An interesting plane dendroid

by

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Abstract. Using inverse limits and geometric techniques, a dendroid M is constructed in the plane with connected set of endpoints, and such that the endpoints and only the endpoints of M are arcwise accessible from the complement of M.

I. Introduction. In this paper, a plane dendroid M is constructed such that only its endpoints, in the sense of Lelek [3], are arcwise accessible from the complement of M. Further, the set E of endpoints of M is a connected set and has a "circle-like" property, namely, that while no point of E separates E, each pair of points of E does separate E. E must be a dense subset of M, so that the closure of E does not separate the plane. When this example was presented in Professor Borsuk's seminar, the following two questions were raised:

1 (Krasinkiewicz). If a plane dendroid has the property that its endpoints are its only arcwise accessible points, must the set of endpoints be connected?

2 (Borsuk). Is M irreducible with respect to the property that its endpoints and its accessible points are the same?

In the last section, examples are described giving a negative answer to each of these questions.

I would like to thank J. Krasinkiewicz and P. Minc for many helpful conversations during the preparation of this paper. The questions leading to this work arose during the seminar on metric continua led by Krasinkiewicz and Minc at Instytut Matematyczny, PAN in Warszawa in the fall of 1975.

The following terminology will be used. The arc in a given dendroid from p to q will be denoted $[p, q]$. $\mathrm{Cl}(A)$ denotes the closure of a set A, in whatever space it is being considered. E^2 is the real Euclidean plane and d its usual metric. If $A \subseteq E^2$, $H(A)$ is the convex hull of A. A *V-arc* is an arc in E^2 which is the union of two noncollinear straight segments with a common endpoint. The *interior* of a *V*-arc V,

denoted $I(V)$, means the interior of $H(V)$. The term *vertex* of a V-arc is self-explanatory. For three non-collinear points, p, q, and r in E^2, $V(p, q, r)$ denotes the V-arc with p and r as opposite endpoints and q as vertex. The *boundary* of a V-arc $V(p, q, r)$, denoted $\mathrm{Bd}(V(p, q, r))$ is the straight segment from p to r and $\mathrm{bd}(V(p, q, r))$ is $\mathrm{Bd}(V(p, q, r))$ with its endpoints deleted. (Thus $H(V)$ is the disjoint union of V, $I(V)$, and $\mathrm{bd}(V)$.)

If K is a continuum in E^2 and $p \in K$, we shall say briefly that p is "accessible in K" to mean that p is arcwise accessible from $E^2 - K$.

The next definition is rather technical. An *exposed V-arc* in a dendroid $K \subseteq E^2$ will mean a V-arc V such that: 1) $V \subseteq K$; 2) $K \cap H(V) = V$; 3) The endpoints of V are also endpoints of K; and 4) The nonendpoints of V are not accessible in $K \cup H(V)$. (It is an immediate consequence of 2) that each point of an exposed V-arc V is accessible in K, by a segment from any point in $I(V)$. Thus, 4) could be restated: The nonendpoints of V are accessible in K only by means of arcs which meet $I(V)$.) A V-arc V which satisfies all of the above except that its vertex may be accessible in $K \cup H(V)$ will be called a *v-exposed V-arc* in K.

In an inverse sequence $\{X_i; h_i\}_{i=1}^\infty$, $h[k, j]$: $X_k \to X_j$ denotes the appropriate multi-step bonding map. X is the inverse limit of the sequence. (Similarly for any other letter in place of X.) If $K \subseteq X$, K_i is the image of K under the projection of X to X_i.

The construction proceeds as follows: First we shall construct a dendroid B which is our "building block". We shall then construct an inverse sequence $\{M_i; h_i\}_{i=1}^\infty$, where each M_i is built up out of specially constructed copies of B. Then we shall prove that M embeds in the plane, using a theorem of Anderson and Choquet to be restated below in a convenient form, and that M is a dendroid with the desired properties.

The following lemmas will smooth the way somewhat.

LEMMA 1. *Any inverse limit of dendroids with monotone bonding maps is a dendroid.*

Proof. Let $\{M_i; h_i\}_{i=1}^\infty$ be such an inverse sequence, and let $\langle p_i \rangle_{i=1}^\infty$, $\langle q_i \rangle_{i=1}^\infty \in M$. Since h_i is monotone and M_i is hereditarily unicoherent for each i, it is easily verified that h_i maps $[p_{i+1}, q_{i+1}]$ monotonically onto $[p_i, q_i]$. Thus, the inverse limit, A, of this sequence of arcs is an arc in M from $\langle p_i \rangle_{i=1}^\infty$ to $\langle q_i \rangle_{i=1}^\infty$. If $W \subseteq M$ is any continuum containing these two points, $[p_i, q_i] \subseteq W_i$ for each i, so that $A \subseteq W$. Thus M is arcwise connected and hereditarily unicoherent and the proof is done.

In an inverse sequence $\{X_i; h_i\}_{i=1}^\infty$ where all the bonding maps are retractions there is a natural embedding k_i: $X_i \to X$ defined by: $k_i(x)$ is that sequence $\langle x_j \rangle_{j=1}^\infty$ such that $x_j = x$ for all $j \geqslant i$. (For $j < i$, x_j is determined by the bonding maps.) If e_{ij}: $X_i \to X_j$ is the inclusion map for $i \leqslant j$, then $k_i = k_j \circ e_{ij}$. Henceforth in such an inverse system each X_j will be identified with its image under k_j to avoid cumbersome notation, and $X_\infty \subseteq X$ will be defined by $X_\infty = \bigcup_{n=1}^\infty X_n$.

LEMMA 2. *Let $\{X_i; h_i\}_{i=1}^\infty$ be an inverse sequence of dendroids and monotone retractions. Then each point of $X - X_\infty$ is an endpoint of X.*

Proof. Let H_i: $X \to X_i$ be the projection. Each H_i is also a monotone retraction. First observe that $X - X_\infty$ contains no arc, for if $[p, q] \subseteq X - X_\infty$ were an arc, then there would be an n such that $H_n(p) \neq H_n(q)$. Then

$$[p, q] \cup H_n^{-1}(H_n\{p, q\}) \cup H_n[p, q]$$

would be a nonunicoherent continuum in X since

$$([p, q] \cup H_n^{-1}(H_n\{p, q\})) \cap H_n[p, q] = \{H_n(p), H_n(q)\}.$$

Next, if $[p, q]$ were an arc with a point $b \in [p, q] - \{p, q\}$ such that $b \in X - X_\infty$, then since $X - X_\infty$ contains no arc, there would be points $r \in [p, b]$ and $s \in [b, q]$ such that $r, s \in X_\infty$. Thus, for some k; $r, s \in X_k$. But then $X_k \cup [r, s]$ would be a nonunicoherent continuum since $X_k \cap [r, s]$ contains both r and s but not b. Therefore an arc in X can meet $X - X_\infty$ only at its endpoints, completing the proof.

