Projective limits of perfect measure spaces *

by

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Abstract. Necessary and sufficient conditions for the existence and perfectness of projective limits of perfect measure spaces are given.

1. Introduction. Projective systems of measure spaces form a natural generalization of product measures. The idea of a projective system of measure spaces was first introduced by Bochner ([1], pp. 118–119), who proved the existence of the projective limit measure space when the spaces were topological and Hausdorff and the measures were approximated by compact sets. Since then projective systems of measure spaces have been the subject of a number of studies (see, e.g. Choksi [3], Metivier [13], Mallory and Sion [10], Mallory [8] and [9], Pachl [18], Parthasarathy [19]). Mainly, projective systems of compact measure spaces (see Marczewski [12]) were considered. Also Scheffer [22] and Topsøe [24] investigated projective limits of measure spaces; however, their results and methods seem to be very different from ours. Dinculeanu [4] proved a sufficient condition for the existence of projective limits in terms of the existence and some regularity properties of conditional probabilities (see also Choksi [3], Theorem 4.1).

In our paper we get necessary and sufficient conditions for the existence and perfectness of projective limits of perfect measure spaces.

Since, as is now known (Mahkamov and Vinokurov [7]), there are perfect but non-compact measures, the theorems we obtain can be applied to a wider class of projective systems than those previously known. However, using our method in the case of an arbitrary projective system of compact measure spaces, we can prove the perfectness of the projective limit measure only.

In contradiction to the approach of other authors, all the presented conditions guaranteeing the existence and perfectness of projective limits are purely measure-theoretic; we do not use either topological properties of measures or any approximating families of sets.

In Section 4 we find necessary and sufficient conditions for the existence and

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* This is a part of the authors investigations for the doctoral thesis written under the supervision of Professor C. Ryll-Nardzewski in 1971. The results were announced in [15] and [17].

perfectness of a projective system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of (perfect) measure spaces provided that all systems $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, J)$ with a countable $J \subset I$ have (perfect) limits. We introduce a new notion of the sequential maximality condition of a system, which is weaker from that introduced by Bochner [1], and does not yield the surjectivity of the mappings f_{ij}. We introduce also a few new conditions of this type, which play an important part in our theory.

Section 5 is devoted to the investigations of projective systems of separable measure spaces. We find a necessary and sufficient condition for the existence and perfectness of the limit of an arbitrary system of separable perfect measure spaces. In particular we prove that each sequentially maximal projective system of separable perfect measure spaces has perfect limit. All previously obtained theorems required different special properties of inner approximating families of sets (cf. Choksi [3], Theorem 3.1, Mallory [8], Theorem 2.4, Metivier [13], Theorem 3.2). Mallory and Sion ([10], Theorem 2.5) proved only the existence of the limit.

In Sections 6 and 7 we examine projective systems of countably generated perfect measure spaces (see Definition 2.2). We show that a projective system of countably generated measure spaces has the perfect limit if and only if it satisfies an atomic sequential almost maximality condition (see Definition 4.7).

In Section 8 we apply the results of Section 7 to the case of arbitrary projective systems of perfect measure spaces.

Finally, let us observe that Musiał [15] and Millington and Sion [14] considered projective systems of group-valued measures also.

2. Preliminaries. Throughout this paper we use the standard set-theoretical notations and terminology. However, by a *countable set* we always mean an infinite set of cardinality $\aleph_0$, and by a *sequence* — a countable one. Moreover, if X is a set and $A \subset X$, then we write A^0 for $X \setminus A$ and A^1 for A. If $\mathscr{F} \subset 2^X$ (all subsets of X) then the set $\{Z \cap F \colon F \in \mathscr{F}\}$ is denoted by $Z \cap \mathscr{F}$. If Y is a set and $f\colon X \to Y$ is a function then $f(X) = \{f(x)\colon x \in X\}$.

If $\mathscr{A}$ is a σ-algebra of subsets of a set X, then $(X, \mathscr{A})$ is said to be a *measurable space*. By a measure on a σ-algebra $\mathscr{A}$ we mean any countably additive set function $\mu\colon \mathscr{A} \to [0, \infty]$ which is σ-finite. However in all the proofs we assume — for the simplicity — that $\mu(X) = 1$. By a measure space we mean a triple $(X, \mathscr{A}, \mu)$ consisting of a measurable space $(X, \mathscr{A})$ and a measure μ defined on $\mathscr{A}$.

If $(X, \mathscr{A}, \mu)$ is a measure space, then by μ_* and μ^* we denote the inner and outer measures on X, respectively, induced by μ. If $\mathscr{B}$ is a sub-σ-algebra of $\mathscr{A}$ and $Z \subset X$, then $\mu|\mathscr{B}$ is the restriction of μ to $\mathscr{B}$ and $\mu|Z$ is the restriction of μ to Z (i.e. $\mu(Z \cap A) = \mu^*(Z \cap A)$ for $A \in \mathscr{A}$). $Z \subset X$ is thick if and only if $\mu_*(X \setminus Z) = 0$.

Two measurable spaces $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are said to be *σ-isomorphic* if and only if there exists a Boolean σ-isomorphism $h\colon \mathscr{A} \to \mathscr{B}$ such that $h(\mathscr{A}) = \mathscr{B}$.

Two measure spaces $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are *σ-isomorphic* if and only if there is a Boolean σ-isomorphism h of $\mathscr{A}$ onto $\mathscr{B}$ such that $\mu(A) = \nu[h(A)]$ for every $A \in \mathscr{A}$.

By $\sigma(\mathscr{F})$ we denote the smallest σ-algebra of subsets of X containing the family $\mathscr{F} \subset 2^X$ ($\sigma(\mathscr{F})$ is said to be *generated* by $\mathscr{F}$).

If $\mathscr{A}$ is a σ-algebra, then $E \in \mathscr{A}$ is an *$\mathscr{A}$-atom* provided $A \in \mathscr{A}$ and $A \subset E$ together imply $A = \emptyset$ or $A = E$. The family of all $\mathscr{A}$-atoms is denoted by $\operatorname{at}(\mathscr{A})$.

If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces, then a function $f\colon X \to Y$ is *$(\mathscr{A}, \mathscr{B})$-measurable* provided $f^{-1}(\mathscr{B}) \subset \mathscr{A}$.

