On spaces admitting openly dyadic compactifications

by

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Abstract. According to B. A. Efimov [Soviet Math. Dokl. 10 (1969), pp. 776–779], a space is called *openly dyadic* if it is the continuous open image of some product of compact metrizable spaces. From known permanence properties of (openly) dyadic spaces it already results that, for fairly general spaces X, the existence of an (openly) dyadic compactification of X does have desirable consequences for the topological structure of many subspaces of X. The main purpose of this note is to give some sufficient and some necessary conditions for the Alexandroff one-point compactifications αL of certain locally openly dyadic spaces L to be openly dyadic; in particular, this is done in the case where L is a locally compact group and, in a sense more generally, in the case where L is of the form $X\times Y$ with αX and Y openly dyadic.

Although dyadic spaces in general, special types of dyadic spaces, and certain of their subspaces have been investigated thoroughly since at least thirty years, it seems to be, except for a few isolated results, rather unknown to date which conditions on a space ([1]) are necessary and/or sufficient in order that it admit a dyadic compactification or a dyadic compactification of a special kind. Among these results are the following: If the Stone–Čech compactification βX of a space X is dyadic, then X is pseudocompact (Engelking–Pełczyński [10; p. 58]). A metrizable space does not admit any non-metrizable, dyadic compactification (Efimov [6]). βX is an irreducible dyadic space for any irreducible, subdyadic, pseudocompact space X (Efimov–Engelking [8; p. 191]). An F'-space does not admit a dyadic compactification unless it is a discrete space with countably many points only (Gait [12]). A non-metrizable space, all of whose non-empty, open subsets have uncountably many points, which is the closed, continuous image of a separable metrizable space, does not admit a dyadic compactification (Arhangel'skiĭ [1]).

From classical permanence properties of dyadic spaces it already results that, for fairly general spaces X, the existence of a dyadic compactification of X does have desirable consequences for the topological structure of many subspaces of X. Our interest in the dyadic compactification problem arose, among other things, from theorems of the following type:

THEOREM 1. *In the space X, let S be a substantial subset, i.e. S a union of arbitrarily many G_δ-subsets of X (that is a $G_{\delta\Sigma}$-set in Engelking's terminology*

([1]) The word *space* will mean completely regular Hausdorff topological space.

[9; p. 288]); *equivalently* (cf. [3; 9.1]), *S a union of arbitrarily many Baire sets of X (that is, members of the σ-algebra generated by the zero-sets of X). Suppose X is of pointwise countable type (i.e. can be covered by compact subsets with countable neighborhood bases) and admits an openly dyadic compactification. Then $\bar{S}$ is a zero-set in X admitting an openly dyadic compactification, and a union of countably many openly dyadic zero-sets of X is dense in S.*

Proof. Use [9; Thm. 6] and an appropriate refinement of the argument of the proof of [9; Thm. 5].

THEOREM 2 [14]. *Suppose X is an absolute Baire space (i.e. a Baire set in one of its compactifications [3; 9.12]) or X is Čech-complete (i.e. homeomorphic with some G_δ-set in a locally compact space), and suppose X admits a dyadic compactification. Let the space Y be of the same sort, and let f: $X\times Y\to Z$ be a separately continuous, sequentially continuous map. Then f is continuous.*

The purpose of this note is to add a few items to the above list by giving some sufficient and some necessary conditions for the Alexandroff one-point compactifications αL of certain locally openly dyadic spaces L to be openly dyadic; in particular, this will be done in the case where L is a locally compact group and, in a sense more generally, in the case where L is of the form $X\times Y$ with αX and Y openly dyadic.

LEMMA 3. *Let $(g_\lambda\colon X_\lambda\to Y_\lambda)_{\lambda\in\Lambda}$ be an infinite family of continuous surjections where the Y_λ are non-empty, compact, and pairwise disjoint. Let $\alpha\Lambda = \Lambda\cup\{\infty\}$ and $\alpha(\coprod Y_\lambda) = \coprod Y_\lambda\cup\{\infty'\}$ be the one-point compactifications of the discrete space Λ and of the topological sum $\coprod Y_\lambda$ of the Y_λ, respectively. Then, by $g(\lambda, (x_\mu)_{\mu\in\Lambda}) = g_\lambda(x_\lambda)$ and $g(\infty, (x_\mu)_{\mu\in\Lambda}) = \infty'$, a continuous surjection $g\colon \alpha\Lambda\times\prod_{\mu\in\Lambda} X_\mu\to\alpha(\coprod_{\mu\in\Lambda} Y_\mu)$ is defined. If all g_λ are open, so is g.*

The proof is straightforward; so is the proof of the following

PROPOSITION 4. *Let Z_λ be a countable family of spaces each member Z_λ of which admits an (openly) dyadic compactification Y_λ. Then the sum $\coprod Z_\lambda$ admits an (openly) dyadic compactification, e.g. $\alpha(\coprod Y_\lambda)$.*

PROPOSITION 5. *Let X be a non-compact, paracompact, locally dyadic space. Then X has a dyadic compactification if and only if X is σ-compact. In this case, the following hold:*

(1) αX is dyadic, whereas βX is not.

