of the $B\langle n, m\rangle$'s. (Of course, some of the exposed V-arcs of M_{n+1} are omitted completely from M'_{n+1}, as can be seen from the deleted portions of $B\langle n, m\rangle$ in Figure 11.)

It is easy to see that all of the argument for the M_n's can be carried out for the M'_n's, defining N'_n to be the union of M'_n with the convex hulls of all its exposed V-arcs. Thus M', the inverse limit of $\{M'_n; h_n|M'_{n+1}\}_{n=1}^{\infty}$, is a proper subcontinuum of M with the same connectedness and accessibility properties for its set of endpoints.

References

[1] R. D. Anderson and G. Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: An application of inverse limits*, Proc. Amer. Math. Soc. 10 (1959), pp. 347–353.

[2] K. Kuratowski, *Topology*, Vol. II, New York–London–Warszawa 1968.

[3] A. Lelek, *On plane dendroids and their endpoints in the classical sense*, Fund. Math. 49 (1961), pp. 301–319.

UNIVERSITY OF DELAWARE
Newark Delaware

Accepté par la Rédaction le 19. 7. 1978

An example concerning automorphisms of generalized cubes

by

Dorothy Maharam (Rochester, N. Y.)

Abstract. An example is given of an involution T on I^m ($m \geqslant c$) with the following properties: (1) $T = T^{-1}$ sends Baire sets to Baire sets, (2) T induces the identity automorphism on both the category algebra and the measure algebra of I^m, (3) T has no fixed points.

Let m be an infinite cardinal, and consider the product I^m of m copies of the unit interval I, both as a topological space (with product topology) and as a measure space (with product Lebesgue measure). It is known ([1]) that every automorphism of the measure algebra of I^m can be "realized" by a point map that can even be required to be a Baire isomorphism; and a similar remark applies to the category algebra ([5]). Some time ago, S. Kakutani asked the author whether the realization would have to be "almost" unique. More specifically, suppose T is a transformation of I^m that induces the identity automorphism of the measure algebra (or of the category algebra) of I^m; need $T(x) = x$ for a "large" set of x's? When $m \leqslant \aleph_0$ the answer is easily seen to be "yes"; we have only to discard, for each member of a countable base $U_1, U_2, \ldots$, for I^m, the negligible set $U_n \Delta T(U_n)$, and T will be the identity transformation on what is left. But we show here that, for $m \geqslant c$, the answer is "no" to both the measure and category forms of the question, and by the same example — even if the transformation is required to be an involution. It is of course enough to consider $m = c$.

THEOREM. *There exists a Baire isomorphism T of I^c, of period 2, such that T induces the identity automorphisms of both the category algebra and the measure algebra of I^c, but such that T has no fixed points* (1).

Proof. We use the following notation. I denotes the closed unit interval $[0, 1]$ as usual; $J = I \cup \{-1\}$, $X = I^c$ regarded as $\prod\{I_j: j \in J\}$, where each $I_j = I$. The jth coordinate of $x \in X$ is $\pi_j(x)$ or $x(j)$. (As this notation suggests, the coordinate $x(-1)$ will play an exceptional role).

Take an arbitrary measure-and-category-preserving Borel isomorphism ξ of I onto itself, without fixed points, but such that ξ^2 = identity map ι_I of I. For instance,

(1) For a simple example of such a T for which the set of fixed points is not measurable (having outer measure 1 and inner measure 0) see [3, p. 702].

define $\xi(t) = t+\frac{1}{2}\bmod 1$ for each irrational $t \in I$; pair the rationals in I in a one-to-one way as α_n, β_n, $n = 1, 2, \ldots$, and define $\xi(\alpha_n) = \beta_n$, $\xi(\beta_n) = \alpha_n$. For convenience, we suppose this done in such a way that $\xi((0, \frac{1}{2})) = [\frac{1}{2}, 1)$.

We define $T\colon X \to X$ by specifying that (for all $x \in X$) $T(x)$ has the same jth coordinate as x, except for $j = x(-1)$; but $\pi_j(T(x)) = \xi(x(j))$ when $j = x(-1)$.

Note that $T(x)$ and x differ in exactly *one* coordinate (the $x(-1)$th): and in particular since $x(-1) \neq -1$ (for of course $x(-1) \in I$) we have

$$\pi_{-1}T(x) = x(-1) \quad (x \in X). \tag{1}$$

An elementary calculation shows that

$$T(T(x)) = x \quad (x \in X), \tag{2}$$

from which it follows that T is a bijection of I^c onto itself. Also, by definition, T has no fixed points.

Next we show

(3) if A is a Baire set in X, then $T(A)$ is Baire, and $A \Delta T(A)$ is of first category and of measure 0.