The letter R is reserved for the real line with the natural topology. If Y is a subset of R, then by $\mathscr{B}_Y$ we denote the σ-algebra of relatively Borel sets. The letter λ is reserved for the Lebesgue measure (on R or on an interval).

The Cartesian product $\prod_{i \in I} X_i$ of a family of sets is denoted by X^I. If $(X_i, \mathscr{C}_i)$, $i \in I$, are measurable spaces, then by $\prod_{i \in I} \mathscr{C}_i$ we mean the smallest algebra of subsets of X^I containing all the sets of the form $A_j \times \prod_{i \neq j} X_i$, with $A_j \in \mathscr{C}_j$, and by $\mathscr{C}^I$ the σ-algebra generated by this algebra, i.e. $\mathscr{C}^I = \sigma(\prod_{i \in I} \mathscr{C}_i)$. If $I = \{i_1, \ldots, i_n\}$, then we use also the notation $\mathscr{C}^{i_1, \ldots, i_n}$.

If $\emptyset \neq J \subset I$, then p^J denotes the canonical projection of X^I onto X^J. In particular, if $J = \{i_1, \ldots, i_n\}$, then we write $p^{i_1 \ldots i_n}$.

DEFINITION 2.1. A measurable space $(X, \mathscr{A})$ (or a σ-algebra $\mathscr{A}$) is *countably generated* if there exists an at most countable family of sets such that $\mathscr{A}$ is generated by this family. If $\mathscr{A}$ is countably generated and contains all points of X, then $(X, \mathscr{A})$ and $\mathscr{A}$ are said to be *separable*.

$(X, \mathscr{A})$ and $\mathscr{A}$ are *quasi-countably generated* (*quasi-separable*) if there exists a countably generated (separable) σ-algebra $\mathscr{B} \subset \mathscr{A}$ with the same atoms as $\mathscr{A}$. Such a $\mathscr{B}$ is called a *basis of* $\mathscr{A}$.

DEFINITION 2.2. A measure space $(X, \mathscr{A}, \mu)$ and μ are *countably generated* (*separable*) (1) if there exist a μ-null set $N \in \mathscr{A}$ and a countably generated (separable) basis $\mathscr{B}$ of $\mathscr{A} \cap N^0$ such that $\mathscr{A} \cap N^0$ is contained in the $\mu|\mathscr{B}$ completion of $\mathscr{B}$.

The following result is essential for our purposes:

PROPOSITION 2.1. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be two measurable spaces and let $f\colon X \to Y$ be an $(\mathscr{A}, \mathscr{B})$-measurable function. Then the graph*

$$G = \{(x, y) \in X \times Y\colon f(x) = y\}$$

belongs to $\sigma(\mathscr{A} \times \mathscr{B})$ if and only if there exists a countably generated σ-algebra $\mathscr{D} \subset \mathscr{B}$ containing all the points of $f(X)$.

Proof. Suppose $G \in \sigma(\mathscr{A} \times \mathscr{B})$. Then there exists an at most countable number of rectangles $A_i \times B_i$, $A_i \in \mathscr{A}$, $B_i \in \mathscr{B}$, such that G belongs to the σ-algebra generated by them.

Let $\mathscr{D}$ be the σ-algebra generated by B_i, $i = 1, 2, \ldots$ and let $\mathscr{C}$ be generated

(1) Almost separable in the terminology of C. Ryll-Nardzewski [22].

by $f^{-1}(\mathscr{D})$ and A_i, $i = 1, 2, \ldots$ Clearly, $\mathscr{C}$ and $\mathscr{D}$ are countably generated and f is $(\mathscr{C}, \mathscr{D})$-measurable.

Let (x, y) be a point of G and let $e_1 \in \mathrm{at}(\mathscr{C})$ and $e_2 \in \mathrm{at}(\mathscr{D})$ be the atoms containing x and y, respectively.

Being $(\mathscr{C}, \mathscr{D})$-measurable, f is constant on e_1, so that

$$e_1 \times \{y\} = (e_1 \times \{y\}) \cap G = (e_1 \times e_2) \cap G \in \sigma(\mathscr{C} \times \mathscr{D}).$$

Since $e_1 \times e_2$ is an atom of $\sigma(\mathscr{C} \times \mathscr{D})$, we have $e_2 = \{y\}$ and since $(x, y) \in G$ was arbitrary, $\mathscr{D}$ contains all points of $f(X)$.

To prove the converse part of the theorem, suppose that there exists a countably generated σ-algebra $\mathscr{D} \subset \mathscr{B}$ containing all points of $f(X)$.

Let $\{D_i\}$ be a sequence generating $\mathscr{D}$ and let

$$\mathscr{D}_n = \{\bigcap_{i=1}^{n} D_i^{\delta_i}\colon \delta_i = 0 \text{ or } 1\}, \quad n = 1, \ldots$$

It is not difficult to see that

$$G = \bigcap_{n=1}^{\infty} \bigcup_{D \in \mathscr{D}_n} f^{-1}(D) \times D \in \sigma(f^{-1}(\mathscr{D}) \times \mathscr{D}) \subset \sigma(\mathscr{A} \times \mathscr{B}).$$

This completes the proof of the theorem.

As simple conclusions we obtain the following:

COROLLARY 2.1. *If $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are measurable spaces with a quasi-separable $\mathscr{B}$, then for every $(\mathscr{A}, \mathscr{B})$-measurable map $f\colon X \to Y$ the graph of f is an element of $\sigma(\mathscr{A} \times \mathscr{B})$.*

COROLLARY 2.2. *Let $\mathscr{A}$ be a σ-algebra on a set X. The diagonal of $X \times X$ belongs to $\sigma(\mathscr{A} \times \mathscr{A})$ if and only if $\mathscr{A}$ is quasi-separable.*

DEFINITION 2.3. A family of sets $(X_i)_{i \in I}$ is a *projective system relative to maps* f_{ij}, $i \in I$, $j \in I$ (we use the notation (X_i, f_{ij}, I) for such a system) if

(i) I is a directed set (the ordering relation is denoted by $\leqslant$; if $i \leqslant j$ and $i \neq j$, then we write $i < j$);

(ii) $f_{ij}\colon X_j \to X_i$ are defined for each i and j from I such that $i \leqslant j$;

(iii) $f_{ik} = f_{ij} \circ f_{jk}$ whenever $i \leqslant j \leqslant k$, and f_{ii} is the identity map.

