (3.3) we gave an example of a monotone mapping from B^3 onto B^3 which is not universal. Thus, the condition in (3.10) that f be a CE-map can not be weakened to merely requiring that f be monotone. Let us note, however, that every monotone mapping from B^2 onto B^2 is universal by (3.2). Moreover: *If M^2 is any compact 2-manifold* (*with or without boundary*) *which has the fixed point property, then every monotone mapping from M^2 onto M^2 is universal.* This fact follows from using (3.8) since every monotone mapping from M^2 onto M^2 is a near-homeomorphism [34]. We note that there are compact 2-manifolds with the fixed point property which are not homeomorphic to B^2—such is the case for real projective 2-space [1, p. 31]. We remark that by letting X and Y be real projective n-space, n even, we see (by using [1, p. 31]) that X and Y satisfy the hypotheses in (3.9); however, X and Y do not satisfy the hypotheses in (3.10). In connection with the discussion above, let us note that there are universal mappings from B^n onto B^n (for any $n = 1, 2, \ldots$) and from the Hilbert cube Q onto Q which are not monotone. This can be seen by letting f be any non-monotone mapping from $[0, 1]$ onto $[0, 1]$, taking the cartesian product of f with itself n times for each $n = 1, 2, \ldots, \infty$, and using [13, 2.5 and 3.1]. Finally, I thank Steve Kaplan for a helpful conversation concerning the proof of (3.9).

In relation to (3.9), it follows immediately from [20, Thm. 4] that there are compact metric absolute neighborhood retracts with the fixed point property whose cartesian product with Q does not have the fixed point property. However, since this can not happen for Q-manifolds, we have the following corollary to (3.9):

(3.12) COROLLARY. *Let M be a compact Q-manifold with the fixed point property. If f is a* CE-*map from M onto M, then f is universal.*

Proof. Since any Q-manifold is homeomorphic to its cartesian product with Q [2, 15.1, p. 22], we see that $M \times Q$ has the fixed point property. Therefore, (3.12) follows from (3.9).

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Accepté par la Rédaction le 12. 10. 1979