The space of ultrafilters on N covered by nowhere dense sets

by

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Abstract. An estimate of a number of nowhere dense subsets of N^* necessary to cover it is given. The main tool for this estimate, so called base matrix, is defined and its properties are investigated. Several consistency results are established.

1. Introduction. Baire number in the Boolean algebraic context. Throughout this article, N^* denotes the space of all uniform ultrafilters on ω with the usual topology; this is the same as the remainder $\beta N - N$ in the Čech–Stone compactification of integers or the Stone representation space of the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$ (i.e. the power set of ω modulo the ideal of finite sets).

Let P be a dense-in-itself topological space. Define $n(P)$, the Baire number of P, to be the minimal cardinality of a family of nowhere dense sets covering the whole P.

The Polish topologists called this cardinal invariant Novák number (cf. [KcS]), since J. Novák was among the first topologists who studied this characterization of general topological spaces ([N]).

The literature concerning the Baire number of various topological spaces is not too extensive. It contains above all the papers proving consistency results for $n(\mathscr{R})$, where $\mathscr{R}$ is the real line ([H], [MS], [VH] and others). In [BV] it is shown, for $\varkappa$ uncountable regular cardinal, under the assumption $\varkappa^+ = 2^\varkappa$, that $U(\varkappa)$, the space of all uniform ultrafilters on $\varkappa$, has $n(U(\varkappa)) = \aleph_1$ (see also [CN]). A general estimation of $n(P)$ is given in [K1S], this result was generalized in [S]. The only non-trivial result independent on additional axioms of set theory that is known to the authors is the theorem stating that $n(P) = \aleph_1$ for P nowhere separable metric space. This was proved by Štěpánek and Vopěnka [ŠV].

Professor J. Mioduszewski asked the question, how small $n(N^*)$ can be.

The aim of the present paper is to discuss the possible cardinalities $n(N^*)$. The estimates are proved and some consistencies are shown. The technique developed in this paper clarifies the role of the set-theoretical assumption $n(N^*) > \mathfrak{c}$ in some constructions. Another motivation for the investigation of $n(N^*)$ stems from the following easy

Fact. $n(N^*) = n(\tau^*)$ for each infinite cardinal τ.

It turns out that Baire number is useful not only as a topological concept, but as a Boolean-algebraic one, too ([RS], [Bu]). Let us translate our mainc result to the Boolean language. (We use the Boolean terminology that of [Si].)

Let B be an atomless Boolean algebra, let $\mathscr{S}t(B)$ be the Stone space of B. Denote by Part (B) the set of all partitions of unity of B. Given nonempty $\mathscr{A} \subseteq \mathrm{Part}(B)$, an ultrafilter $\mathscr{F}$ on B is called to be *$\mathscr{A}$-generic* if $p \cap \mathscr{F} \neq \emptyset$ for every partition $p \in \mathscr{A}$.

Claim. *The minimal cardinal n such that there is a family $\mathscr{A} \subseteq \mathrm{Part}(B)$, $|\mathscr{A}| = n$, for which no ultrafilter on B is $\mathscr{A}$-generic, equals precisely the Baire number of $\mathscr{S}t(B)$.*

Because the genericity property depends only on a dense subset of B, we have immediately $n(\mathscr{S}t(B)) = n(\mathscr{S}t(\mathrm{Comp}(B)))$, where $\mathscr{C}omp(B)$ denotes the completion of B. This observation is slightly generalized in Proposition 4.10.

Now we restrict ourselves to the case $B = \mathscr{P}(\omega)/\mathrm{fin}$ and formulate the main estimation of $n(N^*)$ given in the present paper:

Let $\varkappa$ be the least cardinal for which B is not $(\varkappa, \mathfrak{c})$-distributive. Then the results of 2.5, 2.9, 3.5(i), 3.5(ii), 4.2, 4.5 read as follows:

$\aleph_1 \leqslant \varkappa \leqslant \mathfrak{c}$ and $\varkappa$ is a regular cardinal;
if $\varkappa < \mathfrak{c}$, then $\varkappa \leqslant n(N^*) \leqslant \varkappa^+$;
if $\varkappa = \mathfrak{c}$, then $\mathfrak{c} \leqslant n(N^*) \leqslant 2^{\mathfrak{c}}$;
$\varkappa \leqslant \mathrm{cf}(\mathfrak{c})$;
$\varkappa \leqslant |A|$ for each $A \subset {}^{\omega}\omega$ unbounded.

From now on, the absolute majority of authors of this paper being topologists, all is formulated in a topological language with one exception: Chapter 5 on consistency results. For the unexplained notation, see [CN].

Acknowledgment. The authors would like to express their gratitude to B. Beyrová and M. Dušek, whose support and understanding made this research possible. We are obliged to J. Mioduszewski for his stimulating suggestions and comments.

2. The base matrix lemma. Perhaps nobody can be satisfied with an obvious inequality $\aleph_2 \leqslant n(N^*) \leqslant 2^{\mathfrak{c}}$. To improve it, we need to introduce several notions.

2.1. Definition. Let P be a dense-in-itself topological space. Call a family $\mathscr{G} \subset \mathrm{Open}(P)$ to be an *almost-partition* (of P), if $\mathscr{G}$ is pairwise disjoint and $\bigcup \mathscr{G}$ is dense in P.

If $\mathscr{G}$ and $\mathscr{H}$ are almost-partitions of P, then $\mathscr{G}$ refines $\mathscr{H}$ ($\mathscr{G} \prec \mathscr{H}$) if for each $G \in \mathscr{G}$ there is an $H \in \mathscr{H}$ with $G \subset H$.

Recall that a π-basis for a space P is a collection $\mathscr{B}$ of non-void open subsets of P such that each non-empty open set in P contains at least one member of $\mathscr{B}$.

2.2. Definition. Let P be a dense-in-itself topological space. The family $\Theta \subset \mathscr{P}(\mathrm{Open}(P))$ will be called a *matrix* (for P) if each $\mathscr{G} \in \Theta$ is an almost-partition of P.

A matrix Θ will be called *shattering*, if for each non-void open set $U \subset P$ there is some $\mathscr{G} \in \Theta$ such that U meets at least two members of $\mathscr{G}$.

A matrix Θ will be called *refining*, if the ordering $\succ$ well-orders the whole Θ.

A matrix Θ will be called a *base matrix* (for P) if it is refining, if $\bigcup \Theta$ is a π-basis for P and, if for no refining matrix Θ' with $|\Theta'| < |\Theta|$, $\bigcup \Theta'$ is a π-basis for P.

Indeed, there are spaces where no base matrices exist. We shall show that N^* is not the case (2.11.(c)).

Given two matrices Θ and Θ', then Θ' refines Θ ($\Theta' \prec \Theta$) if for each $\mathscr{G} \in \Theta$ there is an $\mathscr{H} \in \Theta'$ such that $\mathscr{H} \prec \mathscr{G}$. Furthermore, Θ' strongly refines Θ ($\Theta' \prec\prec \Theta$) if $|\Theta| = |\Theta'|$ and there is a bijection $b\colon \Theta \to \Theta'$ such that $b(\mathscr{G}) \prec \mathscr{G}$ for each $\mathscr{G} \in \Theta$.