The set $X_I = \{\{x_i\}_{i \in I} \in X^I\colon f_{ij}(x_j) = x_i$ whenever $i < j\}$ is the *projective limit of* (X_i, f_{ij}, I). We write also $X_I = \varprojlim X_i$.

DEFINITION 2.4. A family $(X_i, \mathscr{C}_i)_{i \in I}$ of measurable spaces is a *projective system relative to maps* f_{ij}, $i \in I$, $j \in I$ (we use the notation $(X_i, \mathscr{C}_i, f_{ij}, I)$ for such a system or $(X_n, \mathscr{C}_n, f_{n,n+1}, N)$ if $I = N = \{1, 2, \ldots\}$) if

(i) (X_i, f_{ij}, I) is a projective system;

(ii) f_{ij} are $(\mathscr{C}_j, \mathscr{C}_i)$-measurable.

If $\mathscr{C}_I$ is the smallest σ-algebra of subsets of X_I relative to which the canonical projections $f_i\colon X_I \to X_i$ given by $f_i(\{x_j\}_{j \in I}) = x_i$ are $(\mathscr{C}_I, \mathscr{C}_i)$-measurable (i.e. $\mathscr{C}_I = \sigma(\bigcup_{i \in I} f_i^{-1}(\mathscr{C}_i))$, then $(X_I, \mathscr{C}_I)$ is called the *projective limit of* $(X_i, \mathscr{C}_i, f_{ij}, I)$.

DEFINITION 2.5. A family of measure spaces $(X_i, \mathscr{C}_i, \mu_i)$, $i \in I$, is a *projective system relative to* $(f_{ij})_{i,j \in I}$ (we use the notation $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$) if

(i) $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a projective system;

(ii) $\mu_j f_{ij}^{-1}(C) = \mu_i(C)$ for every $i \leqslant j$ and $C \in \mathscr{C}_i$.

The system is *convergent* if there is a measure μ_I defined on $(X_I, \mathscr{C}_I)$ such that $\mu_I f_i^{-1}(C) = \mu_i(C)$ for every $i \in I$ and $C \in \mathscr{C}_i$.

$(X_I, \mathscr{C}_I, \mu_I)$ is called the *projective limit of* $(X_i, \mathscr{C}_i, \mu_I, f_{ij}, I)$.

The system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is said to be *sequentially convergent* if for every sequence $J = \{i_n\}$, $i_1 < i_2 < \ldots$ the system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, J)$ is convergent.

All the systems that are used in this paper are projective, so that, in order to simplify the notation, we shall use the word "system" instead of "projective system". Systems of measurable spaces will be called *measurable systems*, and systems of measure spaces — *measure systems*.

For a system (X_i, f_{ij}, I) and a directed subset $J \subset I$ (J will always be directed by the same relation of order as I) we write X_J for the projective limit of the system. Moreover, if systems $(X_i, \mathscr{C}_i, f_{ij}, I)$ and $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ are under consideration, then the symbols $\mathscr{C}_J$ and μ_J have the obvious meaning as well.

If J and K are directed subsets of I and $J \subset K$, then by f_{JK} we mean the canonical projection of X_K into X_J given by $f_{JK}(\{x_i\}_{i \in K}) = \{x_i\}_{i \in J}$. For $K = I$ we write f_J instead of f_{JI}. It is easily seen that we always have $f_J = f_{JK} \circ f_K$ whenever $J \subset K$. Observe also that $f_J = p^J | X_I$. If $J = \{i_1, \ldots, i_n\}$, $i_1 < i_2 < \ldots < i_n$, then we use the notation $X_{i_1 \ldots i_n}$, $\mathscr{C}_{i_1 \ldots i_n}$ and $\mu_{i_1 \ldots i_n}$.

The following two theorems contain a few basic facts about the projective systems which are used in this paper.

PROPOSITION 2.2. *Let $(X_i, \mathscr{C}_i, f_{ij}, I)$ be a measurable system. Then:*

(i) $\sigma(\bigcup_{i \in J} f_i^{-1}(\mathscr{C}_i)) = (p^J)^{-1}(\mathscr{C}^J) \cap X_I$ *for every* $\emptyset \neq J \subset I$;

(ii) $\mathscr{C}_I = \mathscr{C}^I \cap X_I$;

(iii) *if J is a non-empty directed subset of I, then*

$$\sigma(\bigcup_{i \in J} f_i^{-1}(\mathscr{C}_i)) = f_J^{-1}(\mathscr{C}_J);$$

(iv) *if $J \subset I$ is directed and cofinal with I, then the projection $f_J\colon X_I \to X_J$ is a one-to-one surjection and the map $C \to f_J^{-1}(C)$ is a σ-isomorphism of $\mathscr{C}_J$ onto $\mathscr{C}_I$.*

Proof. (i) is an immediate consequence of the equality

$$f_i^{-1}(E) = (p^i)^{-1}(E) \cap X_I$$

for $E \in \mathscr{C}_i$, $i \in I$.

(ii) is a particular case of (i) for $J = I$.

(iii) follows from the equality $f_i = f_{iJ} \circ f_J$, $i \in I$.

(iv) It follows from (iii) that $\sigma\big(\bigcup_{i\in J} f_i^{-1}(\mathscr{C}_i)\big) = f_J^{-1}(\mathscr{C}_J)$ and from the cofinality of J with I it follows that

$$\sigma\big(\bigcup_{i\in J} f_i^{-1}(\mathscr{C}_i)\big) = \mathscr{C}_I .$$

Thus, we have $\mathscr{C}_I = f_J^{-1}(\mathscr{C}_J)$.

To prove that the map $C \to f_J^{-1}(C)$ is a σ-isomorphism of $\mathscr{C}_J$ onto $\mathscr{C}_I$ it is enough to show that f_J is a one-to-one projection of X_I onto X_J, and this can be found in Bourbaki ([2], III, § 1, 12. II).

Proposition 2.3. *Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a measure system. Then:*

(i) *If $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is convergent, then for each non-empty directed set $J \subset I$ the system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, J)$ is also convergent, and the equality $\mu_J = \mu_I f_J^{-1}$ holds.*

In particular, each convergent measure system is sequentially convergent.