It suffices to prove this when A is an "elementary open set" of the form $G_j \times \hat{X}_j$, where $j \in J$ and G_j is an open subset of I_j, and $\hat{X}_j = \prod\{X_t\colon t \in J - \{j\}\}$. For the family of subsets A of X for which the conclusion of (3) is true is a Borel field, which will have to contain the Borel field generated by these sets $G_j \times \hat{X}_j$; but that Borel field is precisely the family of Baire sets in X.

Given, then, $A = G_j \times \hat{X}_j$ (where $j \in J$ and G_j is open in I_j), consider two cases:

(a) If $j = -1$, then from (1) we have $T(A) = A$.

(b) If $j \neq -1$, we have $\hat{X}_j = I_{-1} \times Y$ where $Y = \prod\{X_t\colon t \in I - \{j\}\}$.

Thus $A = I_{-1} \times G_j \times Y = B \cup C$ say, where

$$B = \{j\} \times G_j \times Y \quad \text{and} \quad C = (I_{-1} - \{j\}) \times G_j \times Y.$$

Clearly $T(B) = \{j\} \times \xi(G_j) \times Y$, which is a first category Baire set of measure 0. And it is easy to check that $T(C) = C$, a Baire set. Thus $T(A)$ is Baire; further, $T(A) \Delta A \subset B \cup T(B)$, which is both of first category and of measure 0. This proves (3) when $A = G_j \times \hat{X}_j$: and, as remarked above, (3) follows without restriction.

Since $T^{-1} = T$, (3) implies that T is a Baire isomorphism of X onto itself. Again from (3), if A is a Baire first category set, or a Baire null set, then so is $T(A)$. But an arbitrary first category set is contained in some Baire first category set (see for example [5]), and an arbitrary null set is contained in some Baire null set (from the "completion regularity" of the product measure on I^m; see [2, p. 230] and [4, p. 993]). Hence T (and T^{-1}) sends first category sets to first category sets, and null sets to null sets. From (3), then, we see that T induces the identity automorphisms of the category algebra, and of the measure algebra, and the proof is complete.

It would be interesting to know whether there exists an example of the same phenomenon in which T, instead of (or as well as) being a Baire isomorphism, is a *Borel* isomorphism of I^c. In the present example, T is definitely *not* a Borel isomorphism, as the following argument (for which I am indebted to A. H. Stone) will show.

Choose a non-Borel subset E of the set of irrational numbers in I, and for each $t \in E$ put $G_t = (0, \frac{1}{2}) \subset I_t$; note that $\xi(G_t) = [\frac{1}{2}, 1)$ Define

$$A(t) = (\pi_t)^{-1}(G_t) = I_{-1} \times G_t \times Y(t) \quad \text{where} \quad Y(t) = \prod\{X_i\colon i \in I - \{t\}\}.$$

The same calculation as in the proof of (3) shows that

$$T(A(t)) = (\{t\} \times \xi(G_t) \times Y_{(t)}) \cup ((I_{-1} - \{t\}) \times G_t \times Y_{(t)}).$$

Now put $A = \bigcup\{A(t)\colon t \in E\}$. Clearly A is open in X. Put

$$Z = \{z \in X;\ z_j = \tfrac{3}{4} \text{ for all } j \in [0, 1]\}.$$

It is easy to see that $T(A) \cap Z$ is the set of all points $z \in Z$ having $z_{-1} \in E$. Now if $T(A)$ were Borel in X, $T(A) \cap Z$ would be Borel in Z, and since $\pi_{-1}|Z$ is a homeomorphism of Z onto $I_{-1} (= I)$ it would follow that E is Borel in I, a contradiction. Thus T takes the open set A to a non-Borel set. (Similarly we can construct an open set whose image under T is not even analytic.)

References

[1] J. R. Choksi, *Automorphisms of Baire measures on generalized cubes II*, Z. Wahrscheinlichkeitstheorie verw. Geb. 23 (1972), pp. 97–102.

[2] P. Halmos, *Measure Theory*, Van Nostrand, Princeton N. J. 1950.

[3] D. Maharam, *Automorphisms of products of measure spaces*, Proc. Amer. Math. Soc. 9 (1958), pp. 702–707.

[4] — *On a theorem of von Neumann*, Proc. Amer. Math. Soc. 9 (1958), pp. 987–994.

[5] — *Realizing automorphisms of category algebras of product spaces*, Gen. Top. and Appl. 10 (1979), pp. 161–174.

THE AUSTRALIAN NATIONAL UNIVERSITY, Canberra
and
THE UNIVERSITY OF ROCHESTER, Rochester N. Y.

Accepté par la Rédaction le 31. 7. 1978