2.3. Observation. Let Θ, Θ' be two matrices for P, $\Theta' \prec \Theta$. If Θ is shattering, so is Θ'. If $\bigcup \Theta$ is a π-basis for P, so is $\bigcup \Theta'$. ∎

2.4. Definition. Let P be a dense-in-itself topological space. Define

$$\varkappa(P) = \min\{|\Theta|\colon \Theta \text{ is a shattering matrix for } P\}.$$

Recall that $\pi(P)$, the π-weight of P, denotes the smallest cardinality of a π-basis for P (see e.g. [Ju]).

2.5. Lemma. *$\varkappa(P) \leqslant \pi(P)$ for each regular space P without isolated points. In particular, $\varkappa(N^*) \leqslant \mathfrak{c}$.*

Proof. Indeed, a matrix $\Theta = \{\{B, \mathrm{Int}(P-B)\}\colon B \in \mathscr{B}\}$ is shattering whenever $\mathscr{B}$ is a π-basis for a regular space P. ∎

2.6. Lemma. *Let Θ be a matrix for N^*, $|\Theta| < \varkappa(N^*)$. Then there exists an almost-partition $\mathscr{G}$ of N^* such that $\mathscr{G} \prec \mathscr{H}$ for each $\mathscr{H} \in \Theta$ and such that $\mathscr{G} \subset \mathscr{C}\mathrm{lopen}(N^*)$.*

Proof. We may and shall assume that $\bigcup \Theta \subset \mathscr{C}\mathrm{lopen}(N^*)$. For each non-void $M \in \mathscr{C}\mathrm{lopen}(N^*)$ pick a non-void clopen $B_M \subset M$ such that for each $\mathscr{H} \in \Theta$, $|\{H \in \mathscr{H}\colon H \cap B_M \neq \emptyset\}| = 1$. Such a choice is always possible, otherwise there would be an $M \in \mathscr{C}\mathrm{lopen}(N^*)$ such that the trace of Θ on M be a shattering matrix for M, which contradicts the assumption $|\Theta| < \varkappa(N^*)$ as M is homeomorphic to N^*. Now take a maximal pairwise disjoint subfamily $\mathscr{G}$ of $\{B_M\colon M \in \mathrm{Clopen}(N^*)\}$. Necessarily, $\mathscr{G}$ is the desired almost-partition. ∎

2.7. Lemma. *If Θ is a shattering matrix for N^*, $|\Theta| = \varkappa(N^*)$, then there exists a refining matrix Θ' with $\Theta' \prec\prec \Theta$ such that $\bigcup \Theta' \subset \mathscr{C}\mathrm{lopen}(N^*)$.*

Proof. Well-order $\Theta = \{\mathscr{H}_\xi\colon \xi < \varkappa(N^*)\}$, define $\mathscr{G}_0$ to be an arbitrary almost-partition of N^* consisting of clopen sets and refining $\mathscr{H}_0$. Then, having defined $\mathscr{G}_\xi$ for all $\xi < \eta$ ($\eta < \varkappa(N^*)$), apply Lemma 2.6 to $\Theta_\eta = \{\mathscr{G}_\xi\colon \xi < \eta\} \cup \{\mathscr{H}_\eta\}$ to obtain $\mathscr{G}_\eta$. Set $\Theta' = \{\mathscr{G}_\xi\colon \xi < \varkappa(N^*)\}$. ∎

As an immediate consequence of 2.6, 2.7 and of the well-known fact that each non-void G_δ-set in N^* has a non-void interior, we obtain the following:

2.8. Corollary. $\varkappa(N^*) \geqslant \aleph_1$.

2.9. Corollary. *$\varkappa(N^*)$ is a regular cardinal.*

Proof. Let $\Theta = \{\mathcal{G}_\xi: \xi<\varkappa(N^*)\}$ be a shattering matrix for N^*. By 2.7 and 2.3 we may assume that Θ is shattering and refining. By 2.3, each matrix $\Theta_A = \{\mathcal{G}_\xi: \xi\in A\}$ is shattering whenever A is cofinal in $\varkappa(N^*)$. Since $|\Theta|_A = |A|$ and since $\varkappa(N^*)$ is the minimal cardinality of a shattering matrix, $\mathrm{cf}(\varkappa(N^*)) = \varkappa(N^*)$. ∎

2.10. Corollary. $\varkappa(N^*)\leqslant n(N^*)$.

Proof. Let $\mathcal{D}$ with $|\mathcal{D}| = n(N^*)$ be a family of nowhere dense sets in N^* covering N^*. For $D\in\mathcal{D}$, let $\mathcal{G}_D$ be an almost-partition of N^* by clopen sets with $\bigcup\mathcal{G}_D$ disjoint from D. If U is open and non-empty, then there is some $D\in\mathcal{D}$ which meets U; the regular closedness of members of $\mathcal{G}_D$ implies that U must meet at least two members of $\mathcal{G}_D$. So $\Theta = \{\mathcal{G}_D: D\in\mathcal{D}\}$ is a shattering matrix for N^*, $|\Theta| = n(N^*)$ and $\varkappa(N^*)\leqslant|\Theta|$. ∎

2.11. Base Matrix Lemma.

(a) $\aleph_1\leqslant\varkappa(N^*)\leqslant\mathfrak{c}$.

(b) *$\varkappa(N^*)$ is a regular cardinal.*

(c) *For each shattering matrix Θ with $|\Theta| = \varkappa(N^*)$, there exists a base matrix Θ' with $|\Theta'| = \varkappa(N^*)$ and $\Theta'\lesssim\lesssim\Theta$. Moreover, we can require $\bigcup\Theta'\subset\mathscr{C}\mathrm{lopen}(N^*)$.*

Proof. We need to prove (c) only. Let Θ be a shattering matrix for N^* with $|\Theta| = \varkappa(N^*)$. In virtue of 2.7 and 2.3 it suffices to show that there exists a matrix $\Theta'\lesssim\lesssim\Theta$ such that $\bigcup\Theta'$ is a π-basis for N^*. By 2.7, we may assume Θ to be shattering and refining, $\bigcup\Theta\subset\mathscr{C}\mathrm{lopen}(N^*)$ and $\Theta = \{\mathcal{G}_\xi: \xi<\varkappa(N^*)\}$ such that $\mathcal{G}_\xi\lesssim\mathcal{G}_\eta$ whenever $\eta<\xi<\varkappa(N^*)$.

Claim. *For each nonempty clopen set $M\subset N^*$ there exists some $\bar\xi<\varkappa(N^*)$ such that M meets $\mathfrak{c}$ members of $\mathcal{G}_{\bar\xi}$.*

Fix a set M. There exists a sequence $\xi_1<\xi_2<\ldots<\xi_n<\ldots$ such that

(i) if $f\in{}^n2$, then there exists a $G_f\in\mathcal{G}_{\xi_n}$ such that $G_f\cap M\neq\emptyset$;

(ii) if $f,f'\in{}^n2$, $f\neq f'$, then $G_f\neq G_{f'}$,

(iii) if $f\in{}^n2$, $h\in{}^m2$, $n>m$ and $h = f\restriction m$, then $G_f\subset G_h$.