(2) If αX is openly dyadic, so is αY for every open, perfect image Y of X.

(3) αX is openly dyadic provided X can be covered by countably many open and openly dyadic subsets.

(4) Suppose X is locally openly dyadic and totally disconnected. Then αX is openly dyadic. Moreover, if $\mathfrak{m}$ denotes the weight of X, then X is a perfect, open image of the sum of countably many copies of $\boldsymbol{D}^{\mathfrak{m}}$.

Proof. Suppose X admits a dyadic compactification K. Then X satisfies the Suslin condition since K does. So, being the sum of σ-compact subspaces [5; p. 241], X is necessarily σ-compact. Conversely, suppose X is σ-compact. Then some locally finite sequence D_n of dyadic subsets covers X. The canonical map $\coprod D_n\to X$ is perfect, hence has a continuous extension $\alpha(\coprod D_n)\to\alpha X$. As $\alpha(\coprod D_n)$ is dyadic by Proposition 4, so is αX. If βX were dyadic, X would be pseudocompact by the Engelking–Pełczyński Theorem quoted above and, therefore [13; p. 79], compact as it is realcompact [13; p. 115], a contradiction. As for (2), observe that, for any perfect, open map $X\to Y$, the extension $\alpha X\to\alpha Y$ is still open. Suppose now some sequence X_n of open, openly dyadic subsets covers X. Then

$$X = X_1\cup\bigcup_{n=1}^{\infty}\Big(\big(\bigcup_{i=1}^{n+1} X_i\big)\setminus\big(\bigcup_{i=1}^{n} X_i\big)\Big)$$

is a representation of X as a sum of countably many pairwise disjoint, open, openly dyadic subspaces, whence αX is openly dyadic by Proposition 4. Finally, suppose X is locally openly dyadic and totally disconnected. Then X is zero-dimensional [15; p. 12] and one is, therefore, in the situation of (3). So, let $X = \coprod Y_n$ for some sequence Y_n of non-empty, open, openly dyadic subspaces of X. By a theorem of Efimov [7], for every n, there is a continuous, open surjection $\boldsymbol{D}^{\mathfrak{m}}\to Y_n$. The rest is clear.

Remarks 6. (a) For a locally compact, non-paracompact space X, βX may be openly dyadic: Let $\mathfrak{m}>\aleph_0$ and X the Cantor space $\boldsymbol{D}^{\mathfrak{m}}$ minus one point; then $\boldsymbol{D}^{\mathfrak{m}} = \beta X$ [8; Lemma 4, Theorem 5].

(b) If Y is locally openly dyadic and zero-dimensional, but not σ-compact, αY may be dyadic without being openly dyadic, as simple examples show.

A well-known theorem of Kuzminov asserts that every compact group is dyadic; in fact, any such group is homeomorphic with a quotient group of some product of compact, metrizable groups [20; p. 39] and, consequently, is openly dyadic. We will show now, using the available structure theory, that locally compact, σ-compact groups and many of their coset spaces admit openly dyadic compactifications.

THEOREM 7. *Let G be a locally compact group. Then G admits an openly dyadic compactification if and only if G is σ-compact.*

Proof. Since locally compact groups are paracompact, necessity of the σ-compactness of G is clear in view of Proposition 5. Suppose now, conversely, that G is σ-compact. Let G_0 denote the connected component of the identity. As G/G_0 is zero-dimensional, the quotient map $G\to G/G_0$ admits a cross section according to a theorem of Mostert [18; Theorem 8], for which reason G is homeomorphic with $G/G_0\times G_0$. By a theorem of Iwasawa [17; p. 549], G_0, in turn, is homeomorphic with some product $\boldsymbol{R}^n\times C$ of a Euclidean space and any maximal compact subgroup C of G_0. Since G/G_0 has an open, compact subgroup [15; p. 61], it follows from Proposition 5 (4) that $\alpha(G/G_0)$ is openly dyadic. $\boldsymbol{R}^n$, obviously, admits an openly dyadic compactification as well, and so the proof is complete.