LEMMA 3. *If S is a complete metric space and $\{M_i\}_{i=1}^\infty$ is an inverse sequence of compact subsets of S and monotone retractions, and if for each $\varepsilon > 0$ there exists an n such that for each $p \in M_n$,*

$$\mathrm{dia}\left[\bigcup_{i=n}^\infty (h[i, n])^{-1}(p)\right] < \varepsilon$$

and if for each n the family $\{h[i, n]\}_{i=1}^\infty$ is uniformly equicontinuous, then H: $M \to S$, defined by $H[\langle p_i \rangle_{i=1}^\infty] = \lim_{i \to \infty} p_i$ is a homeomorphism of M onto $\mathrm{Cl}[\bigcup_{i=1}^\infty M_i]$. Furthermore (with the identifications made above before Lemma 2), $H|M_i$ is the identity on M_i.

Proof. Except for the last sentence, this is a special case of Theorem 1 of [1, p. 348]. The last sentence is clear.

II. Construction of the example. Let X denote the Cantor set consisting of all those numbers in $[0, 1]$ which can be expanded base 5 using only the digits 0, 2, and 4. (This set may be obtained by dividing the interval into fifths and removing the second and fourth fifths; that is, the open intervals $(\frac{1}{5}, \frac{2}{5})$ and $(\frac{3}{5}, \frac{4}{5})$ and then recursively repeating this process on the remaining intervals at each step.) Observe that $\frac{1}{2} \in X$ and that $\frac{1}{2}$ is not an endpoint of a deleted interval.

Then there is a map μ: $X \to [0, 1]$ defined by

$$\mu\left[\sum_{k=1}^\infty a_k \cdot 5^{-k}\right] = \sum_{k=1}^\infty \left[\tfrac{1}{2} a_k\right] \cdot 3^{-k},$$

where it is understood that each a_k is even. The properties of μ are similar to those of the Cantor ternary function. The facts we need about μ are: 1) μ is surjective; 2) μ has a continuous extension $\bar{\mu}$: $[0, 1] \to [0, 1]$ which is constant on each of the

components of $[0, 1] - X$; 3) If x is an endpoint of one of the intervals deleted in the construction of X, then $\mu(x)$ is a triadic rational; and 4) $\mu(x) = x$ if $x \in \{0, \frac{1}{2}, 1\}$.

First consider a copy, B_0, of the mapping cylinder of μ, embedded in E^2 as follows: B_0 is the union of all straight line segments from $(x, 0)$ to $(\mu(x), 1)$ for $x \in X$ (see Fig. 1.)
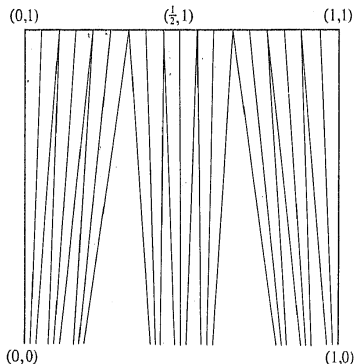

Fig. 1. B_0

B_0 is a dendroid and has a v-exposed V-arc corresponding to each deleted interval in the construction of X. For technical reasons, define also a slightly different dendroid, C_0, as follows: C_0 is the union of the two sets

$$\{(x + \tfrac{1}{2}, y) | \ (x, y) \in B_0 \text{ and } x \leqslant \tfrac{1}{2}\}$$

and

$$\{(x - \tfrac{1}{2}, y) | \ (x, y) \in B_0 \text{ and } x \geqslant \tfrac{1}{2}\}.$$

Geometrically, we have cut B_0 in two along the vertical line segment at $x = \frac{1}{2}$ and sewn it back together, identifying the vertical segments at $x = 0$ and $x = 1$ together. C_0 is a dendroid since B_0 contains the vertical segment from $(x, 0)$ to $(x, 1)$ for $x = 0, \frac{1}{2},$ or 1. The important thing to observe about C_0 and B_0 here is that the vertices of the v-exposed V-arcs in C_0 have x-coordinates of the form $a \pm \frac{1}{2}$, where a is a triadic rational, while those of B_0 are themselves triadic rationals. Hence no vertex of one of these V-arcs in B_0 is a vertex of one in C_0. This last statement will also hold for B_1 and C_1, for B and C, and finally for B and $\bar{C}$, all to be constructed below.

The retraction $r_0 : B_0 \rightarrow [0, 1] \times \{1\}$, defined, for $t \in [0, 1]$ and $x \in X$, by

$$r_0\big(t(x, 0) + (1 - t)(\mu(x), 1)\big) = (\mu(x), 1)$$

is just the map which sends each point on the line segment from $(x, 0)$ to $(\mu(x), 1)$ to the point $(\mu(x), 1)$. This is just the standard retraction of a mapping cylinder

onto its range space. Observe that r_0 is a monotone retraction, as will be every map which we denote by the letter r with or without subscripts or other means of distinction. The reader may check this in each case.

Let $D_0 = [0, 1] \times [0, 1]$. r_0 has a continuous extension $\bar{r}_0 : D_0 \rightarrow [0, 1] \times \{1\}$ defined by $\bar{r}_0\big(t(x, 0) + (1 - t)(\bar{\mu}(x), 1)\big) = (\bar{\mu}(x), 1)$, where now $t, x \in [0, 1]$. $\bar{r}_0$ is constant on the convex hull of each v-exposed V-arc in B_0.

B_0, C_0, and D_0 have been constructed where they are easy to visualize; they will now be moved to where they are easier to work with. Let $f : E^2 \rightarrow E^2$ be defined by $f(x, y) = (2x - 1, y - 1)$, and let $B_1 = f(B_0)$; $C_1 = f(C_0)$; and $D_1 = f(D_0)$. The retractions $f \circ r_0 \circ (f^{-1}|B_1)$ and $f \circ \bar{r}_0 \circ (f^{-1}|D_1)$ will be called r_1 and $\bar{r}_1$, respectively.

B, C, and D are now obtained from B_1, C_1, and D_1 by setting:

$$B = \{(x, y) \in B_1 | \ y \geqslant |x| - 1\},$$
$$C = \{(x, y) \in C_1 | \ y \geqslant |x| - 1\},$$
$$D = \{(x, y) \in D_1 | \ y \geqslant |x| - 1\}.$$

Geometrically, this means that B (respectively C, D) is the set of points of B_1 (respectively C_1, D_1) on or above $V((-1, 0), (0, -1), (1, 0))$ (see Figure 3 or the lower half of Figure 2). The retractions $\bar{r}_1|D$ and $r_1|B$ will be called simply $\bar{r}$ and r, respectively. Again, r is a restriction of $\bar{r}$, and the range of each is $[-1, 1] \times \{0\}$, and $\bar{r}$ is constant on the convex hull of each of the v-exposed V-arcs of B.

Let M_1 denote the union of B with the reflection of C through the x-axis; that is, $M_1 = B \cup \bar{C}$, where $\bar{C} = \{(x, y)| \ (x, -y) \in C\}$ (see Fig. 2). It is clear that M_1 and B are dendroids. Observe also the following facts about M_1 and B.