Induction. ξ_1 already exists since Θ is shattering. Suppose ξ_{n-1} be defined, let $h\in{}^{n-1}2$. Θ is shattering and refining, thus there exists some $\xi_h>\xi_{n-1}$ such that the set $G_h\cap M$ meets at least two members of $\mathcal{G}_{\xi_h}$. Then in $\mathcal{G}_{\xi_n}$, where $\xi_n = \sup\{\xi_h: h\in{}^{n-1}2\}$ the desired family $\{G_f: f\in{}^n2\}$ can be found, since $\mathcal{G}_{\xi_n}$ refines all $\mathcal{G}_{\xi_h}$.

The ordinal $\bar\xi = \sup\{\xi_n: n \text{ a positive integer}\}$ is smaller than $\varkappa(N^*)$ because $\mathrm{cf}(\varkappa(N^*))>\omega$ by 2.8 and 2.9. If $f\in{}^\omega2$, one can choose a decreasing sequence $G_{f\restriction n}$ with each $G_{f\restriction n}\in\mathcal{G}_{\xi_n}$. The intersection $\bigcap_{n\in\omega}G_{f\restriction n}$ has non-empty interior, which must meet at least one member of $\mathcal{G}_{\bar\xi}$, call this member G_f. An almost partition $\mathcal{G}_{\bar\xi}$ refines all $\mathcal{G}_{\xi_n}$, it follows that if $f,f'\in{}^\omega2$, $f\neq f'$, then $G_f\neq G_{f'}$. Obviously every G_f meets M. The claim is proves.

Let $\xi<\varkappa(N^*)$, denote by $\mathscr{A}_\xi$ the family of all clopen subsets of N^* which meet $\mathfrak{c}$ members of $\mathcal{G}_\xi$. If $\mathscr{A}_\xi = \emptyset$, let $\mathscr{H}_\xi = \mathcal{G}_\xi$. If $\mathscr{A}_\xi\neq\emptyset$, one can (using an easy transfinite recursion) find a one-to-one mapping $\varphi_\xi: \mathscr{A}_\xi\to\mathcal{G}_\xi$ such that $A\cap\varphi_\xi(A)\neq\emptyset$ for each $A\in\mathscr{A}_\xi$. For $G\in\mathcal{G}_\xi$, let $\mathscr{H}_G$ be an almost partition of G such that $\mathscr{H}_G$ contains some H with $H\subset A$ in each case when $G = \varphi_\xi(A)$. Let $\mathscr{H}_\xi = \{H\in\mathscr{H}_G: G\in\mathcal{G}_\xi\}$.

The matrix $\Theta' = \{\mathscr{H}_\xi: \xi<\varkappa(N^*)\}$ has the desired properties. Obviously $\Theta'\lesssim\lesssim\Theta$, and in order to prove that $\bigcup\Theta'$ is a π-basis it suffices to show that for each non-void clopen $A\subset N^*$ there is some $\bar\xi<\varkappa(N^*)$ with $A\in\mathscr{A}_{\bar\xi}$, that is, A has to meet $\mathfrak{c}$ members of $\mathcal{G}_{\bar\xi}$ — and this is precisely the statement in the claim. ∎

2.12. Remarks. (a) Lemma 2.6 is a mere translation of an assertion that the Boolean algebra $\mathscr{P}(\omega)/_{\mathrm{fin}}$ is $|\Theta|$-distributive for $|\Theta|<\varkappa(N^*)$.

(b) Lemma 2.6 does not hold for the space N^* only. It can be extended e.g. to those topological spaces P having a π-basis consisting of homeomorphic images of P. Similar comment applies to the Lemmas 2.7 and 2.9. Assuming certain additional conditions concerning the cardinal invariants of a space P an analogue of the whole Base Matrix Lemma can be formulated and proved.

(c) It was noticed by Kulpa and Szymański that the matrices Θ for which $\bigcup\Theta$ is a π-basis are of utmost importance for estimating the Baire number. We shall begin the next paragraph with a reformulation of their theorem together with a sketch of the proof.

3. Covering N^* by nowhere dense sets.

3.1. Theorem ([KlS]). *Let P be a topological space, let β be a regular infinite cardinal and suppose that there exists a matrix Θ for P satisfying*

(a) *$\bigcup\Theta$ is a π-basis of P,*

(b) *to each $G\in\bigcup\Theta$ one can assign a family $\{G(\eta): \eta<\beta\}$ of pairwise disjoint nonempty open subsets of G,*

(c) $|\Theta|<\beta$.

Then there exists a family $D_0\subset D_1\subset\ldots\subset D_\eta\subset\ldots$ $(\eta<\beta)$ of nowhere dense sets covering P.

Proof. Let $D_\eta = P-\bigcup\{G(\iota): \eta<\iota<\beta,\ G\in\bigcup\Theta\}$. Each D_η is obviously closed. If U is a non-void open subset of P, then there is some $G\subset\bigcup\Theta$ with $G\subset U$. Clearly any $G(\iota)$ with $\iota>\eta$ misses D_η and is contained in U, so D_η is nowhere dense. If $x\in P$, define $\eta_x = \sup\{\iota<\beta: x\in G(\iota) \text{ for some } G\in\bigcup\Theta\}$, $\sup\emptyset$ is understood to be 0. Since $|\{\iota<\beta: x\in G(\iota) \text{ for some } G\in\bigcup\Theta\}|\leqslant|\Theta|$, β is regular, and $|\Theta|<\beta$, we have $\eta_x<\beta$. By the definition, $x\in D_{\eta_x}$. Thus $\bigcup\{D_\eta: \eta<\beta\} = P$. ∎

3.2. Definition. Let Θ be a matrix for a dense-in-itself topological space P. A family $\mathscr{C}$, which is centered, contained in $\bigcup\Theta$, and maximal with respect to those two properties, will be called a *chain* in Θ. The number $|\mathscr{C}|$ is called a *length* of the chain $\mathscr{C}$; and the chain $\mathscr{C}$ is said to be *long* if $|\mathscr{C}| = |\Theta|$.

3.3. Lemma. *Let Θ, Θ' be two matrices for N^*. If $\Theta'\lesssim\lesssim\Theta$, then Θ' does not contain longer chains than Θ does.* (Obvious.)

3.4. Lemma. *$\varkappa(N^*) = n(N^*)$ if and only if there exists a shattering matrix Θ without long chains, $|\Theta| = \varkappa(N^*)$.*

Proof. Let Θ be a shattering matrix without long chains, $|\Theta| = \varkappa(N^*)$, let $D_{\mathscr{G}} = X - \bigcup \mathscr{G}$ for each $\mathscr{G} \in \Theta$. Each $D_{\mathscr{G}}$ is clearly nowhere dense and $\bigcup \{D_{\mathscr{G}}: \mathscr{G} \in \Theta\} = N^*$: the existence of a point in $N^* - \bigcup \{D_{\mathscr{G}}: \mathscr{G} \in \Theta\}$ implies the existence of a long chain in Θ. Thus $n(N^*) \leqslant \varkappa(N^*)$, but $\varkappa(N^*) \leqslant n(N^*)$ by 2.10.