THEOREM 8. *Let G be a locally compact group and H some closed (not necessarily invariant) subgroup such that the left coset space $G:H = \{gH|\ g \in G\}$ is σ-compact. Then, the following hold:*

(1) *$G:H$ admits a dyadic compactification.*

(2) *If H contains the connected component G_0 of the identity, then $\alpha(G:H)$ is openly dyadic.*

(3) *$\alpha(G:H)$ is openly dyadic provided G_0 and H are compact.*

Proof. (1) Since G is locally dyadic by Theorem 7, so is $G:H$. Thus, quote Proposition 5.

(2) Suppose $G_0 \subseteq H$. Then $gG_0 \mapsto gH$: $G/G_0 \to G:H$ is an open, continuous surjection. Therefore, $G:H$ is zero-dimensional and locally openly dyadic as G/G_0 is. Thus, Proposition 5 (4) does the rest.

(3) Let G_0 and H be compact. Since the canonical quotient map $G \to G:H$ is an open, perfect map, then, in view of Proposition 5 (2), it suffices to show that αG is openly dyadic. This, however, is an immediate consequence of Proposition 5 (3) as G is homeomorphic with $G_0 \times G/G_0$ and, thus, is the sum of countably many open and openly dyadic subspaces.

COROLLARY 9. *Let X be a locally compact, σ-compact, totally disconnected space admitting a group of homeomorphisms which has a dense orbit and is uniformally equicontinuous with respect to some compatible uniformity of X. Then, for every substantial subset S of X, $\bar{S}$ is a zero-set in X and admits an openly dyadic compactification.*

Proof. By a theorem of Itzkowitz [16; p. 134], there exists a locally compact group G and a closed subgroup H of G such that $G:H$ is homeomorphic with X. As X is totally disconnected, necessarily $H \supseteq G_0$. So, quote Theorem 8 (2) and Theorem 1.

That every locally compact, σ-compact group G admits some openly dyadic compactification has been a rather direct consequence of the product decomposition $G \approx R^n \times C \times G/G_0$ with C a compact group. The question of whether or not αG is openly dyadic seems to be more delicate. Before we return to this question, we will consider the general problem of deciding open dyadicity of $\alpha(X \times Y)$ where αX and Y are openly dyadic.

LEMMA 10. *Let X be a non-compact, locally compact space, and let Y, Z be compact spaces. Suppose there exists a continuous map φ: $X \times Y \to Z$ with the following properties:*

(a) *For every $x \in X$, the map $y \mapsto \varphi(x, y)$: $Y \to Z$ is injective.*

(b) *The filter base consisting of the sets $\varphi((X \setminus C) \times Y)$, as C runs through the collection of all compact subsets of X, converges in Z.*

Then Φ: $(x, y) \mapsto (x, \varphi(x, y))$ embedds $X \times Y$ into $\alpha X \times Z$ in such a way that the closure of $\Phi(X \times Y)$ in $\alpha X \times Z$ is the one-point compactification of $\Phi(X \times Y)$.

Proof. Due to (a) and the compactness of Y, Φ is an embedding map. As X is not compact, $\Phi(X \times Y)$ is not compact. So, the closure of $\Phi(X \times Y)$ in $\alpha X \times Z$ contains at least one more point, which we claim is necessarily the point (∞, z_0), where $\{\infty\} = \alpha X \setminus X$ and where z_0 is the limit of $\{\varphi((X \setminus C) \times Y)|\ C \subseteq X \text{ compact}\}$ in Z: Let the point $(a, z) \in \alpha X \times Z$ be different from (∞, z_0) and the limit of a convergent net $\Phi(x_\lambda, y_\lambda)$. Assume $a = \infty$. Then $z \neq z_0$ and there exists a closed neighborhood V of z_0 in Z not containing z. Pick a compact subset C in X with $\varphi((X \setminus C) \times Y) \subseteq V$. Since $x_\lambda \to \infty$, then $\varphi(x_\lambda, y_\lambda) \in V$ eventually. On the other hand, as $\varphi(x_\lambda, y_\lambda) \to z$ in Z, then $\varphi(x_\lambda, y_\lambda) \in Z \setminus V$ eventually, a contradiction. Thus, $a \in X$. Let $y_{\lambda_\mu} \to y$ be a convergent subnet of y_λ in Y. Then $\Phi(x_{\lambda_\mu}, y_{\lambda_\mu})$ converges to $\Phi(a, y)$ and to (a, z) in $\alpha X \times Z$, whence $(a, z) \in \Phi(X \times Y)$.