(i) The accessible points of M_1 consist of its endpoints together with the points of countably many exposed V-arcs in M_1. (The v-exposed V-arcs of B and of $\bar{C}$ are exposed V-arcs in M_1.)

(ii) The only points of $[-1, 1] \times \{0\}$ which are accessible in B by an arc lying entirely in the closed lower half-plane are the endpoints $(-1, 0)$ and $(1, 0)$ and the vertices of the v-exposed V-arcs in B. In particular, $(0, 0)$ is not accessible by such an arc. $((0, 0)$ corresponds to $(\frac{1}{2}, 1)$ in B_0.)

The dendroids M_i for $i \geqslant 2$ will be constructed as unions of M_1 with particular copies of B used to "cover up" the exposed V-arcs of M_1. No further use will be made of C apart from its place in M_1.

For $1 \geqslant \varepsilon > 0$, let $B\langle\varepsilon\rangle$ (respectively $D\langle\varepsilon\rangle$) denote that portion of B (respectively D) on or above $V((-1, 0), (0, -\varepsilon), (1, 0))$; that is,

$$B\langle\varepsilon\rangle = \{(x, y) \in B| \ y \geqslant |\varepsilon x| - \varepsilon\}$$

and similarly for $D\langle\varepsilon\rangle$ (see Fig. 3).

Let $r\langle\varepsilon\rangle = r|B\langle\varepsilon\rangle$; $\bar{r}\langle\varepsilon\rangle = \bar{r}|D\langle\varepsilon\rangle$. By choosing ε sufficiently small, the maximum diameter of point inverses under $\bar{r}\langle\varepsilon\rangle$ and $r\langle\varepsilon\rangle$ can be made arbitrarily
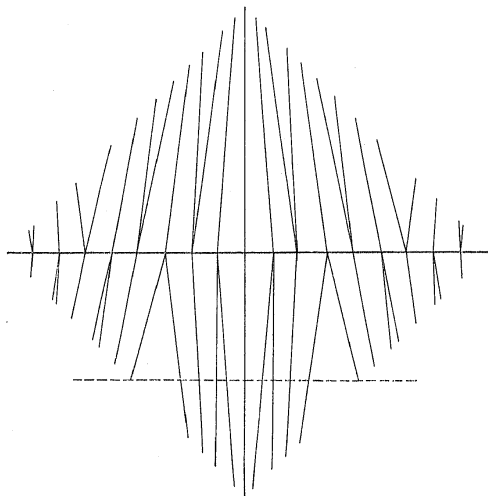
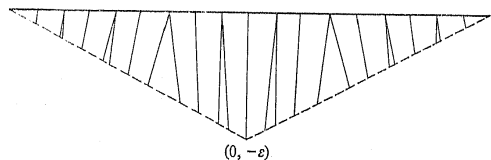Fig. 2. M_1 (N_1 is the square indicated by the dashed lines.) The horizontal dashed line is used in Section VI

Fig. 3. $B\langle\varepsilon\rangle$ is that portion of B above the dashed V-arc

small. Next define, for each fourtuple $[a, b, c, d]$ of positive reals, the homeomorphism $g[a, b, c, d]$ of E^2 by:

$$g[a, b, c, d](x, y) = \begin{cases} (ax, y+bx) & \text{if} \quad x \leqslant 0, \\ (cx, y-dx) & \text{if} \quad x \geqslant 0. \end{cases}$$

Whenever convenient, α will denote a fourtuple of positive reals. Each $g[\alpha]$ is the identity on the y-axis and is linear in the left and right closed half planes, and bends both halves of the x-axis downward, to slopes of b/a and $-d/c$ respectively. The image under $g[\alpha]$ of any V-arc V which misses the y-axis is a V-arc, and $g[\alpha](I(V)) = I(g[\alpha](V))$. In particular, this is true of the v-exposed V-arcs in each $B\langle\varepsilon\rangle$.

Denote by $V[\alpha]$ the V-arc $g[\alpha]([-1, 1] \times \{0\})$. By appropriate choice of α, $V[\alpha]$ can be made congruent with any preassigned V-arc in E^2.

Let $B[\alpha, \varepsilon] = g[\alpha](B\langle\varepsilon\rangle)$ and $D[\alpha, \varepsilon] = g[\alpha](D\langle\varepsilon\rangle)$, and define $r[\alpha, \varepsilon]: B[\alpha, \varepsilon] \to V[\alpha]$ and $\bar{r}[\alpha, \varepsilon]: D[\alpha, \varepsilon] \to V[\alpha]$ in the obvious way, by conjugating $r\langle\varepsilon\rangle$ and $\bar{r}\langle\varepsilon\rangle$ with $g[\alpha]$. If $\alpha = [a, b, c, d]$, it will always be assumed when $B[\alpha, \varepsilon]$ or $D[\alpha, \varepsilon]$ is written that $\varepsilon < \min\{b, d\}$. This insures that $D[\alpha, \varepsilon] - V[\alpha]$ lies in the interior of $V[\alpha]$, as does $B[\alpha, \varepsilon] - V[\alpha]$. This will be referred to as the "concavity restriction on ε" (see Fig. 4).
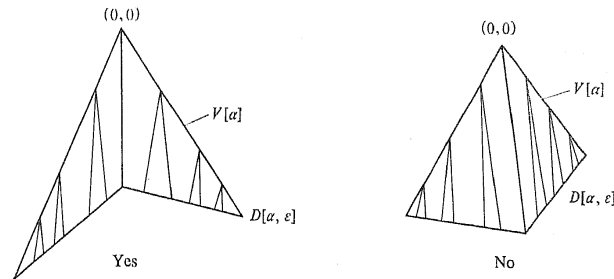

Fig. 4. Concavity restriction on ε

Observe that condition (ii), above, can be restated for $B[\alpha, \varepsilon]$ as follows:

(iii) The accessible points of $B[\alpha, \varepsilon]$ are its endpoints, the points of $V[\alpha]$, and the points of its v-exposed V-arcs. The only points of $V[\alpha]$ which are accessible in $B[\alpha, \varepsilon]$ by an arc lying entirely in $H(V[\alpha])$ are the endpoints of $V[\alpha]$ and the vertices of the v-exposed V-arcs of $B[\alpha, \varepsilon]$. In particular, $(0, 0)$ is not accessible by such an arc.

Observe also:

(iv) The convex hulls of the v-exposed V-arcs in $B[\alpha, \varepsilon]$ are pairwise disjoint.

(v) If v is the vertex of a v-exposed V-arc in $B[\alpha, \varepsilon]$ and A is an arc from $E^2 - B[\alpha, \varepsilon]$ to v such that $A \subseteq H(V[\alpha])$, then $A \cap I(V) \neq \emptyset$. (This is clear from Figure 4.)

Suppose now that the dendroid M_n has been constructed such that M_n has no accessible points except its endpoints and the points of countably many exposed V-arcs, $\{V\langle n, m\rangle\}_{m=1}^{\infty}$, whose convex hulls are pairwise disjoint.