Let Θ' be the matrix given in the proof of 2.10. Apply 2.7 to Θ' to obtain a refining matrix Θ with $\bigcup \Theta \subset \mathscr{C}\mathrm{lopen}(N^*)$. Then $|\Theta| = n(N^*) (= \varkappa(N^*)$ by the assumption). The matrix Θ is shattering and contains no long chain by compactness and by $\bigcup \Theta \subset \mathscr{C}\mathrm{lopen}(N^*)$. ■

3.5. Main Theorem. *The numbers $\varkappa(N^*)$ and $n(N^*)$ are related as follows:*

(i) *If $\varkappa(N^*) < \mathfrak{c}$, then $\varkappa(N^*) \leqslant n(N^*) \leqslant \varkappa(N^*)^+$;*

(ii) *if $\varkappa(N^*) = \mathfrak{c}$, then $\varkappa(N^*) \leqslant n(N^*) \leqslant 2^{\mathfrak{c}}$;*

(iii) *$\varkappa(N^*) = n(N^*)$ if and only if there is a shattering matrix Θ for N^* without long chains, and with $|\Theta| = \varkappa(N^*)$;*

(iv) *moreover, if $n(N^*) \leqslant \mathfrak{c}$, then the covering family of nowhere dense sets of cardinality $n(N^*)$ can be chosen monotonically increasing.*

Proof. (ii) is a consequence of 2.10, (i) follows from 2.10 and 3.1, since for $\varkappa(N^*) < \mathfrak{c}$, the Kulpa–Szymański Theorem may be applied with $\beta = \varkappa(N^*)^+$. (iii) is then implied by 3.4.

For (iv), assume $n(N^*) \leqslant \mathfrak{c}$. If $\varkappa(N^*) < n(N^*)$, then (iv) holds by 3.1, so suppose $\varkappa(N^*) = n(N^*)$. By 3.4, 2.11(c) and 3.3, there is a base matrix Θ without long chains, $|\Theta| = \varkappa(N^*)$. Then $\bigcup \{D_{\mathscr{G}}: D_{\mathscr{G}} = N^* - \bigcup \mathscr{G}, \mathscr{G} \in \Theta\}$ covers N^* and the family is monotonically increasing, because Θ is refining and if $\mathscr{H}$ refines $\mathscr{G}$, then $D_{\mathscr{H}} \supset D_{\mathscr{G}}$. ■

3.6. Corollary. *If $\mathfrak{c}$ is a singular cardinal, then $n(N^*) < \mathfrak{c}$.* ■

(More will be shown in 4.2.)

Recall that a point $p \in N^*$ (= non-principal ultrafilter p on N) is selective, if for each partition $\mathscr{R} = \{R_n: n < \omega\}$ of N either there is an $n < \omega$ with $R_n \in p$ or there is a set $U \in p$ with $|U \cap R_n| \leqslant 1$ for each $n < \omega$. A point p is called $P(\mu)$-point (μ cardinal) if any intersection of less than μ neighborhoods of p is again a neighborhood of p.

3.7. Theorem. *If $n(N^*) > \mathfrak{c}$, then there exist at least $n(N^*)$ selective $P(\mathfrak{c})$-points in N^*.*

Proof. For each countable partition $\mathscr{R}$ of N, let $\mathscr{P}_{\mathscr{R}}$ be a maximal almost disjoint family in N such that $\mathscr{P}_{\mathscr{R}} \supset \mathscr{R}$ and, if $P \in \mathscr{P}_{\mathscr{R}} - \mathscr{R}$, then $|P \cap R| \leqslant 1$ whenever $R \in \mathscr{R}$. Let $\mathscr{G}_{\mathscr{R}} = \{P^*: P \in \mathscr{P}_{\mathscr{R}} \,\&\, P^* \neq \emptyset\}$. The matrix $\Theta = \{\mathscr{G}_{\mathscr{R}}: \mathscr{R}$ is a countable partition of $N\}$ is clearly shattering and $|\Theta| = \mathfrak{c}$.

By 3.5 and by the assumption $n(N^*) > \mathfrak{c}$, $\varkappa(N^*) = \mathfrak{c}$, $|\Theta| = \varkappa(N^*)$, so 2.11(c) applies: There exists a base matrix Θ' strongly refining Θ with $\bigcup \Theta' \subset \mathscr{C}\mathrm{lopen}(N^*)$. By 3.5, Θ' contains long chains.

Let $\mathscr{C} \subset \bigcup \Theta'$ be a long chain in Θ'. Then $\bigcap \mathscr{C} \neq \emptyset$ by compactness.

Let $x \in \bigcap \mathscr{C}$, then x is a selective $P(\mathfrak{c})$-point: If $\mathscr{R}$ is a countable partition of N, pick a neighborhood H of x from $\mathscr{H} \cap \mathscr{C}$, where $\mathscr{H} \in \Theta'$, $\mathscr{H} \precneqq \mathscr{G}_{\mathscr{R}}$. If P is a member of $\mathscr{P}_{\mathscr{R}}$ for which $H \subset P^*$, then $P \in x$ and either $P \in \mathscr{R}$ or $|P \cap R| \leqslant 1$ for each $R \in \mathscr{R}$. Hence x is selective. By the same argument and by the fact that Θ' is refining, $\mathscr{C}$ is a monotonically decreasing neighborhood basis for x. $|\mathscr{C}| = \mathfrak{c}$ and $\mathfrak{c} = \varkappa(N^*)$, thus by 2.11(b) $\mathfrak{c}$ is regular, so x is a $P(\mathfrak{c})$-point.

Finally, put $D_{\mathscr{G}} = N^* - \bigcup \mathscr{G}$ for each $\mathscr{G} \in \Theta'$. Clearly

$$N^* = \bigcup \{D_{\mathscr{G}}: \mathscr{G} \in \Theta'\} \cup \bigcup \{\bigcap \mathscr{C}: \mathscr{C} \text{ is a long chain in } \Theta'\},$$

but this gives us a covering of N^* by nowhere dense sets. $|\Theta'| = \mathfrak{c}$ and $n(N^*) > \mathfrak{c}$, so $|\{\bigcap \mathscr{C}: \mathscr{C}$ is a long chain in $\Theta'\}| \geqslant n(N^*)$ and the theorem follows. ■

Thus following theorem was proved under MA by Kucia and Szymański [KcS]. We shall show that the statement holds without any superfluous set-theoretical assumption.

3.8. Theorem. *Let S be a set of all selective $P(\mathfrak{c})$-points in N^*. Then $N^* - S$ can be covered by at most $\mathfrak{c}$ nowhere dense subsets of N^*.*

Proof. If $n(N^*) \leqslant \mathfrak{c}$, there is nothing to prove. If $n(N^*) > \mathfrak{c}$, then, using the notation from the proof of 3.7, $S \supset \{\bigcap \mathscr{C}: \mathscr{C}$ is a long chain in $\Theta'\}$. Thus $\{D_{\mathscr{G}}: \mathscr{G} \in \Theta'\}$ is a family of nowhere dense sets in N^* covering the set $N^* - S$. ■

3.9. Remarks. (a) The method of the proof of 3.7 seems to be quite general and we hope that the assumption $n(N^*) > \mathfrak{c}$ can be used instead of MA also in other constructions, where MA is usually used. Unfortunately, not in all — ask e.g. for the existence of non-selective P-point under $n(N^*) > \mathfrak{c}$.