LEMMA 11. *Let X be a non-compact, locally compact space for which αX is openly dyadic, let Y, Z be openly dyadic spaces, and suppose φ: $X \times Y \to Z$ is a separately continuous, sequentially continuous map satisfying (a) and (b) in Lemma 10 and such that $\varphi(\{x_0\} \times Y)$ is substantial in Z, for every $x_0 \in X$. Then $\alpha(X \times Y)$ is openly dyadic and homeomorphic with some zero-set in $\alpha X \times Z$, provided one of the following two conditions is satisfied:*

(1) *For every $x_0 \in X$, $S(x_0) = \{x \in X|$ for every $y \in Y$, there is some $y' \in Y$ with $\varphi(x, y') = \varphi(x_0, y)\}$ is substantial in X.*

(2) *There exists a space N in which every closed subset is substantial (e.g. N perfectly normal) and some quotient map q: $X \to N$ such that, whenever $q(x) = q(x')$ for some $x, x' \in X$, then $\varphi(x, y) = \varphi(x', y)$ for all $y \in Y$.*

Proof. By Theorem 2, φ is continuous. In view of Lemma 10 and Theorem 1, it suffices to show that $\{(x, \varphi(x, y))|\ x \in X, y \in Y\}$ is a substantial subset of $X \times Y$. In case (1), this is clear because the latter set is the union of the sets $S(x_0) \times \varphi(\{x_0\} \times Y)$ as x_0 runs through X. Suppose now (2) holds. Since Y is (locally) compact, the product map $q \times \mathrm{id}$: $X \times Y \to N \times Y$ of q with the identity map on Y is a quotient map according to a theorem of J. H. C. Whitehead [24]. Consequently, there exists a continuous map ψ: $N \times Y \to Z$ with $\varphi = \psi(q \times \mathrm{id})$. For every $x_0 \in X$, then $T(x_0) = \{t \in N|$ for every $y \in Y$, there exists some $y' \in Y$ with $\psi(t, y') = \varphi(x_0, y)\}$ is closed in N by compactness of Y, hence is a substantial subset of N by hypothesis on N, whence $S(x_0) = q^{-1}(T(x_0))$ is substantial in X.

Let us call a space Z *substantially contractible* if there exists a continuous map F: $[0, 1[\times Z \to Z$ such that the filter base $\{F(]1-1/n, 1[\times Z)|\ n = 1, 2, \ldots\}$ converges in Z and such that, for every $0 \leqslant t < 1$, the map $z \mapsto F(t, z)$ embedds Z onto a substantial subset of Z.

Clearly, for any compact space Y, the *cone over* Y, i.e. the quotient space $\mathrm{cone}(Y)$ obtained from $[0, 1] \times Y$ by identifying the points of $\{1\} \times Y$, is substantially contractible; it is, likewise, clear that $\mathrm{cone}(Y)$ is nothing but the one-point compactification of $[0, 1[\times Y$.

THEOREM 12. *Let X be a non-compact, locally compact space for which αX is openly dyadic, and let Y be a compact space that is homeomorphic with some sub-*

stantial subset of a substantially contractible, openly dyadic space. Then $\alpha(X\times Y)$ is openly dyadic.

Proof. By hypothesis, we may assume that Y is a substantial subspace of an openly dyadic space Z for which there is a continuous map F as above. Since X is σ-compact, there exists a continuous map $p\colon \alpha X\to [0,1]$ with $p^{-1}(1)=\alpha X\setminus X$. Then $q\colon x\mapsto p(x)$ is a quotient map from X onto the subspace $N=p(X)$ of $[0,1]$ and $\varphi\colon (x,y)\mapsto F(q(x),y)\colon X\times Y\to Z$ is a continuous map satisfying (a) and (b) in Lemma 10. Y is openly dyadic by Theorem 1. Now, Lemma 11 does the rest.