For each $V\langle n, m\rangle$, choose an $\alpha(n, m)$ such that $V[\alpha(n, m)]$ is isometric with $V\langle n, m\rangle$, and choose an $\varepsilon(n, m) > 0$, satisfying the concavity restriction, such that for all $p \in V[\alpha(n, m)]$,

(vi) $$\text{dia}\left((\bar{r}[\alpha(n, m), \varepsilon(n, m)])^{-1}(p)\right) < \min\{1/n, 1/m\}.$$

Then the same inequality is true if $\bar{r}$ is replaced by r.

Let $G\langle n, m\rangle: D[\alpha(n, m), \varepsilon(n, m)] \to E^2$ be an isometric embedding such that $G\langle n, m\rangle(V[\alpha(n, m)]) = V\langle n, m\rangle$, and set

$$D\langle n, m\rangle = G\langle n, m\rangle(D[\alpha(n, m), \varepsilon(n, m)])$$

and

$$B\langle n, m\rangle = G\langle n, m\rangle(B[\alpha(n, m), \varepsilon(n, m)]).$$

(This "fills in" part of $I(V\langle n, m\rangle)$.) Define $\bar{r}\langle n, m\rangle$ and $r\langle n, m\rangle$ in the obvious way, conjugating through $G\langle n, m\rangle$. Then define:

$$M_{n+1} = M_n \cup \left[\bigcup_{m=1}^{\infty} B\langle n, m\rangle\right].$$

It is also necessary to define the sequence of continua $\{N_i\}_{i=1}^{\infty}$ by: $N_1 = H(M_1)$ and $N_{n+1} = M_n \cup \left[\bigcup_{m=1}^{\infty} D\langle n, m\rangle\right]$. Then define functions $h_n: N_{n+1} \to M_n$ and $\bar{h}_n: M_{n+1} \to M_n$ by:

$$\bar{h}_n | M_n \text{ is the identity},$$
$$\bar{h}_n | D\langle n, m\rangle = \bar{r}\langle n, m\rangle \text{ and}$$
$$h_n = \bar{h}_n | M_{n+1}.$$

Observe that $h_n | B\langle n, m\rangle = r\langle n, m\rangle$ for each m.

The verification that M_{n+1} satisfies the induction hypotheses imposed on M_n is relatively straightforward. M_{n+1} (also N_{n+1}) is compact since if $\varepsilon > 0$ only finitely many of the $B\langle n, m\rangle$'s (also $D\langle n, m\rangle$'s) contain points at a distance greater than ε from M_n by (vi), above. M_{n+1} is clearly a dendroid, and it is tedious but easy to check, using (iii), (iv), and (v), above, applied (with minor notational alterations) to the $B\langle n, m\rangle$'s, that the v-exposed V-arcs of the $B\langle n, m\rangle$'s are exposed V-arcs of M_{n+1} and that the set of accessible points of M_{n+1} consists of these V-arcs together with the endpoints of M_{n+1} (see Fig. 5).
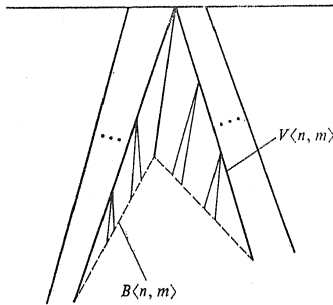

Fig. 5. Small section of M_{n+1}, showing $B\langle n, m\rangle$ attached to $V\langle n, m\rangle$

The example, M, then, is just the inverse limit of $\{M_i; h_i\}_{i=1}^{\infty}$. To prove that it is planar, it is necessary to verify the hypotheses of Lemma 3, the Anderson–Choquet Theorem.

III. The Anderson–Choquet hypotheses. The following lemmas are relatively easy observations about the collections $\{M_i\}_{i=1}^{\infty}$, $\{N_i\}_{i=1}^{\infty}$, $\{h_i\}_{i=1}^{\infty}$, and $\{\bar{h}_i\}_{i=1}^{\infty}$. The proofs of the less obvious ones are given or indicated.

LEMMA 4. *Each N_i is the union of M_i with the convex hulls of its exposed V-arcs.*

LEMMA 5. *Each h_n (respectively $\bar{h}_n$ is a continuous monotone retraction of M_{n+1} (respectively N_{n+1}) onto M_n.*

Proof. The continuity follows easily from (vi), above, and a convergent sequence argument. Once the continuity has been established, it is self-evident that h_n (respectively $\bar{h}_n$) is a monotone retraction since each $r\langle n, m\rangle$ (respectively $\bar{r}\langle n, m\rangle$) is.

LEMMA 6. *Every endpoint of M_n is also an endpoint of M_i for all $i \geqslant n$.*

LEMMA 7. *For each n, $N_{n+1} \subseteq N_n$; $M_n \subseteq N_n$; and $M_n \subseteq M_{n+1}$.*

Proof. $N_{n+1} = M_n \cup \left[\bigcup_{m=1}^{\infty} D\langle n, m\rangle\right]$. However, $D\langle n, m\rangle \subseteq H(V\langle n, m\rangle)$. Thus the first statement follows from Lemma 4. The other two statements are obvious.

LEMMA 8. *If n and k are positive integers, $M_n \subseteq N_k$.*

Proof. Immediate from Lemma 7.

LEMMA 9. $\mathrm{Cl}\left[\bigcup_{n=1}^{\infty} M_n\right] = \bigcap_{k=1}^{\infty} N_k.$

Proof. $\mathrm{Cl}\left[\bigcup_{n=1}^{\infty} M_n\right] \subseteq \bigcap_{k=1}^{\infty} N_k$ as a consequence of Lemma 8. Conversely, if $x \in N_{k+1}$, $d(x, M_k) < k^{-1}$. Thus, if $x \in \bigcap_{k=1}^{\infty} N_k$, $d[x, \bigcup_{n=1}^{\infty} M_n] = 0$.

LEMMA 10. *If $x \in M_k - M_j$, then x lies in the interior of one of the exposed V-arcs in M_j.*

Proof. $x \in M_k \subseteq N_{j+1} = M_j \cup \left[\bigcup_{i=1}^{\infty} D\langle j, i\rangle\right]$, but $x \notin M_j$, so $x \in D\langle j, i\rangle$ for some i. $x \notin V\langle j, i\rangle$, so by the concavity restriction, $x \in D\langle j, i\rangle - V\langle i, j\rangle \subseteq I(V\langle j, i\rangle)$.

LEMMA 11. *$\bar{h}_n$ is constant on the convex hull of each exposed V-arc in M_{n+1}.*

Proof. Each $\bar{r}\langle n, m\rangle$ is constant on the convex hull of each exposed V-arc in M_{n+1} where it is defined.

LEMMA 12. *If V is an exposed V-arc in M_k and $1 \leqslant n \leqslant k$, V lies in (and hence also $H(V)$) the convex hull of some L, where L is an exposed V-arc of M_n.*

LEMMA 13. *If V is an exposed V-arc in M_k and $1 \leqslant n < k$, $\bar{h}_n$ is constant on $H(V)$.*

Proof. This is immediate from Lemmas 11 and 12.