(b) In Chapter 5, the model of ZFC will be described where $n(N^*) = 2^{\mathfrak{c}}$ and $2^{\mathfrak{c}} > \mathfrak{c}^+$, so there is no hope to prove that $n(N^*)$ equals either $\varkappa(N^*)$ or $\varkappa(N^*)^+$ as in the case $\varkappa(N^*) < \mathfrak{c}$.

4. Estimations for $\varkappa(N^*)$. The Gleason space of N^*. Let us start with a lemma, which enables us to strengthen 2.5 and 3.6.

4.1. Lemma. *Let $\mathscr{B} \subset \mathscr{C}\mathrm{lopen}(N^*)$, $|\mathscr{B}| < \mathfrak{c}$. Then there exists an almost-partition $\mathscr{G}$ of N^* such that each $B \in \mathscr{B}$ meets at least two members of $\mathscr{G}$.*

Proof. For each non-void $M \in \mathscr{C}\mathrm{lopen}(N^*)$ there exists a non-void clopen $G_M \subset M$ such that for each $B \in \mathscr{B}$, $B - G_M \neq \emptyset$. (Suppose the contrary: There is a non-void $M \in \mathscr{C}\mathrm{lopen}(N^*)$ such that each clopen $G \subset M$ contains some member of $\mathscr{B}$. Thus $\mathscr{B}$ contains a π-basis for M, yet $|\mathscr{B}| < \mathfrak{c}$ — a contradiction.) Let $\mathscr{G}$ be a maximal disjoint subfamily of $\{G_M: M \in \mathscr{C}\mathrm{lopen}(N^*)\}$. Clearly $\mathscr{G}$ has the desired properties. ■

4.2. Theorem. $\varkappa(N^*) \leqslant \mathrm{cf}(\mathfrak{c})$. (Compare with 2.5.)

Proof. Let $\tau = \mathrm{cf}(\mathfrak{c})$, let $\mathscr{B}_\xi \subset \mathscr{C}\mathrm{lopen}(N^*)$ be chosen for each $\xi < \tau$ such that $|\mathscr{B}_\xi| < \mathfrak{c}$ and $\bigcup \{\mathscr{B}_\xi: \xi < \tau\} = \mathscr{C}\mathrm{lopen}(N^*)$. Using Lemma 4.1, denote by $\mathscr{G}_\xi$ the almost-partition which "shatters" $\mathscr{B}_\xi$. The matrix $\Theta = \{\mathscr{G}_\xi\colon \xi<\tau\}$ is evidently shattering, so $\tau \geqslant \varkappa(N^*)$. ■

4.3. COROLLARY (Compare with 2.6.) *If $\mathfrak{c}$ is singular, then $n(N^*) \leqslant \mathrm{cf}(\mathfrak{c})^+$.* ■

4.4. DEFINITION. Consider ${}^\omega\omega$ with the order $f<g$ iff $f(n)<g(n)$ for all but finitely many $n\in\omega$. Define $\lambda = \inf\{|H|\colon H\subset {}^\omega\omega \;\&\; H$ has no upper bound$\}$.

4.5. THEOREM. $\varkappa(N^*)\leqslant\lambda$.

Proof. If $A\in[N]^\omega$, $A=\{x_n\colon n<\omega\}$ with $x_n<x_{n+1}$ for each $n<\omega$, we shall call the mapping $g_A\in{}^\omega\omega$ defined by $g_A(n)=x_n$ the enumeration of A. Obviously, if $\emptyset\neq A^*\subsetneqq B^*\subset N^*$, then $g_A>g_B$; if $A\in[N]^\omega$ and if $f\in{}^\omega\omega$, then there exists a $B\in[N]^\omega$ with $B^*\subset A^*$ and $g_B>f$.

In order to show that $\varkappa(N^*)\leqslant\lambda$ it suffices to prove that $\eta<\lambda$ whenever $\eta<\varkappa(N^*)$. So let $\eta<\varkappa(N^*)$, take any $S=\{f_\xi\colon \xi<\eta\}\subset{}^\omega\omega$ and let

$$\Theta = \{\mathscr{G}_\xi\colon \xi<\varkappa(N^*)\}$$

be an arbitrary shattering matrix for N^*. Let us define a matrix Θ_0 by the following. For $\xi<\eta$, let $\mathscr{H}_\xi$ be an almost-partition of N^* refining $\mathscr{G}_\xi$, consisting of clopen sets and such that for each $H\in\mathscr{H}_\xi$ we have $g_A>f_\xi$ whenever $A\in[N]^\omega$, $A^*=H$. For $\xi\geqslant\eta$, $\xi<\varkappa(N^*)$, let $\mathscr{H}_\xi=\mathscr{G}_\xi$.

Find a base matrix $\Theta' \lessdot\!\!\lessdot \Theta_0$ using 2.11(c); $\Theta'=\{\mathscr{F}_\xi\colon \xi<\varkappa(N^*)\}$. Let $F\in\mathscr{F}_\eta$, let $A\subset N$ be such that $\emptyset\neq A^*\subsetneqq F$. Since $\mathscr{F}_\eta$ refines all $\mathscr{H}_\xi$, $\xi<\eta$, there is a set $A_\xi\subset N$ with $A_\xi^*\supsetneqq A^*$, $A_\xi^*\in\mathscr{H}_\xi$ for each $\xi<\eta$. Then $g_A>g_{A_\xi}>f_\xi$, that is, S has an upper bound, hence $\eta<\lambda$. ■

4.6. COROLLARY. *If $\varkappa(N^*)=\mathfrak{c}$, then each subset of ${}^\omega\omega$ of cardinality less than $\mathfrak{c}$ has an upper bound.* ■

4.7. Remark. *The equality $\varkappa(N^*)=\lambda$ cannot be proved in* ZFC.

4.8. DEFINITION. Let P be a topological space, $\mathrm{Ro}(P)$ the Boolean algebra of all regular open subsets of P. The Gleason space of P, $G(P)$, is $\mathscr{S}t(\mathrm{Ro}(P))$, the Stone representation space of $\mathrm{Ro}(P)$.

4.9. Remark. Notice that [G] defines the notion of the Gleason space for compact Hausdorff spaces only. We are mainly interested in $G(N^*)$. Recall that the completion of the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$ is precisely $\mathrm{Ro}(N^*)$. It may be useful to characterize $G(N^*)$ using $\mathrm{Ro}(P)$ for some P non-compact (see 4.12).