COROLLARY 13. *Let X be as in Theorem 12, and let Y be openly dyadic. If* cone(Y) *is openly dyadic, then so is $\alpha(X\times Y)$. The converse holds if X admits a perfect, open map onto $[0,1[$ or if there exists a continuous injection $f\colon [0,1[\to X$ with $\lim_{n\to\infty} f(]1-1/n,1[)=\infty$ in αX and $f([0,1[)$ a substantial subset of X.*

COROLLARY 14. *Let X be as in Theorem 12. Then $\alpha(X\times Y)$ is openly dyadic for every compact, metrizable space Y.*

In view of Theorem 12, the question arises for non-metrizable, openly dyadic, substantially contractible spaces. That non-metrizable, substantially contractible spaces are not so rare is suggested by the following

LEMMA 15. (1) *Let Z be a bounded subset of a Hausdorff locally convex topological vector space for which there exists a point z_0 such that $\{tz_0+(1-t)z|\ z\in Z\}$ is a substantial subset of Z, for every $0\leqslant t<1$. Then Z is substantially contractible.*

(2) *Let Z be a compact, convex set that is stable, i.e.* [2] *whose midpoint map $(x,y)\mapsto\frac{1}{2}(x+y)\colon Z\times Z\to Z$ is open. Then Z is substantially contractible provided it has at least one G_δ-point.*

(3) *Let Y be a compact space and then $P(Y)$ the space of all Radon probability measures on Y with the vague topology. Then $P(Y)$ is substantially contractible if Y has at least one G_δ-point.*

Proof. Verification of (1) is straightforward. Under the hypotheses of (2), there is a point z_0 in Z with a countable open neighborhood base $\{U_n|\ n=1,2,\ldots\}$ such that $U_{n+1}\subseteq U_n$ for all n. Fix $0\leqslant t<1$. Then [2], $\tau\colon (x,y)\mapsto tx+(1-t)y\colon Z\times Z\to Z$ is an open map, whence $\bigcap\tau(U_n\times Z)$ is a G_δ-set in Z. By compactness of Z, however, the latter set is nothing but $\{tz_0+(1-t)z|\ z\in Z\}$. As for (3), it is known that $P(Y)$ is stable, being a Bauer simplex [19]. Moreover, if a is a G_δ-point in Y, then $\varepsilon_a\colon f\mapsto f(a)$ is a G_δ-point in $P(Y)$; in fact, if $g\colon Y\to[0,1]$ is a continuous map with $g(y)<1=g(a)$ for all $y\neq a$, then

$$\Gamma\colon \mu\mapsto \int g\,d\mu\colon P(Y)\to \boldsymbol{R}$$

is a continuous map with $\Gamma(\mu)<\Gamma(\varepsilon_a)$ for all $\mu\in P(Y)$ with $\mu\neq\varepsilon_a$.

Non-metrizable, openly dyadic, convex sets — besides those that are already in product form, such as cubes — seem to be completely unknown. In fact, the only corresponding dyadic examples known to us are the spaces $P(S)$ with S a Dugundji space [20; p. 34] of weight $\aleph_1$ (e.g. $S=\boldsymbol{D}^{\aleph_1}$) which, by a theorem of Ditor and Haydon [4], are absolute retracts, hence are dyadic. The space $P(\boldsymbol{D}^{\aleph_1})$ might be a good test case to consider and if it turned out to be openly dyadic, then, for any totally disconnected, openly dyadic space Y of weight $\aleph_1$, cone $P(Y)$ would be openly dyadic: namely, the sum $S\cup Y$, where S is singleton, is a continuous open image of $\boldsymbol{D}^{\aleph_1}$ [7], whence $P(S\cup Y)$ is a continuous, open image of $P(\boldsymbol{D}^{\aleph_1})$ [22], and $P(S\cup Y)$ is easily seen to be homeomorphic with cone $P(Y)$.

By Corollaries 13 and 14, the problem of proving open dyadicity of αG, for G a non-compact, locally compact, σ-compact group, can be reduced in various ways; a few examples:

COROLLARY 16. *αG is openly dyadic if G/G_0 is compact and if* cone(Y) *is openly dyadic in each of the following cases:*