Recall the notation $h[n, k]$, above

LEMMA 14. *If* $k \geqslant n$, $h[k, n] = \bar{h}_n | M_k$.

Proof. For $k = n$, both these functions are the identity on M_k. For $k = n+1$, $h[n+1, n] = h_n$, and $h_n = \bar{h}_n | M_{n+1}$ by definition. Suppose the lemma be established for some value $k \geqslant n+1$. Then if $x \in M_{k+1}$, either $x \in M_k$ or $x \notin M_k$. If $x \in M_k$, $h[k+1, n](x) = h[k, n] \circ h_k(x) = h[k, n](x) = h_n(x)$ by the induction hypothesis. If $x \notin M_k$, $h[k+1, n](x) = h[k, n] \circ h_k(x) = \bar{h}_n(h_k(x))$ by the induction hypothesis. However, $h_k(x)$ and x both lie in $H(V\langle k, m\rangle)$ for some m, and by Lemma 13, since $n < k$, $\bar{h}_n$ is constant on $H(V\langle k, m\rangle)$. Thus, $\bar{h}_n(h_k(x)) = \bar{h}_n(x) = h[k+1, n](x)$, and $h[k+1, n](x) = \bar{h}_n(h_k(x)) = \bar{h}_n(x)$, and the proof is complete.

LEMMA 15. *If* $x \in M_n$, $\mathrm{dia}[\bigcup_{k=n}^{\infty} (h[k, n])^{-1}(x)] < 1/n$.

Proof. By Lemma 14, $(h[k, n])^{-1}(x) \subseteq (\bar{h}_n)^{-1}(x)$ for all $k > n$. Thus,

$$\mathrm{dia} \bigcup_{k=n}^{\infty} (h[k, n])^{-1}(x) \leqslant \mathrm{dia}\, (\bar{h}_n)^{-1}(x) < 1/n\,.$$

LEMMA 16. *For any* n, *the collection* $\{h[k, n]|\ k \geqslant n\}$ *is an equi-uniformly continuous family.*

Proof. By Lemma 14, all are restrictions of the same uniformly continuous function, $\bar{h}_n$.

Lemmas 15 and 16 given us precisely the hypotheses of the Anderson–Choquet theorem, our Lemma 3. Hence M is topologically the closure of the union of the M_i's.

IV. The properties of M. We have three different representations of M, one as the inverse limit of the sequence $\{M_i; h_i\}$, another as $\mathrm{Cl}[\bigcup_{n=1}^{\infty} M_n]$, and another as $\bigcap_{n=1}^{\infty} N_n$. The relationship between the first two of these, given in Lemma 3, is used strongly here. The inverse limit M, will now be identified with $\mathrm{Cl}[\bigcup_{i=1}^{\infty} M_i]$ to simplify the terminology.

LEMMA 17. *If* x *is an endpoint of some* M_k, x *is an endpoint of* M.

Proof. This is an easy consequence of the fact that no arc is extended beyond its original endpoints in the construction of M_{n+1} from M_n.

THEOREM 1. *No nonendpoint of* M *is arcwise accessible in* M.

Proof. A nonendpoint, x, of M lies in M_∞ and hence in some M_k, by Lemma 2. Then x is a nonendpoint of M_k by Lemma 17, and

$$M_k \subseteq M_{k+1} \subseteq M_{k+2} \subseteq M\,.$$

The reader can easily check that no nonendpoint of M_k is accessible in M_{k+2}, so that x is not accessible in M_{k+2} and hence not in M.

Before proceeding to the proof that each endpoint of M is accessible in M, the following technical lemma will be proven.

LEMMA 18. *If* $\langle x_i \rangle_{i=1}^{\infty} \in M - M_\infty$, *then for each* i, $x_{i+1} \neq x_i$, *and* x_i *is the vertex of one of the exposed* V-*arcs in* M_{i+1}.

Proof. Suppose for some i, $x_i = x_{i+1}$. Since $\langle x_i \rangle_{i=1}^{\infty}$ is not eventually constant, there is a smallest $k > i$ such that $x_k \neq x_i$. We have $k \geqslant i+2$. Then since $x_{k-1} = x_i$, $x_{k-1} \neq x_k$, and this means that x_{k-1} lies on one of the exposed V-arcs in M_{k-1}, since no other points have non-degenerate preimages under h_{k-1}. If $k-1 > i+1$, this exposed V-arc contains no point of M_i, which is impossible since $x_{k-1} = x_i \in M_i$. Thus $k-1 \leqslant i+1$, or $k \leqslant i+2$. Since $k \geqslant i+2$, we must have $k = i+2$. Thus $x_i = x_{i+1}$ and $x_{i+1} \neq x_{i+2}$. Since x_{i+1} must lie on one of the exposed V-arcs in M_{i+1}, and must belong to M_i, x_{i+1} is a vertex of one of the exposed V-arcs of M_{i+1}. But then x_{i+1} is not the vertex of a V-arc in M_{i+2}, and $(h_{i+1})^{-1}(x_{i+1})$ is a segment. Thus x_{i+2} does not lie on an exposed V-arc of M_{i+2}, and none of the exposed V-arcs of M_j for $j \geqslant i+2$, can contain x_{i+2}, so that for $j \geqslant i+2$, $x_{i+2} = x_j$, and the sequence is eventually constant and so not in $M - M_\infty$. Thus for each i, $x_i \neq x_{i+1}$.

Since for any i, x_{i+1} must lie on an exposed V-arc of M_{i+1}, and $h_i(x_{i+1})$ is the vertex of that V-arc, x_i is the vertex of an exposed V-arc in M_{i+1}.

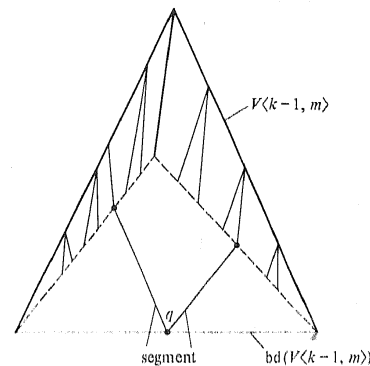THEOREM 2. *Every endpoint of* M *in arcwise accessible in* M.

Fig. 6

Proof. If x is an endpoint of some M_k, let k be the smallest integer such that $x \in M_k$. If $k = 1$, a vertical segment to x from above or below shows that x is accessible in N_1 and hence in $M \subseteq N_1$. If $k > 1$, x is an endpoint of $B\langle k-1, m\rangle$ for some m. Let q be a point on $\mathrm{bd}\,(V\langle k-1, m\rangle)$. The straight line segment from q to x misses $N_k - \{x\}$, so that x is accessible in N_k and hence in M since $M \subseteq N_k$ (see Fig. 6).