Concerning the Baire number there is actually no difference between N^* and $G(N^*)$, to wit:

4.10. PROPOSITION. *Let P be a compact Hausdorff space without isolated points, $G(P)$ its Gleason space. Then $n(P)=n(G(P))$.*

The proof follows from the fact that there is an irreducible continuous map from $G(P)$ onto P. ■

It is known that

a) $G(N^*\times N^*)$ is homeomorphic to $G(G(N^*)\times G(N^*))$,

b) N^* is not homeomorphic to $N^*\times N^*$.

We shall show that $G(N^*)$ may be homeomorphic to $G(N^*\times N^*)$ in the next proposition, whose assumption is satisfied e.g. if $\varkappa(N^*)=\aleph_1$ or if MA holds. The proposition is a mere corollary of the forthcoming Theorem 4.12.

4.11. PROPOSITION. *Suppose that there exists a base matrix Θ for N^* all chains in which are long. Then*

(a) *$G(N^*)$ is homeomorphic to $G((N^*)^\alpha)$ for each $\alpha\leqslant\varkappa(N^*)$,*

(b) *in particular, $G(N^*)$ is homeomorphic to $G(N^*\times N^*)$,*

(c) *hence $n(N^*)=n(N^*\times N^*)$.* ■

4.12. THEOREM. *Denote $\varkappa=\varkappa(N^*)$. There exists a base matrix Θ for N^* all chains in which are long if and only if the spaces $G(N^*)$ and $G((\mathfrak{c}^\varkappa)_{(\varkappa)})$ are homeomorphic.*

($(\mathfrak{c}^\varkappa)_{(\varkappa)}$ denotes the product of $\varkappa$ copies of a discrete space of cardinality $\mathfrak{c}$ endowed with the $\varkappa$-product topology, see [CN], p. 69.)

Proof. Suppose $\Theta=\{\mathscr{G}_\xi\colon \xi<\varkappa(N^*)\}$ is a base matrix for N^* with long chains only. We may assume without any loss of generality that $|\{G'\in\mathscr{G}_\eta\colon G'\subset G\}|=\mathfrak{c}$ holds for any $G\in\mathscr{G}_\xi$ and $\xi<\eta<\varkappa(N^*)$. Thus it is possible to label all G's from $\mathscr{G}_\xi$ by functions from $\xi+1$ to $\mathfrak{c}$ $(\xi<\varkappa(N^*))$ such that the following hold:

(a) if $f,g\in{}^{\xi+1}\mathfrak{c}$, $f\neq g$, then $G_f\cap G_g=\emptyset$,

(b) if $\xi<\eta$, $G_f\in\mathscr{G}_\xi$, $G_g\in\mathscr{G}_\eta$ and $G_f\supset G_g$, then $f=g\restriction\xi+1$.

The space $(\mathfrak{c}^\varkappa)_{(\varkappa)}$ has a basis $\mathscr{B}$ consisting of all sets $U(f,\xi)$, $f\in{}^{\xi+1}\mathfrak{c}$, $\xi<\varkappa$, where $U(f,\xi)=\{g\in{}^\varkappa\mathfrak{c}\colon g\restriction\xi+1=f\}$. Thus it is easy to check that the mapping $\psi\colon \bigcup\Theta\to\mathscr{B}$ defined by the rule $\psi(G_f)=U(f,\xi)$ whenever $G_f\in\mathscr{G}_\xi$ extends to a Boolean algebra isomorphism $\tilde\psi\colon \mathrm{Ro}(N^*)\to\mathrm{Ro}((\mathfrak{c}^\varkappa)_{(\varkappa)})$. So, by Stone representation theorem, $G(N^*)$ is homeomorphic to $G((\mathfrak{c}^\varkappa)_{(\varkappa)})$.

If $G(N^*)$ and $G((\mathfrak{c}^\varkappa)_{(\varkappa)})$ are homeomorphic, then there must be an isomorphism $\psi\colon \mathrm{Ro}((\mathfrak{c}^\varkappa)_{(\varkappa)})\to\mathrm{Ro}(N^*)$. Since each set $U(f,\xi)$ $(f\in{}^{\xi+1}\mathfrak{c},\ \xi<\varkappa)$ is regular open, we may define $\mathscr{G}_\xi=\{\psi(U(f,\xi))\colon f\in{}^{\xi+1}\mathfrak{c}\}$, $\Theta=\{\mathscr{G}_\xi\colon \xi<\varkappa\}$. The verification that Θ is a base matrix for N^* all chains in which are long is straightforward. ■

Another consequence of 4.12 is stated in the following

4.13. COROLLARY. *Suppose that there exists a base matrix Θ for N^* each chain in which is long, let τ be an infinite cardinal. If $\tau<\varkappa(N^*)$, then $2^\tau=\mathfrak{c}$.* ■

4.14. PROBLEMS. (a) We do not know whether $\varkappa(N^*)=n(N^*)$ may happen.

(b) Put $\varkappa_n=\varkappa((N^*)^n)$. Clearly $\varkappa_n\geqslant\varkappa_{n+1}$, hence there is a p such that $\varkappa_p=\varkappa_r$ for all $r\geqslant p$. Proposition 4.11 shows that the sequence may be constant. We do not know any further properties of this sequence. (The Baire number of $(N^*)^n$ represents an open problem as well.)

(c) We have shown (3.8) that N^*-S can be covered by at most $\mathfrak{c}$ sets which are nowhere dense in N^*. On the other hand, consider N^*-S endowed with the subspace topology. Then the value of $n(N^*-S)$ is an open problem.

(d) Is it possible that $G(N^*)$ and $G(N^*\times N^*)$ are not homeomorphic?

5. Several consistency results and comments. In this chapter we shall introduce several examples of models of ZFC in order to show some possible values of $\varkappa$ and n for N^*. We do not construct any new forcing conditions; for the generic extensions we use only those Boolean algebras or partially ordered sets which are familiarly known. The proofs are rather brief; for the method of forcing the reader is referred to [Je] or [VH]. The distributivity of Boolean algebras in connection with the properties of generic extensions is studied e.g. in [Bu] or [VH].

In what follows, $\mathfrak{M}$ will denote some countable transitive model of ZFC. If G is a generic ultrafilter on $(P, \leqslant) \in \mathfrak{M}$ over $\mathfrak{M}$, then $\mathfrak{M}[G]$ denotes the generic extension of $\mathfrak{M}$ obtained by adding G to $\mathfrak{M}$.

5.1. Case $\varkappa(N^*)$.

I. $\varkappa = \aleph_1$.

Let us consider a generic extension $\mathfrak{M}[G]$ adding to $\mathfrak{M}$ at least $\aleph_1$ mutually Cohen reals ([Co]). That is, the set of forcing conditions is the set

$$\{f\colon f \in {}^u\{0,1\},\ u \subset \tau,\ |u| < \aleph_0\} \quad \text{for some } \tau \geqslant \aleph_1,$$

ordered by the inverse inclusion and constructed in $\mathfrak{M}$.

In $\mathfrak{M}[G]$, $\lambda = \aleph_1$, hence by 4.5 $\varkappa = \aleph_1$, too.

These types of models show the consistency of "$\varkappa(N^*) = \aleph_1$ & c is arbitrarily large". The same holds in the familiary known models used by Kunen [Ku] to obtain the consistency of "there is no selective ultrafilter on ω".