(a) *Y is a product of simple and simply connected compact Lie groups.*

(b) *Y is a product of copies of the dual group of the discrete rationals.*

(c) *Y is a Cantor space $\boldsymbol{D}^{\mathfrak{m}}$.*

Proof. By what has been said in the proof of Theorem 7, G is homeomorphic with $\boldsymbol{R}^n\times C\times G/G_0$, where C is any maximal compact subgroup of G_0. Thus, in order that αG be openly dyadic, it is good enough, by Corollary 13, that cone$(C\times G/G_0)$ be openly dyadic. Now, G/G_0, being openly dyadic and zero-dimensional, is the continuous, open image of a Cantor space. C is known to be the quotient group of a product $A\times\Sigma^*$, where A is a compact connected Abelian group and where Σ^* is a product of simple and simply connected compact Lie groups [23; p. 91]. The dual group $\hat{A}$ of A is torsion-free [15; p. 385], hence [11; p. 108, Ex. 1] embeddable into some torsion-free divisible group and, consequently [11; p. 104], embeddable into a direct sum of copies of the discrete rationals $\boldsymbol{Q}$; by duality, A is the continuous, homomorphic image of a product of copies of $\hat{\boldsymbol{Q}}$. Thus, $C\times G/G_0$ is the open, continuous image of a product of three spaces each of which is either of type (a), or (b), or (c). It therefore suffices to apply the following lemma with $X=[0,1[$: Let X be a non-compact, locally compact, σ-compact, metrizable space, and let Y_λ be a countable family of compact spaces; then $\alpha(X\times\prod Y_\lambda)$ is homeomorphic with some G_δ-subspace of $\prod\alpha(X\times Y_\lambda)$.

COROLLARY 17. *αG is openly dyadic if one of the following two conditions is satisfied:*

(1) *There is a closed subgroup H and a compact, metrizable subgroup K of G with $G=HK$ and αH openly dyadic.*

(2) *G is connected, and there exists a compact, invariant subgroup K whose connected component of the identity K_0 is metrizable and such that αH_0 is openly dyadic, where H is the centralizer of K in G.*

Proof. (1) Using arguments similar to those of the standard proof of the Open Mapping Theorem for locally compact, σ-compact groups [15; p. 42], one shows that μ: $(h, k) \mapsto hk$: $H \times K \to G$ is an open map. μ is perfect as K is compact. So, one may quote Corollary 14 and Proposition 5 (2).

(2) By a theorem essentially due to Iwasawa [21; 1.4], $G = H_0 K_0$.

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Universal mappings and weakly confluent mappings

by

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Abstract. All spaces are compact Hausdorff. Let f: $X \to Y$ be continuous; f is *universal* provided f has a coincidence with every map from X into Y, and f is *weakly confluent* provided every subcontinuum of Y is the image under f of a subcontinuum of X. This paper discusses relationships between these types of maps. Some results are: (1) *Universal maps onto locally connected metric continua are weakly confluent*; (2) *If* $H^1(X) \approx 0$, *then weakly confluent maps from* X *onto the* 2-*dimensional disk* B^2 *are universal* (*hence, have fixed points if* $X \subset B^2$); (3) CE-*maps between compact metric absolute retracts are universal.* Result (1) generalizes theorems of Grispolakis and Tymchatyn and of Mazurkiewicz. In relation to (1), it is shown that universal maps between metric continua need not be weakly confluent (which answers a question of Grispolakis and Tymchatyn). Result (2) strengthens theorems of Hamilton and of the author. Examples show that (2) would be false if $H^1(X) \not\approx 0$ and that the analogue of (2) for B^3 is false. Result (3) is shown to be false for monotone maps. Various applications to fixed point theorems are given.

1. Introduction. For each $n = 1, 2, \ldots$, let B^n denote the closed unit ball in Euclidean n-space R^n, $B^n = \{v \in R^n: ||v|| \leqslant 1\}$, and let $S^{n-1} = \{v \in B^n: ||v|| = 1\}$. A mapping (= continuous function) f from a topological space Z into B^n is AH-*essential* (this terminology comes from [14, p. 156]) provided that

$$f|f^{-1}(S^{n-1}): f^{-1}(S^{n-1}) \to S^{n-1}$$

can not be extended to a mapping defined on all of Z into S^{n-1}. AH-essential mappings are usually simply called essential mappings, and their existence is equivalent to Z having covering dimension $\geqslant n$ [25, 5.1, p. 44]. In this paper, *essential mapping* means a mapping which is not homotopic to any constant mapping; an *inessential mapping* is a mapping which is not essential. A mapping f from a topological space Z_1 to a topological space Z_2 is *universal* [12] provided that for any mapping g: $Z_1 \to Z_2$ there exists a point $p \in Z_1$ such that $g(p) = f(p)$. A *continuum* is a compact connected Hausdorff space. Let X and Y be compact Hausdorff spaces. A mapping f: $X \to Y$ is *weakly confluent* [16] provided that for each continuum $B \subset Y$ there is a continuum $A \subset X$ such that $f[A] = B$. A mapping f: $X \to Y$ is *monotone* [33, p. 127] provided that $f^{-1}(y)$ is connected for each $y \in Y$. Note that, by [33, 2.2, p. 138], monotone mappings between compact Hausdorff spaces are weakly confluent.