Suppose then that $x \in M - M_\infty$ (Figure 7 may be helpful here). Then $x = \langle x_i \rangle_{i=1}^\infty$ where $x_i \neq x_{i+1}$ for all i, and each x_i is a vertex of an exposed V-arc, V_i, in M_{i+1}. Let V_0 be the exposed V-arc in M_1 on which x_1 lies, and let q_i denote the midpoint of $\mathrm{Bd}(V_i)$ for each $i \geqslant 0$. Then, for $i > 1$, q_i and x_i both belong to $(\bar{h}_i)^{-1}(x_i)$, so that
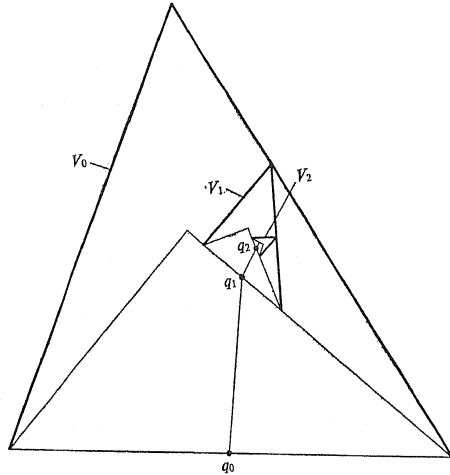

Fig. 7

$d(q_i, x_i) < i^{-1}$. Then since $\langle x_i \rangle$ converges to x, $\langle q_i \rangle$ converges to x also. The straight line segment $[q_i, q_{i+1}]$ misses N_{i+2} and hence misses M. Then, the closure of $\bigcup_{i=0}^\infty [q_i, q_{i+1}]$ is an arc from q_0 to x missing M except for the point x, so that x is accessible in M.

V. The connectedness properties of the set of endpoints of M. The next lemma is a fairly simple generalization of Theorem 6 of [2, p. 129]. I am indepted to J. Krasinkiewicz for calling it to my attention.

LEMMA 19. *If S is a connected one-manifold and E is a set contained in any Euclidean space and $f: S \to E$ is surjection of Baire class I such that for any open interval $(a, b) \subseteq S$, $\{a, b\} \subseteq \mathrm{Cl}(f(a, b))$, then E is connected.*

The following technical set — theoretic lemma is needed in the proof of the main result of this section.

LEMMA 20. *Suppose $\{A_i\}_{i=1}^\infty$ is a countable collection of pairwise disjoint countable sets, and suppose that $g: \bigcup_{i=2}^\infty A_i \to \bigcup_{i=1}^\infty A_i$ is a function such that $g(A_{i+1}) \subseteq A_i$*

for each i. Then the set $\bigcup_{i=1}^\infty A_i$ can be arranged in a sequence such that for every $a \in \bigcup_{i=2}^\infty A_i$, $g(a)$ preceeds a in the sequence.

Proof. Let h be a one to one function from $\bigcup_{i=1}^\infty A_i$ to the set of prime positive integers. Define $\eta: \bigcup_{i=1}^\infty A_i \to N$, where N is the set of positive integers, recursively on i, by:

$$\eta(a) = \begin{cases} h(a) & \text{if } a \in A_1, \\ h(a) \cdot \eta(g(a)) & \text{if } \eta \text{ has been defined on } A_i \text{ and } a \in A_{i+1}. \end{cases}$$

Then η is a one to one function from $\bigcup_{i=1}^\infty A_i$ into N such that $\eta(a) > \eta(g(a))$ whenever $g(a)$ is defined. Let $\langle n_j \rangle_{j=1}^\infty$ be the numbers in $\eta[\bigcup_{i=1}^\infty A_i]$ arranged in increasing order and define the desired sequence $\langle a_j \rangle_{j=1}^\infty$ by $a_j = \eta^{-1}(n_j)$. The required properties are easily checked.

Let E denote the set of endpoints of M.

THEOREM 3. *E is connected, as is $E - \{p\}$ for any $p \in E$. However, if $p, q \in E$ with $p \neq q$, then $E - \{p, q\}$ is not connected.*

Proof. To prove that $E - \{p, q\}$ is not connected, let A be an arc from p to q missing M except for p, q and let B be the arc $[p, q]$ in M. Then $A \cup B$ is a simple closed curve, and so separates the plane. It is readily seen that both complementary domains of $A \cup B$ must contain points of E, so that $E - \{p, q\}$ is disconnected.

To prove the first two assertions of the theorem, Lemma 19 is used. Let $\langle V_n \rangle_{n=1}^\infty$ be an enumeration of the $V \langle k, m \rangle$'s such that if $V \langle k+1, m \rangle \subseteq H(V \langle k, j \rangle)$, then $V \langle k, j \rangle$ preceeds $V \langle k+1, m \rangle$ in the enumeration. This is possible by Lemma 20. Observe that the boundary of $D \langle k, m \rangle$ is the union of $V \langle k, m \rangle$ and another V-arc which we shall designate $U \langle k, m \rangle$, and that $U \langle k, m \rangle$ and $V \langle k, m \rangle$ meet only at their endpoints (see Fig. 4, 5, or 6). Let $u \langle k, m \rangle$ denote the vertex of $U \langle k, m \rangle$.

Let $\langle U_n \rangle_{n=1}^\infty$ be the enumeration of the $U \langle k, m \rangle$'s corresponding to the enumeration $\langle V_n \rangle_{n=1}^\infty$ of the $V \langle k, m \rangle$'s; that is, $U_n = U \langle k, m \rangle$ if and only if $V_n = V \langle k, m \rangle$, and let u_n be the vertex of U_n for each n.

Let S denote the boundary of N_1. Define a sequence of functions $\langle f_n \rangle_{n=1}^\infty$, $f_n: S \to E^2$, as follows: f_1 is the identity on S. Suppose f_n has been defined such that $\mathrm{Bd}(V_k) \subseteq f_n(S)$ and $f_n(S) \cap I(V_k) = \varnothing$ for $k \geqslant n$. Let v_n be the midpoint of $\mathrm{Bd}(V_n)$. Define $e_n: f_n(S) \to E^2$ by:

$e_n(x) = x$ if $x \notin \mathrm{bd}(V_n)$;

$e_n(v_n) = u_n$; and

e_n is linear on the closure of each component of $\mathrm{Bd}(V_n) - \{v_n\}$.