(ii) *If J is cofinal with I, then $(X_I, \mathscr{C}_I, \mu_I)$ exists iff $(X_J, \mathscr{C}_J, \mu_J)$ does. Moreover, the map $C \to f_J^{-1}(C)$ is a σ-isomorphism of $(X_J, \mathscr{C}_J, \mu_J)$ onto $(X_I, \mathscr{C}_I, \mu_I)$.*

Proof. (i) is a direct consequence of Proposition 2.2 (iii). (ii) follows easily from (i) and Proposition 2.2 (iv).

Our further considerations concerning the convergence of measure systems are based on the following simple observation:

Proposition 2.4. *Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a system of probability spaces. Then there exists a unique finitely additive set function $\tilde{\mu}_0$ on $\prod_{i\in I} \mathscr{C}_i$ such that*

$$\tilde{\mu}_0[(p^J)^{-1}(E)] = \mu_J(E \cap X_J)$$

for every non-empty finite directed set $J \subset I$ and every $E \in \mathscr{C}^J$.

Proof. It is enough to verify that $\tilde{\mu}_0$ given by the above formula is properly defined. But this is an obvious consequence of the equality $\mu_J = \mu_K f_{JK}^{-1}$, where $J \subset K$ is an arbitrary directed subset of I (see Proposition 2.3 (i) with $I = K$). The uniqueness of $\tilde{\mu}_0$ is a consequence of the uniqueness of μ_J's.

Since for each system (X_i, f_{ij}, I) the set X_I is a subset of X^I, in order to establish the convergence of a probability system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ we shall try first to guarantee the countable additivity of $\tilde{\mu}_0$ on the product algebra $\prod_{i\in I} \mathscr{C}_i$ and then, using its unique extension $\tilde{\mu}$ to $(X^I, \mathscr{C}^I)$, we shall examine the restriction of $\tilde{\mu}$ to X_I.

It follows from the properties of $\tilde{\mu}_0$ and Proposition 2.2 (i) that if such a restriction is a probability measure then it coincides with μ_I.

As was observed by Mallory and Sion [10], if I is uncountable then X_I need not be an element of $\mathscr{C}^I$. This explains many of the difficulties encountered by measure systems.

Proposition 2.5 [10]. *If I is uncountable and X_i contains at least two points for uncountably many $i \in I$, then $X_I \notin \mathscr{C}^I$.*

Proof. Clearly, it is enough to prove that $X_I \notin \mathscr{C}^I$ if $\operatorname{card} X_i \geqslant 2$ for each $i \in I$. But in such a case we have $X_{ij} \neq X^{ij}$ for all pairs (i, j) with the property $i<j$, so that X_I depends on uncountably many $i \in I$.

Since each element of $\mathscr{C}^I$ is of the form $(p^J)^{-1}(E)$ where $E \in \mathscr{C}^J$ and J is at most countable, the set X_I does not belong to $\mathscr{C}^I$.

3. Perfect measures. We recall in this section a few more or less known facts about perfect measures. More information can be found in [21] and [20].

Definition 3.1 (Gnedenko and Kolmogorov [5]). A measure space $(X, \mathscr{A}, \mu)$ (or a measure μ) is *perfect* if and only if for each $(\mathscr{A}, \mathscr{B}_R)$-measurable function $f\colon X \to R$ there is a set $B \in \mathscr{B}_R$, such that $B \subset f(X)$ and $\mu[X \setminus f^{-1}(B)] = 0$.

It is easily seen that if μ and ν are measures on $(X, \mathscr{A})$ and μ is absolutely continuous with respect to ν ($\mu \ll \nu$), then μ is perfect if ν is such. Moreover, if $(X, \mathscr{A}, \mu)$ is perfect, $Y \in \mathscr{A}$ and $\mathscr{B}$ is a sub-σ-algebra of $\mathscr{A}$, then $(Y, Y \cap \mathscr{A}, \mu|Y)$ and $(X, \mathscr{B}, \mu|\mathscr{B})$ are perfect. It is also known that each measure on $\mathscr{B}_R$ is regular, hence also perfect.

Proposition 3.1 (Ryll-Nardzewski [21]). *$(X, \mathscr{A}, \mu)$ is perfect if and only if for each countably generated σ-algebra $\mathscr{B} \subset \mathscr{A}$ the measure $\mu|\mathscr{B}$ is perfect.*

Proposition 3.2. *Let $(X, \mathscr{A})$ be a quasi-countably generated (quasi-separable) measurable space. If μ is a perfect measure on $(X, \mathscr{A})$, then μ is countably generated (separable).*

Proof. Let $\{E_n\} \subset \mathscr{A}$ be a sequence of sets separating the $\mathscr{A}$-atoms (points) and let h be the Marczewski function of $\{E_n\}$ [11]:

$$h(x) = 2\sum_{n=1}^{\infty} (1/3^n)\chi_{E_n}(x) .$$

If $\mathscr{B} = \sigma(\{E_n\})$, then h is a one-to-one $(\mathscr{B}, \mathscr{B}_R)$-measurable function.

Take an arbitrary $E \in \mathscr{A}$. Then, since $\mu|E$ and $\mu|X\setminus E$ are perfect, there are sets $B, C \in \mathscr{B}_R$, $B \subset h(E)$ and $C \subset h(X \setminus E)$ such that

$$\mu(E \setminus h^{-1}(B)) = \mu[(X \setminus E) \setminus h^{-1}(C)] .$$

It follows that

$$h^{-1}(B) \subset E \subset X \setminus h^{-1}(C)$$

and

$$\mu h^{-1}(B) = \mu[X \setminus h^{-1}(C)] .$$

This proves that E belongs to the $\mu|\mathscr{B}$-completion of $\mathscr{B}$ and so μ is countably generated (separable).

THEOREM 3.1. *Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be measure spaces and let $g\colon Y \to X$ be a $(\mathscr{B}, \mathscr{A})$-measurable map such that $\mu(A) = \nu[g^{-1}(A)]$ whenever $A \in \mathscr{A}$. Then the perfectness of ν yields the perfectness of μ and $\mu|g(Y)$.*

Proof. Let $f\colon X \to R$ be an $(\mathscr{A}, \mathscr{B}_R)$-measurable function. Then $f\circ g\colon Y \to R$ is $(\mathscr{B}, \mathscr{B}_R)$-measurable and the perfectness of ν yields the existence of a set $B \in \mathscr{B}_R$ such that $B \subset f[g(Y)]$ and $\nu[Y \setminus g^{-1}f^{-1}(B)] = 0$.