II. $\varkappa = c$.

One of the well-known consequences of MA says that the non-void intersection of less than c clopen subsets of N^* has a non-void interior (cf. [MS]). Consequently $\varkappa = c$.

III. $\aleph_1 < \varkappa < c$.

Let $\mathfrak{M}$ be a model of ZFC in which $c = \aleph_{\omega_2}$ and $\aleph_{\omega_2}^{\aleph_1} = \aleph_{\omega_2}$ hold. Let $B \in \mathfrak{M}$ be a complete Boolean algebra obtained by the canonical construction used by Solovay and Tennenbaum [ST] in their proof of the consistency of $\mathrm{MA}_{\aleph_1}$.

Then in $\mathfrak{M}[G]$, where G is a generic ultrafilter on B, the cardinals and the cofinalities are preserved and $\mathfrak{M}[G] \models 2^{\aleph_0} = \aleph_{\omega_2}$ & $\mathrm{MA}_{\aleph_1}$.

By 4.2 we have $\varkappa \leqslant \mathrm{cf}(c)$. Now $\mathrm{MA}_{\aleph_1}$ implies $\varkappa \geqslant \aleph_2$, hence $\mathfrak{M}[G] \models \varkappa(N^*) = \aleph_2$.

5.2. Case $n(N^*)$.

IV. $\varkappa < c$ & $n = \varkappa^+$.

If the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$ has a $\varkappa$-closed dense subset, then $n > \varkappa$. This situation arises in the models I and III from 5.1. In particular, the statement "$n(N^*) = \aleph_2$ & c is arbitrarily large" is consistent. Notice that there are no selective $P(c)$-points in N^* in Cohen and Solovay models for $c = \aleph_2$ mentioned in I. ([Ru], [Ku].) Thus $n(N^*) \leqslant c$ by 3.7 in these models.

V. $\varkappa = c$ & $c^+ < 2^c$ & $n = c^+$.

Suppose GCH takes place in $\mathfrak{M}$. Denote by $B_1 \in \mathfrak{M}$ the Solovay–Tennenbaum's Boolean algebra used for the consistency of $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$. The algebra B_1 satisfies c.c.c. Let $B_2 \in \mathfrak{M}$ be an algebra determined by the partial ordering $(P, \leqslant)$, where

$$P = \{f\colon f \in {}^u\{0,1\},\ |u| \leqslant \aleph_0,\ u \subset \omega_4\} \text{ and } f \leqslant g \quad \text{iff} \quad f \supset g\,.$$

The set P is countably closed dense subset of B_2. Let G be a generic ultrafilter on the free product $B = B_1 \times B_2$ over $\mathfrak{M}$. Then $G = G_1 \times G_2$ and $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$. Moreover, the Easton lemma is applicable. Hence

(i) all cardinals and cofinalities are preserved,

(ii) if $f \in \mathfrak{M}[G]$, $f\colon \omega \to \mathfrak{M}$, then $f \in \mathfrak{M}[G_1]$, thus the set $\mathscr{P}(\omega)$ is the same in $\mathfrak{M}[G]$ as in $\mathfrak{M}[G_1]$,

(iii) $\mathfrak{M}[G] \models 2^{\aleph_0} = \aleph_2$ & $2^{\aleph_1} = 2^c = \aleph_4$,

(iv) $\mathfrak{M}[G_1] \models \mathrm{MA}$ & $2^c = \aleph_3$ & B_2 satisfies $\aleph_2$-chain condition.

By (ii), the algebra $\mathscr{P}(\omega)/\mathrm{fin}$ is the same in $\mathfrak{M}[G]$ as in $\mathfrak{M}[G_1]$. Since $\mathrm{MA} + c = \aleph_2$ holds in $\mathfrak{M}[G_1]$, there is a base matrix Θ in $\mathfrak{M}[G_1]$ with long chains only. Hence $\mathfrak{M}[G_1] \models n = \aleph_3$. We shall show that also $\mathfrak{M}[G] \models n = \aleph_3$.

At first, notice that $\mathfrak{M}[G] \models n \leqslant \aleph_3$. Otherwise $\mathfrak{M}[G] = \mathfrak{M}[G_1][G_2]$ would contain a generic ultrafilter on $\mathscr{P}(\omega)/\mathrm{fin}$ over $\mathfrak{M}[G_1]$ which is impossible by the choice of B_2.

The following Easton-type lemma gives the estimation $n \geqslant \aleph_3$ in $\mathfrak{M}[G]$, if we take $\mathfrak{M}[G_1]$ as a ground model, $\tau = \aleph_1$, C_1 and C_2 as the completion in $\mathfrak{M}[G_1]$ of $\mathscr{P}(\omega)/\mathrm{fin}$ and B_2, respectively.

LEMMA. *Let τ be an infinite cardinal. Let C_1, C_2 be two complete Boolean algebras such that C_1 has a τ^+-closed dense subset and C_2 satisfies τ^+-chain condition. If G is a generic ultrafilter on C_2, then $n(C_1) > \tau^+$ in $V[G]$. (V is the ground universe of sets.)*

Proof. It is enough to show that for each matrix S in $V[G]$ consisting of partitions of unity of C_1 and such that $|S| \leqslant \tau^+$ there is a matrix $W \in V$ with $|W| \leqslant \tau^+$ and $W \prec\prec S$.

Let us therefore assume that $S = \{H_\alpha\colon \alpha < \tau^+\}$ is a system of partitions of $\mathbf{1}_{C_1}$ in $V[G]$. There exists a mapping $f\colon \tau^+ \times C_1 \to C_2$ such that

(i) $f \in V$,

(ii) for each $\alpha < \tau^+$, $u \in H_\alpha$ if and only if $f(\alpha, u) \in G$.

Hence, the formula

$$g(\alpha, u) = f(\alpha, u) - \bigvee\{f(\alpha, v)\colon v \neq u \;\&\; v \wedge u \neq \mathbf{0}\}$$

defines a set g that satisfies (i) and (ii), too. Moreover,

(iii) $u \wedge v \neq \mathbf{0}$ implies $g(\alpha, u) \wedge g(\alpha, v) = \mathbf{0}$.

Define $\mathscr{A} = \{A_\alpha\colon \alpha < \tau^+\}$ by the following $A_\alpha = \{u \in C_1 - \{\mathbf{0}\}\colon g(\alpha, u) \neq \mathbf{0}\}$. Then $\mathscr{A} \in V$ and $\bigvee A_\alpha = \mathbf{1}$ for each $\alpha < \tau^+$.

Fix α. Using the properties of C_1 and C_2 we can find a partition R_α of unity of C_1 such that every $r \in R_\alpha$ is compatible with at most τ members of A_α. This fact and the τ-distributivity of C_1 guarantees the existence of a partition W_α of unity which refines not only R_α, but also all partitions $\{u, -u\}$ whenever $u \in A_\alpha$.