4*

Then $\mathrm{Bd}(V_n)$ is mapped piecewise linearly onto U_n, the midpoint going to the vertex, and otherwise $f_n(S)$ is undisturbed. f_{n+1} is defined to be $e_n \circ f_n$. (Referring to Figure 8 may be helpful.) Then each f_n is continuous and one-to-one. Now let $p \in S$. If for some n, $f_n(p)$ is an endpoint of some M_k, it follows that for all $m > n$, $f_m(p) = f_n(p)$, so that $\lim_{m \to \infty} f_m(p) = f_n(p) \in E$.
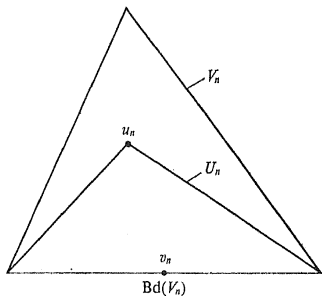

Fig. 8

If $f_n(p)$ is not an endpoint of any M_k for any n, then for each n there is a unique k_n such that $f_n(p) \in \mathrm{Bd}[V_{k_n}]$. We have $k_n \geqslant n$, and if $k_n > n$, then $k_{n+1} = k_n$. Thus, for infinitely many n, $k_n = n$. Let $\langle n_i \rangle_{i=1}^{\infty}$ be the increasing sequence of those and only those n for which $k_n = n$. Then $f_{n_i}(p) \in \mathrm{Bd}[v_{n_i}]$, so that $f_{1+n_i}(p) \in U_{n_i}$, and $f_{1+n_i}(p) \in \mathrm{Bd}[V_{k_{1+n_i}}] \subseteq U_{n_i}$. Furthermore

$$\mathrm{dia}\left[f_{1+n_i}^{-1}(\mathrm{Bd}(V_{k_{1+n_i}}))\right] < \tfrac{1}{2}\mathrm{dia}\, f_{n_i}^{-1}(\mathrm{Bd}(V_{n_i}))$$

for each i, so that p is the only point in $\bigcap_{n=1}^{\infty} f_n^{-1}(\mathrm{Bd}(V_{k_n}))$. Let $\langle x_i \rangle_{i=1}^{\infty}$ be the point in M such that x_i is the vertex of $V_{n_{i+1}}$ for each i. (Observe that $V_{n_{i+1}}$ is an exposed V-arc of M_{i+1} for each i and that there is such a point $\langle x_i \rangle_{i=1}^{\infty}$. Refer to Figure 9.)

Then, for $i > 1$, $d(x_i, f_{1+n_i}(p)) < 1/(i-1)$, since both belong to $h_{i-1}^{-1}(x_{i-1})$. Thus $\langle f_{n_i}(p) \rangle_{i=1}^{\infty}$ converges to $\lim_{i \to \infty} x_i$, which is an endpoint of M. Furthermore, if $q \in S$ and $q \neq p$, then for some i, $q \notin f_{n_i}^{-1}(\mathrm{Bd}(V_{n_i}))$ so that $\lim_{n \to \infty} f_n(p) \neq \lim_{n \to \infty} f_n(q)$. Thus, if $f: S \to E^2$ is defined by $f(x) = \lim_{n \to \infty} f_n(x)$, f is one-to-one and $f(S) \subseteq E$. However, if $x \in E$, either $x \in M_k$ for some k, or $x \in M - M_{\infty}$. If $x \in M_k$ for some k, either $k = 1$ or $k \neq 1$. If $k = 1$, $x \in S$ and $f_n(x) = x$ for each n so that $f(x) = x$. If $k > 1$, x lies on $U\langle k-1, m \rangle$ for some m and $U\langle k-1, m \rangle = U_j$ for some j. Let $y = f_{j+1}^{-1}(x)$; $y \in S$, and for all $k \geqslant j+1$, $f_k(y) = x$, so that $f(y) = x$. In case $x \in M - M_{\infty}$ there is a unique subsequence $\langle V_{n_i} \rangle_{i=1}^{\infty}$ of $\langle V_n \rangle_{n=1}^{\infty}$ such that: 1) V_{n_i} is an exposed V-arc of M_i, 2) $V_{n_{i+1}} \subseteq H(V_{n_i})$; and 3) $x \in H(V_{n_i})$ for each i. Then, by

the fact that $\bigcap_{i=1}^{\infty} f_{n_i}^{-1}(\mathrm{Bd}(V_{n_i})) \neq \varnothing$, there is a point q such that $f(q) = \lim_{n \to \infty} f_n(q) = x$. Hence $f(S) = E$.

Let $p \in E$ and let $q = f^{-1}(p)$. If $p \in M - M_{\infty}$, q is a point of continuity of f, since $f^{-1}(I(V_i))$ is open for each i and some subsequence here will form a neighborhood base at p. If $p \in M_k$ for some k, then in M_{k+1}, p is a limit of a sequence of
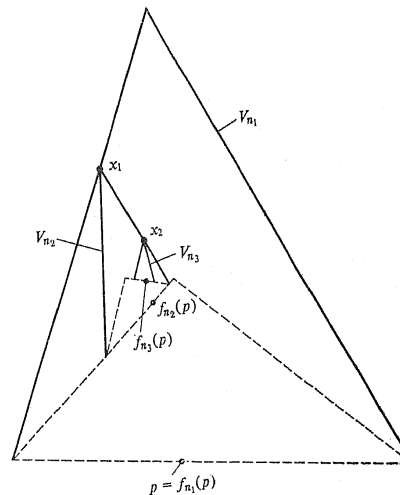

Fig. 9. Observe that $f_{n_1}(p) = p$, $f_{n_2}(p) = f_{1+n_1}(p)$, $f_{n_3}(p) = f_{1+n_2}(p)$, ...

other endpoints of M_{k+1} from "both sides", so that the function f satisfies the hypotheses of Lemma 19, and $f(S) = E$ is a connected set. (f is of Baire class I, since it is a pointwise limit of continuous functions.)

If $p \in E$ and $f(q) = p$, then $S - \{q\}$ is also a connected one-manifold and again Lemma 19 applies, so that $E - \{p\}$ is a connected set.

VI. Modifications of the example: The questions of Krasinkiewicz and Borsuk (See Introduction). To give an example of a plane dendroid whose endpoints and arcwise accessible points coincide, but whose set of endpoints is not connected, let $K = \{(x, y) \in M \mid y \leqslant 0\}$, and call the arc $[-1, 1] \times \{0\}$ the *edge* of K, and if K' is a homeomorphic copy of K, call the corresponding arc the edge of K'. Consider the following plane dendroid W_0, due to P. Minc,

$$W_0 = A \times [-1, 1] \cup F_+ \cup F_-$$

where $A = \{0\} \cup \{1/n : n \text{ an integer}, n \neq 0\}$; F_+ is the union of all straight line segments from $(0, 2)$ to $(1/n, 1)$ for odd integers n, and the segment from $(0, 2)$

to $(0, 1)$; and F_- is the union of all segments from $(0, -2)$ to $(1/n, -1)$ for n even and the segment from $(0, -2)$ to $(0, -1)$. For $n \neq 0$, let

$$a_n = \begin{cases} \left(\dfrac{1}{2n-1}, -1\right), & \text{if} \quad n>0, \\[2ex] \left(\dfrac{1}{2n+1}, -1\right), & \text{if} \quad n<0, \end{cases}$$