Hence $\mu[X \setminus f^{-1}(B)] = 0$ and $B \subset f[g(Y)] = f(X)$.

This proves the perfectness of μ.

The perfectness of $\mu|g(Y)$ follows from the first part of the theorem if we put $X = g(Y)$.

As a corollary we get

PROPOSITION 3.3. *If $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a convergent measure system, J is a non-empty directed subset of I and μ_I is perfect, then μ_J is perfect as well.*

Proof. In virtue of Proposition 2.3 (i) μ_J does exist. Its perfectness is a consequence of Theorem 3.1 with $g = f_J$.

LEMMA 3.1 (Sikorski [23]). *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be countably generated measurable spaces. If $h\colon \mathscr{A} \to \mathscr{B}$ is a σ-homomorphism of $\mathscr{A}$ onto $\mathscr{B}$, then there exists a $(\mathscr{B}, \mathscr{A})$-measurable map $g\colon Y \to X$ such that $h(A) = g^{-1}(A)$ for every $A \in \mathscr{A}$.*

Proof. If $e \in \operatorname{at}(\mathscr{A})$, then $h(e)$ is an atom of $\mathscr{B}$ or it is the empty set.

It is easy to see that for each $y \in Y$ there is only one $\mathscr{A}$-atom $e(y)$ such that $y \in h[e(y)]$.

Take for each $e \in \operatorname{at}(\mathscr{A})$ a single point $x_e \in e$ and then put $g(y) = x_{e(y)}$ for every $y \in Y$.

g is a map of Y into X and clearly $h(A) = g^{-1}(A)$ for every $A \in \mathscr{A}$.

THEOREM 3.2. *If $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ are σ-isomorphic measure spaces, then ν is perfect if and only if μ is perfect.*

Proof. Let $h\colon \mathscr{A} \to \mathscr{B}$ be a σ-isomorphism of $\mathscr{A}$ onto $\mathscr{B}$ such that $\nu[h(A)] = \mu(A)$ whenever $A \in \mathscr{A}$. We shall prove, for instance, that the perfectness of ν yields the perfectness of μ.

In view of Proposition 3.1 it is enough to show that μ is perfect on each countably generated σ-algebra $\mathscr{C} \subset \mathscr{A}$.

Let $\mathscr{D} = h(\mathscr{C})$.

By Lemma 3.1, there exists a map $g\colon Y \to X$ such that $h(C) = g^{-1}(C)$ for all $C \in \mathscr{C}$.

Hence, $\mu(A) = \nu[g^{-1}(A)]$, whenever $A \in \mathscr{C}$, and in view of Theorem 3.1 the measure $\mu|\mathscr{C}$ is perfect.

THEOREM 3.3. *If $(X, \mathscr{A}, \mu)$ is a perfect measure space, $(Y, \mathscr{B})$ is a measurable space and $h\colon \mathscr{A} \to \mathscr{B}$ is a σ-homomorphism of $\mathscr{A}$ onto $\mathscr{B}$, then a set function $\nu\colon \mathscr{B} \to R$, given by $\nu[h(A)] = \mu(A)$, for all $A \in \mathscr{A}$, is a perfect measure if and only if for each countably generated σ-algebra $\mathscr{D} \subset \mathscr{A}$ there exists a set $N \in \mathscr{D}$ such that $\mu(N) = 0$ and the restriction of h to $\mathscr{D} \cap N^0$ is a σ-isomorphism of $\mathscr{D} \cap N^0$ onto $h(\mathscr{D}) \cap h(N^0)$.*

Proof. Suppose that ν is perfect and take a countably generated $\mathscr{D} \subset \mathscr{A}$. By Lemma 3.1 there is a map $g\colon Y \to X$ such that $h(D) = g^{-1}(D)$ for each $D \in \mathscr{D}$.

Let $f\colon X \to R$ be such that $f^{-1}(\mathscr{B}_R) \subset \mathscr{D}$. Since $f\circ g\colon Y \to R$ is $(\mathscr{B}, \mathscr{B}_R)$-measurable, there exists a set $B \in \mathscr{B}_R$, such that $B \subset f[g(Y)]$ and $\nu[Y \setminus g^{-1}f^{-1}(B)] = 0$.

Put $N = X \setminus f^{-1}(B)$. Clearly, $\mu(N) = 0$ and $g^{-1}f^{-1}(x) \neq \emptyset$, whenever $x \in B$. This yields $h(D) = g^{-1}(D) \neq \emptyset$ for every $D \in \mathscr{D} \cap N^0$, and so the restriction of h to $\mathscr{D} \cap N^0$ is a σ-isomorphism.

Conversely, take an arbitrary countably generated σ-algebra $\mathscr{C} \subset \mathscr{B}$ and a countably generated σ-algebra $\mathscr{D} \subset \mathscr{A}$ such that $h(\mathscr{D}) = \mathscr{C}$.

Let $N \in \mathscr{D}$ be such that $\mu(N) = 0$ and the restriction of h to $\mathscr{D} \cap N^0$ is a σ-isomorphism of $\mathscr{D} \cap N^0$ onto $h(\mathscr{D} \cap N^0) = \mathscr{C} \cap h(N^0)$.

Since $h(A) \neq \emptyset$ for each $A \in \mathscr{A}$ with $\mu(A) > 0$, ν is a measure on $\mathscr{B}$.

Since $(N^0, N^0 \cap \mathscr{D}, \mu|\mathscr{D} \cap N^0)$ is perfect, it follows from Theorem 3.2 that the restriction of ν to $h(N^0) \cap \mathscr{C}$ is perfect as well.

Since $\mathscr{C}$ was arbitrary, this proves (in view of Proposition 3.1) the perfectness of ν.

Remark 3.1. Observe that, in general, the perfectness of ν does not yield the existence of a set $N \in \mathscr{A}$ such that $\mu(N) = 0$ and $h(A) \neq \emptyset$ for every non-empty set $A \in N^0 \cap \mathscr{A}$. However, it is the case for countably generated $\mathscr{A}$.

Remark 3.2. If I is a directed set and J is a countable directed subset of I which is cofinal with I, then there is also an increasing sequence that is cofinal with I.

If $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a measure system, then by Proposition 2.3 (ii) the measure spaces $(X_I, \mathscr{C}_I, \mu_I)$ and $(X_J, \mathscr{C}_J, \mu_J)$ are σ-isomorphic.