Clearly $W_\alpha \precsim H_\alpha$, hence $\{W_\alpha\colon \alpha < \tau^+\}$ is the desired matrix. ∎

The inequality $\aleph_1 < \mathfrak{c} < 2^{\aleph_1}$ holds in the model just described. Proposition 4.13 tells us that in this model no base matrix for N^* contains long chains only. Nevertheless $G(N^*)$ and $G(N^* \times N^*)$ are homeomorphic in $\mathfrak{M}[G]$ (compare with 4.11).

VI. $\varkappa = \mathfrak{c}\ \&\ \mathfrak{c}^+ < 2^{\mathfrak{c}}\ \&\ n = 2^{\mathfrak{c}}$.

Suppose $\mathfrak{M} \models$ GCH. The set of forcing conditions

$$P = \{f\colon f \in {}^u\omega_1,\ |u| \leqslant \aleph_0,\ u \subset \omega_3\}$$

ordered by the inverse inclusion has at most $\aleph_1$ mutually incompatible elements. The completion of $\mathscr{P}(\omega)/\mathrm{fin}$ is isomorphic to the complete Boolean algebra generated by the dense subset $\{f\colon f \in {}^u\omega_1,\ |u| \leqslant \aleph_0,\ u \subseteq \omega_1$ (e.g. by 4.12).

In $\mathfrak{M}[G]$ (G is generic on P) all cardinals are absolute, no new subset of ω is added, and $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_3$.

Moreover, for an arbitrary system in $\mathfrak{M}[G]$ of partitions of unity in $\mathscr{P}(\omega)/\mathrm{fin}$ whose cardinality is less than $\mathfrak{c}$, there exists in $\mathfrak{M}[G]$ an ultrafilter on $\mathscr{P}(\omega)/\mathrm{fin}$ which is generic with respect to this system of partitions. Thus $\mathfrak{M}[G] \models n = 2^{\mathfrak{c}}$.

From the last model we immediately obtain that $n(N^*)$ can happen to be a singular cardinal, but $\mathrm{cf}(n(N^*)) > 2^{\aleph_0}$.

The question whether $\mathrm{cf}(n(N^*))$ may equal to $\aleph_0$ is open.

5.3. Now let us briefly summarize the properties of a generic extension $\mathfrak{M}[G]$, where the Boolean algebra the generic ultrafilter is taken on is $\mathscr{P}(\omega)/\mathrm{fin}$.

$\mathfrak{M}[G] \models \mathfrak{c} = \varkappa^{\mathfrak{M}}$, moreover, all bounded subsets of $\varkappa^{\mathfrak{M}}$ are in $\mathfrak{M}$, and $\mathfrak{c}^{\mathfrak{M}}$ collapses to $\varkappa^{\mathfrak{M}}$.

In $\mathfrak{M}[G]$, the Rothberger's assertion Q holds, i.e. each subset of ${}^\omega\omega$ of cardinality smaller than $\mathfrak{c}$ has an upper bound (see [Ro]). Thus, by Ketonen's result [K], there are many P-ultrafilters on N in $\mathfrak{M}[G]$. Moreover, the filter

$$\mathscr{F} = \{u \in \mathscr{P}^{\mathfrak{M}}(N)\colon [u] \in G\}$$

is a selective $P(\mathfrak{c})$-ultrafilter in $\mathfrak{M}[G]$.

As a consequence of Q, the following is valid in $\mathfrak{M}[G]$ too:

Every infinite maximal almost-disjoint system of subsets of N has the cardinality $\mathfrak{c}$. Thus for each uniform ultrafilter on N there exists a Comfort system ([CH]), i.e. an almost-partition of N^* such that each member of the ultrafilter contains a member of the almost-partition. The existence of a Comfort system for a given $p \in N^*$ is equivalent to the statement that p is a $\mathfrak{c}$-point in N^*. For the last assertion and other consequences of the assumption that every infinite maximal almost-disjoint family on N contains $\mathfrak{c}$ members, see [R] or [Ma_1, Ma_2].

5.4. Throughout the whole paper, the notion of a base matrix was exploited mainly for studying of $n(N^*)$. The following two examples indicate that the base matrices may serve as a tool in a wider area.

The assertion (c) of Base Matrix Lemma was used for a topological construction of examples in the theory of E-compact spaces in [PŠ]. The space constructed there is another example disproving Efimov's conjecture that a compact Hausdorff space has to contain either a convergent sequence or a copy of N^*.

W. W. Comfort has raised the question whether N^* and the space $U(\aleph_\omega)$ of all uniform ultrafilters on $\aleph_\omega$, may be homeomorphic. Though we do not know the answer, we can show that the Gleason spaces of the spaces in question may be homeomorphic.

Added in proof. Recently an interesting paper by S. H. Hechler (*Generalizations of almost disjointness, c-sets, and the Baire number of $\beta N - N$*, Gen. Top. and its Appl. 8 (1978), pp. 93–110) has appeared. Using another method, he studies the number $n(N^*)$, too.

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Accepté par la Rédaction le 19. 6. 1978

Equivariant embeddings of finite abelian group actions in euclidean space

by

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Abstract. Let X be a finite dimensional compact metric space and let G be a finite abelian group which acts on X. This paper shows that X equivariantly embeds in a euclidean space with an orthogonal G-action. Moreover, a minimum dimension for the euclidean space is obtained.

1. Introduction. Mostow [8] first showed that every action of a compact Lie group with a finite number of non-conjugate isotropy subgroups on a finite dimensional, separable, metric space can be equivariantly embedded in a linear action of the group on some euclidean space. However, Mostow's theorem said nothing about the required dimensions of the euclidean space. Copeland and de Groot [3] went on to find dimensions for the euclidean space in the case of an action of a cyclic group of prime order. Kister and Mann [7] extended this result to actions of compact abelian Lie groups with a finite number of distinct isotropy subgroups.

In [1] the present author obtained improvements on the results of Copeland and de Groot using methods different from those employed previously. A consequence is that, if X is a compact n-dimensional metric space with a free Z_p-action, then X equivariantly embeds in R^{2n+1} with an orthogonal Z_p-action.

The present work provides improvements on the results of Kister and Mann in the case of a finite dimensional compact metric space with a finite abelian group acting on it. The methods used here are extensions of the ideas found in [1]. An important corollary of this work is contained in the theorem stated below.

THEOREM (1.1). *Let $G = R_1 \oplus \ldots \oplus R_r \oplus H_1 \oplus \ldots \oplus H_s$ be a finite abelian group, where the R's are cyclic groups of order $\neq 2$ and the H's are all of order 2. Suppose X is a compact n-dimensional metric space with a free G-action. Then X equivariantly embeds in an orthogonal G-action on R^N, where $N = \max\{2n+1, 2r+s\}$.*

2. Equivariant spaces and equivariant maps. Throughout the remainder of this paper let G denote a finite abelian group and let X be a compact metric space. If G acts on X, then X is called a *G-space*, frequently denoted as (X, G). An equivariant map $f\colon (X, G) \to (Y, G)$ between two G-spaces is an equivariant ε-map if $\operatorname{diam} f^{-1}y < \varepsilon$ for every $y \in fX$. If (X, G) is a compact metric G-space and (Y, G) is a separable metric G-space, then $(Y, G)^{(X,G)}$ is the subspace of the metric space Y^X