$$b_n = \left(\frac{1}{2n}, 1\right).$$

W is now obtained from W_0 by attaching a homeomorphic copy of K along its edge to each of the arcs: $[a_{-1}, a_1]$, $[a_n, a_{n+1}]$, $[b_{-1}, b_1]$ and $[b_n, b_{n+1}]$ in W_0 on the side such that this is possible, and such that: 1) these copies of K are disjoint except where they necessarily meet along their edges; 2) these copies of K meet W_0 only along their edge; and 3) for any $\varepsilon > 0$ only finitely many of these copies of K contain points at a distance greater than ε from W_0. The pattern can be seen in Figure 10, where the attached copies of K are indicated (K). The idea is to "cover up" all accessible non-endpoints of W_0. W is the union of W_0 and the countably
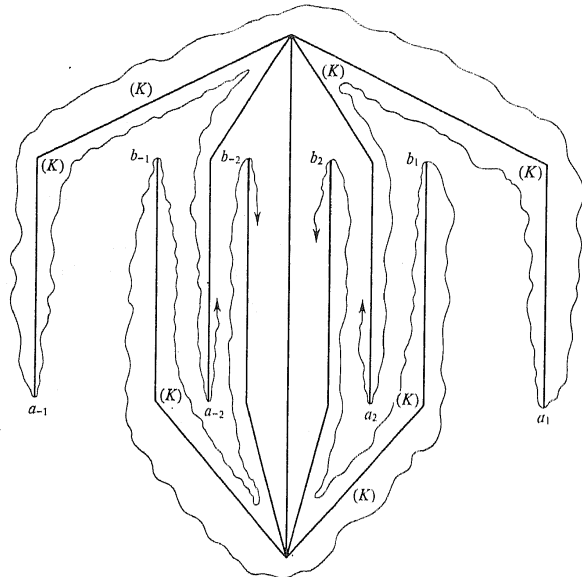

Fig. 10. W_0 is pictured in heavy lines, the remainder of the construction is pictured schematically in thinner lines

many attached copies of K. W is a dendroid and the endpoints and accessible points of W are the same; however, the set of endpoints of W is not connected since W is separated by the segment $\{0\} \times [-2, 2]$, which contains no accessible points, and this yields a separation of the set of accessible points of W.

Next, to show that M is not irreducible with respect to its stated properties, define a sequence $\langle M_i' \rangle_{i=1}^{\infty}$, where M_i' is a subcontinuum of M_i for each i such that h_i maps M_{i+1}' to M_i' in such a way that the arguments concerning the properties of M can be repeated. This can be done as follows:

Let $M_1' = \{(x, y) \in M_1 \mid y \geqslant -\frac{10}{19}\}$ (The value $-\frac{10}{19}$ is chosen here only so that the line used to truncate M_1 does meet $\mathrm{bd}(V)$ for any exposed V-arc V. It does meet the "outer" endpoints of the two broadest exposed V-arcs in M_1 (see Fig. 2).

If M_n' has been defined, consider any exposed V-arc V in M_n. It will be a subarc of an exposed V-arc $V\langle n, m \rangle$ in M_n. In order to define M_{n+1}', simply attach a portion
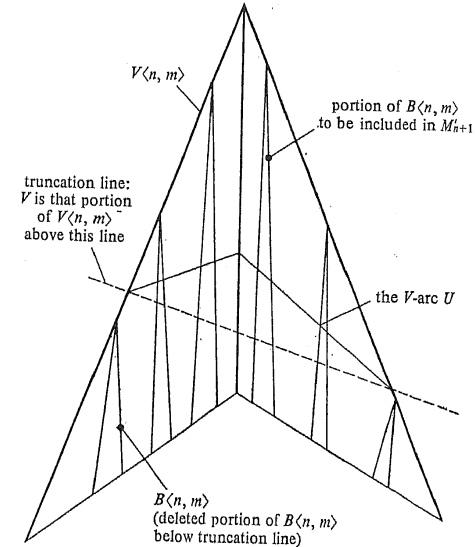

Fig. 11. The V-arc U becomes in each case the truncation line for the V-arcs at the next stage of the construction. If the V-arc V is all of $V\langle n, m \rangle$, then all of $B\langle n, m \rangle$ is included in M_{n+1}

of each $B\langle n, m \rangle$, determined by truncating $B\langle n, m \rangle$ with a V-arc U with the same endpoints as V and lying in $H(V)$ and having its vertex on the arc in $B\langle n, m \rangle$ from the vertex of V (see Fig. 11).

When this truncation and cutting out of part of $B\langle n, m \rangle$ has been done for each exposed V-arc of M_n', then M_{n+1}' is the union of M_n' and the remaining portion

of the $B\langle n, m\rangle$'s. (Of course, some of the exposed V-arcs of M_{n+1} are omitted completely from M'_{n+1}, as can be seen from the deleted portions of $B\langle n, m\rangle$ in Figure 11.)

It is easy to see that all of the argument for the M_n's can be carried out for the M'_n's, defining N'_n to be the union of M'_n with the convex hulls of all its exposed V-arcs. Thus M', the inverse limit of $\{M'_n; h_n | M'_{n+1}\}_{n=1}^{\infty}$, is a proper subcontinuum of M with the same connectedness and accessibility properties for its set of endpoints.

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An example concerning automorphisms of generalized cubes

by

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Abstract. An example is given of an involution T on I^m ($m \geqslant c$) with the following properties: (1) $T = T^{-1}$ sends Baire sets to Baire sets, (2) T induces the identity automorphism on both the category algebra and the measure algebra of I^m, (3) T has no fixed points.

Let m be an infinite cardinal, and consider the product I^m of m copies of the unit interval I, both as a topological space (with product topology) and as a measure space (with product Lebesgue measure). It is known ([1]) that every automorphism of the measure algebra of I^m can be "realized" by a point map that can even be required to be a Baire isomorphism; and a similar remark applies to the category algebra ([5]). Some time ago, S. Kakutani asked the author whether the realization would have to be "almost" unique. More specifically, suppose T is a transformation of I^m that induces the identity automorphism of the measure algebra (or of the category algebra) of I^m; need $T(x) = x$ for a "large" set of x's? When $m \leqslant \aleph_0$ the answer is easily seen to be "yes"; we have only to discard, for each member of a countable base $U_1, U_2, \ldots$, for I^m, the negligible set $U_n \triangle T(U_n)$, and T will be the identity transformation on what is left. But we show here that, for $m \geqslant c$, the answer is "no" to both the measure and category forms of the question, and by the same example — even if the transformation is required to be an involution. It is of course enough to consider $m = c$.

THEOREM. *There exists a Baire isomorphism T of I^c, of period 2, such that T induces the identity automorphisms of both the category algebra and the measure algebra of I^c, but such that T has no fixed points* ([1]).

Proof. We use the following notation. I denotes the closed unit interval $[0, 1]$ as usual; $J = I \cup \{-1\}$, $X = I^c$ regarded as $\prod \{I_j : j \in J\}$, where each $I_j = I$. The jth coordinate of $x \in X$ is $\pi_j(x)$ or $x(j)$. (As this notation suggests, the coordinate $x(-1)$ will play an exceptional role).

Take an arbitrary measure-and-category-preserving Borel isomorphism ξ of I onto itself, without fixed points, but such that $\xi^2 = $ identity map ι_I of I. For instance,

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([1]) For a simple example of such a T for which the set of fixed points is not measurable (having outer measure 1 and inner measure 0) see [3, p. 702].