It is clear that countable additivity is preserved by σ-isomorphisms. Moreover, it has been proved in Theorem 3.2 that the perfectness of measure spaces is preserved by σ-isomorphisms.

In this paper we are interested in the existence and perfectness of projective limits of measure systems and so, from our point of view, instead of measure systems with countable cofinal directed index subsets we may consider only systems directed by positive integers.

4. Convergence of sequentially convergent measure systems. Throughout this section we assume that I is a directed set and Ξ is the collection of all increasing subsequences of I: $\Xi = \{\{i_n\}_{n=1}^{\infty}\colon i_1 < i_2 < \ldots \text{ and } i_n \in I \text{ for all } n\}$.

We begin our considerations with two examples.

EXAMPLE 4.1. Let I be the collection of all at most countable subsets of the interval $[0, 1]$, ordered by inclusion. Setting $X_i = [0, 1] \setminus i$, $\mathscr{C}_i = \mathscr{B}_{Xi}$ and $f_{ij}\colon X_j \to X_i$ to be the identity injection, we get a system of measurable spaces $(X_i, \mathscr{C}_i)$, $i \in I$. If λ_i, $i \in I$, is the restriction of λ to the space $(X_i, \mathscr{C}_i)$, then $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a system of perfect measure spaces.

Since $X_I = \emptyset$, the limit measure space $(X_I, \mathscr{C}_I, \lambda_I)$ does not exist.

On the other hand, for every $J = \{i_n\} \in \Xi$, the system $(X_i, \mathscr{C}_i, \lambda_i, f_{ij}, J)$ has a limit; as can easily be seen, it is equal to $([0,1]\setminus\bigcup_{n=1}^{\infty} i_n, \mathscr{B}_{X_J}, \lambda|\mathscr{B}_{X_J})$. It is obvious that the measure $\lambda|\mathscr{B}_{X_J}$ is perfect.

EXAMPLE 4.2. Let A be a subset of $[0, 1]$ such that $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$. Moreover, let I be the collection of all at most countable subsets of the set $[0,1]\setminus A$ ordered by inclusion, $X_i = [0,1]\setminus i$, $\mathscr{C}_i = \mathscr{B}_{X_i}$, $\lambda_i = \lambda|\mathscr{C}_i$ and $f_{ij}(x) = x$, whenever $x \in X_j$.

In the same manner as in the first example, it follows that for every $J = \{i_n\} \in \Xi$, the system $(X_i, \mathscr{C}_i, \lambda_i, f_{ij}, J)$ has a perfect limit $(X_J, \mathscr{B}_{X_J}, \lambda|X_J)$ with $X_J = [0,1]\setminus\bigcup_{n=1}^{\infty} i_n$.

It is easy to see that $(A, \mathscr{B}_A, \lambda|A) = (X_I, \mathscr{C}_I, \lambda_I)$. However, it follows from the properties of A that $\lambda|A$ is not perfect.

The two systems $(X_i, \mathscr{C}_i, \lambda_i, f_{ij}, I)$ which have been considered above have the following common property:

for each $J \in \Xi$ there exists a perfect measure space $(X_J, \mathscr{C}_J, \mu_J)$. In the first case $(X_I, \mathscr{C}_I, \mu_I)$ does not exist and in the second one the limit exists but is not perfect.

It is the aim of this section to give necessary and sufficient conditions for the existence and perfectness of limits $(X_I, \mathscr{C}_I, \mu_I)$ provided that all measure spaces $(X_J, \mathscr{C}_J, \mu_J)$ with $J \in \Xi$ exist.

Our first theorem gives a necessary and sufficient condition for the convergence of an arbitrary sequentially convergent measure system.

THEOREM 4.1. *For each sequentially convergent measure system* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *the following conditions are equivalent:*

(i) *the system is convergent to* $(X_I, \mathscr{C}_I, \mu_I)$;

(ii) *for each* $J \in \Xi$ *the set* $f_J(X_I)$ *is thick in* $(X_J, \mathscr{C}_J, \mu_J)$.

Proof. (i→ii) If $J \in \Xi$, then, in virtue of Proposition 2.3 (i), we have $\mu_J(C) = \mu_I f_J^{-1}(C)$ for every $C \in \mathscr{C}_J$.

If $D \in \mathscr{C}_J$ is disjoint with $f_J(X_I)$ then clearly $f_J^{-1}(D) = \emptyset$, and hence we have $\mu_J(D) = 0$, which proves (ii).

(ii→i) First of all we shall prove that by setting $\mu f_i^{-1}(C) = \mu_i(C)$, for arbitrary $C \in \mathscr{C}_i$, $i \in I$, we get a well defined non-negative and finitely additive set function μ on the algebra $\bigcup_{i\in I} f_i^{-1}(\mathscr{C}_i)$.

Indeed otherwise there would exist an i_0 and a $C \in \mathscr{C}_{i_0}$ such that $f_{i_0}^{-1}(C) = \emptyset$ and $\mu_{i_0}(C) \neq 0$.

If $J = \{i_n\}_{n=0}^{\infty} \in \Xi$, then $(X_{i_n}, \mathscr{C}_{i_n}, \mu_{i_n})_{n=0}^{\infty}$ is a system relative to the maps $f_{i_n, i_{n+1}}$, $n = 0, 1, \ldots$

Since $(X_J, \mathscr{C}_J, \mu_J)$ exists, it follows that $f_{i_0J}^{-1}(C) \neq \emptyset$ and $\mu_J f_{i_0J}^{-1}(C) > 0$.

It follows from the thickness of $f_J(X_I)$ that $f_{i_0J}^{-1}(C) \cap f_J(X_I) \neq 0$, and hence $f_{i_0}^{-1}(C) \neq \emptyset$.

This contradiction shows that μ is well defined on $\bigcup_{i\in I} f_i^{-1}(\mathscr{C}_i)$.

Clearly it is also non-negative and finitely additive.

To show that $\mu = \mu_I$ it is now enough to prove the σ-additivity of μ.

Clearly it is sufficient to prove the σ-additivity of μ on each algebra $\bigcup_{n=1}^{\infty} f_{i_n}^{-1}(\mathscr{C}_{i_n})$, where $J = \{i_n\} \in \Xi$, which in virtue of Proposition 2.2 (iii) generates $f_J^{-1}(\mathscr{C}_J)$.

But $f_J(X_I)$ is thick in $(X_J, \mathscr{C}_J, \mu_J)$, and therefore, setting

$$\hat{\mu}[f_J^{-1}(D)] = \mu_J(D) \quad \text{for} \quad D \in \mathscr{C}_J,$$

we get a measure $\hat{\mu}$ on $f_J^{-1}(\mathscr{C}_J)$. Since $\hat{\mu} f_{i_n}^{-1}(C) = \mu f_{i_n}^{-1}(C)$ whenever $C \in \mathscr{C}_{i_n}$, the restriction of $\hat{\mu}$ to the algebra $\bigcup_{n=1}^{\infty} f_{i_n}^{-1}(\mathscr{C}_{i_n})$ coincides with μ.

Thus, μ is countably additive on $\bigcup_{n=1}^{\infty} f_{i_n}^{-1}(\mathscr{C}_{i_n})$, and this completes the proof.

The next example shows that the assumption of the surjectivity of f_{ij} does not imply in general the existence of the projective limit of any sequentially convergent measure system.

EXAMPLE 4.3. Let I be the collection of all finite subsets of a set T whose cardinality is greater than the continuum. We assume that I is ordered by inclusion.

Moreover, let D_i be the diagonal of the n-dimensional cube $[0,1]^i$, $n = 2, 3, \ldots$ where $\operatorname{card}(i) = n$.

Then, put

$$X_i = \{[0,1]^n \setminus \{(s_1, \ldots, s_n) \in [0,1]^n\colon\ s_p = s_q \text{ for some } p \neq q\}\} \cup D_n,$$

where $\operatorname{card}(i) = n$, $\mathscr{C}_i = \mathscr{B}_{X_i}$ and $\mu_i = \lambda^n|X_i$, where λ^n is the n-dimensional Lebesgue measure.

If $f_{ij}\colon X_j \to X_i$ are the usual projections, then $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ forms a system in which f_{ij} are surjections. It is easy to see that it is a sequentially convergent system (all μ_J being perfect measures) and $f_i(X_I) = D_i$, which is a set of measure zero in X_i if $\operatorname{card}(i) > 1$.

It follows not only that μ_I does not exist but also that it is impossible to define a finitely additive set function μ on $\bigcup_{i\in I} f_i^{-1}(\mathscr{C}_i)$ such that $\mu f_i^{-1}(C) = \mu_i(C)$ whenever $C \in \mathscr{C}_i$.

PROBLEM 4.1. Is every sequentially convergent system in which $f_i\colon X_I \to X_i$ are surjections convergent?

DEFINITION 4.1. Let $(X, \mathscr{C})$ be a measurable space. A subset of X is said to be *thick in* $(X, \mathscr{C})$ if it is thick with respect to each non-atomic measure on $\mathscr{C}$ (e.g. $A \subset [0,1]$ is thick in $([0,1], \mathscr{B}_{[0,1]})$ if and only if $[0,1]\setminus A$ does not contain any uncountable Borel set).

Using this notion, we can formulate the following condition ensuring at least the existence of projective limits of non-atomic measure spaces.

COROLLARY 4.1. *Let $(X_i, \mathscr{C}_i, f_{ij}, I)$ be a measurable system. If for every $J \in \Xi$ the set $f_J(X_I)$ is thick in $(X_J, \mathscr{C}_J)$, then each sequentially convergent system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ with non-atomic μ_i has a (non-atomic) limit.*

Now the question is: what changes should be made in the formulation of Theorem 4.1, in order to guarantee the perfectness of $(X_I, \mathscr{C}_I, \mu_I)$ in the case of a sequentially convergent system of measure spaces? The next theorem gives an answer to this question.

DEFINITION 4.2. Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a sequentially convergent system. If for each $J \in \Xi$ the space $(X_J, \mathscr{C}_J, \mu_J)$ is perfect, then we say that $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a *sequentially convergent system with perfect limits.*

THEOREM 4.2. *For each sequentially convergent system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ with perfect limits the following conditions are equivalent:*

(i) *there exists a perfect measure space $(X_I, \mathscr{C}_I, \mu_I)$;*

(ii) *for each $J \in \Xi$, the set $f_J(X_I)$ is thick in $(X_J, \mathscr{C}_J, \mu_J)$ and the measure $\mu_J | f_J(X_I)$ is perfect;*

(iii) *for each $J \in \Xi$ and each (quasi-) countably generated σ-algebra $\mathscr{D} \subset \mathscr{C}_J$, there is a set $D_0 \in \mathscr{D}$ with the following properties:*

(a) $\mu_J(X_J \setminus D_0) = 0$,

(b) $f_J(X_I) \cap e \neq \emptyset$ *for every $\mathscr{D}$-atom $e \subset C_0$.*

Proof. (i→ii) It follows from Theorem 4.1 that $f_J(X_I)$ is thick, and Theorem 3.2 and Proposition 4.1 imply the perfectness of $\mu_J | f_J(X_I)$.

(ii→iii) Take a $J \in \Xi$ and a (quasi-) countably generated σ-algebra $\mathscr{D} \subset \mathscr{C}_J$. Since the measure $\mu_J | \mathscr{D} \cap f_J(X_I)$ is perfect, there is — in virtue of Theorem 3.3 — a set $N \in \mathscr{D}_J$ such that $\mu_J(N) = 0$ and $e \cap f_J(X_I) \neq \emptyset$ for every $\mathscr{D}$-atom $e \subset X_J \setminus N$.

Thus, it is sufficient to take $D_0 = X_J \setminus N$.

(iii→i). It is easy to see that assuming (iii) we get the thickness of $f_J(X_I)$, and hence, in virtue of Theorem 4.1, there exists a space $(X_I, \mathscr{C}_I, \mu_I)$; we have only to prove that it is perfect.

Clearly, it is enough to show the perfectness of μ_I on every σ-algebra $f_J^{-1}(\mathscr{D})$, where $\mathscr{D} \subset \mathscr{C}_J$ is countably generated and $J \in \Xi$.

If $J \in \Xi$ and $\mathscr{D} \subset \mathscr{C}_J$ is countably generated, then it follows from condition (b) that the map

$$D \cap D_0 \to f_J^{-1}(D \cap D_0), \quad D \in \mathscr{D},$$

is a σ-isomorphism of $D \cap \mathscr{D}$ onto $f_J^{-1}(\mathscr{D}) \cap f_J^{-1}(D_0)$.

Thus, in view of Theorem 3.2, the restriction of μ_I to $f_J^{-1}(\mathscr{D} \cap D_0)$ is perfect. It follows that $\mu_I | f_J^{-1}(\mathscr{D})$ is perfect as well.

This completes the proof of the theorem.

DEFINITION 4.3. A system (X_i, f_{ij}, I) $((X_i, \mathscr{C}_i, f_{ij}, I)$ or $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I))$ is *sequentially maximal* (s.m.) if, for each sequence $J = \{i_n\}_{n=1}^{\infty} \in \Xi$ and each sequence $\{x_n\}_{n=1}^{\infty} \in X_J$, there exists an $x \in X_I$ such that $f_{i_n}(x) = x_n$, $n = 1, 2, \ldots$

Observe that this notion of sequential maximality is weaker from that given by Bochner [1], and, in general, it does not yield the surjectivity of the canonical mappings f_{ij} and f_i.

If f_{ij} are surjections, the two notions coincide.

It is clear that (X_i, f_{ij}, I) is s.m. iff $f_J(X_I) = X_J$ for every $J \in \Xi$.

DEFINITION 4.4. If (X_i, f_{ij}, I) is a system and $J \subset I$ is directed, then we say that a family $\{A_i : i \in J\}$ of sets $A_i \subset X_i$ is *ascending* (*descending*) if $f_{ij}^{-1}(A_i) \subset A_j$ $(f_{ij}^{-1}(A_i) \supset A_j)$ whenever $i < j$.

DEFINITION 4.5 [14]. A system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is *sequentially almost maximal* (*s.a.m.*) if for every $\varepsilon > 0$ and every $J = \{i_n\}_{n=1}^{\infty} \in \Xi$, there exists a descending sequence of sets $A_n \in \mathscr{C}_{i_n}$ with the following properties:

(a) $\mu_{i_n}(X_{i_n} \setminus A_n) < \varepsilon$, $n = 1, 2, \ldots$

(b) for each sequence $\{x_n\}_{n=1}^{\infty} \in X_J$ such that $x_n \in A_n$, $n = 1, 2, \ldots$, there exists a point $x \in X_I$ such that $f_{i_n}(x) = x_n$, for all n.

An idea of the almost maximality condition becomes clear due to the following:

PROPOSITION 4.1. *A sequentially convergent system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is s.a.m. if and only if for every $J \in \Xi$ there exists a set $D \in \mathscr{C}_J$ such that $D \subset f_J(X_I)$ and $\mu_J(X_J \setminus D) = 0$.*

Proof. Assume that the system is s.a.m. and take a $J \in \Xi$.

Then, for $\varepsilon = 1/k$, $k = 1, 2, \ldots$, there exist sets $A_n^k \in \mathscr{C}_{i_n}$ satisfying the assumptions of Definition 4.5.

Setting

$$D = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} f_{i_n J}^{-1}(A_n^k),$$

we get the desired result.

Conversely, suppose that, for a given $J \in \Xi$, we have $D \in \mathscr{C}_J$ such that $D \subset f_J(X_I)$ and $\mu_J(X_J \setminus D) = 0$.

Then, for a given $\varepsilon > 0$ there is an ascending sequence of sets $B_n \in \mathscr{C}_{i_n}$ such that

$$X_J \setminus D \subset \bigcup_{n=1}^{\infty} f_{i_n J}^{-1}(B_n)$$

and

$$\mu_J\Big[\bigcup_{n=1}^{\infty} f_{i_n J}^{-1}(B_n)\Big] < \varepsilon .$$

Setting $A_n = X_{i_n} \setminus B_n$, $n = 1, \ldots$, we get the sequential almost maximality of the system.

As direct consequences of Theorems 4.1 and 4.2 we get the following results:

THEOREM 4.3. *Each sequentially convergent measure system (with perfect limits) which is s.a.m. or s.m. is convergent (to a perfect measure space).*

Remark 4.1. Since the system in Example 4.2 is not s.a.m., the sequential almost maximality of a sequentially convergent measure system is not necessary for the existence of projective limits.

COROLLARY 4.2. *A sequentially convergent measure system* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *with one-to-one surjections* f_{ij} *is convergent. If all* μ_J, $J \in \Xi$, *are perfect, then* μ_I *is perfect as well.*

Proof. The assertion follows from Theorem 4.3 because (X_i, f_{ij}, I) is s.m.

The example given below shows that in Corollary 4.2 the assumption that the system is sequentially convergent is essential.

EXAMPLE 4.4. Let $I = (0, 1)$ with the usual ordering and let $X_i = (0, 1)$, $i \in I$. Moreover, let $\mathscr{C}_i$ be the σ-algebra generated by all the intervals $(0, j)$ with $j \leqslant i$, and let μ_i be a probability measure on $\mathscr{C}_i$ such that $\mu_i([i, 1)) = 1$.

If f_{ij} are the identity maps, then $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a system such that f_i and f_{ij} are one-to-one surjections.

Clearly, the above system is not sequentially convergent because

$$\bigcap_{n=1}^{\infty} [1-1/n, 1) = \emptyset .$$

COROLLARY 4.3. *Let* $(X_i, \mathscr{C}_i, f_{ij}, I)$ *be a measurable system. If, for every* $J \in \Xi$, *the set* $X_J \setminus f_J(X_I)$ *is universally null (i.e. it is* μ*-null for each non-atomic* μ *on* $\mathscr{C}_J$*), then each sequentially convergent system* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *of non-atomic measure spaces and with perfect limits is convergent to a perfect measure space.*

It is easy to construct a system $(X_i, \mathscr{C}_i, f_{ij}, I)$ which is not s.m. and for which all sets $X_J \setminus f_J(X_I)$ $(J \in \Xi)$ are universally null.

Now we shall introduce two notions which in the case of countably generated spaces are more useful than s.a.m. and s.m.

DEFINITION 4.6 [15]. A system $(X_i, \mathscr{C}_i, f_{ij}, I)$ of quasi-countably generated measurable spaces is *atomic sequentially maximal* (*a.s.m.*) if for each $J = \{i_n\} \in \Xi$ and each descending sequence of $\mathscr{C}_{i_n}$-atoms e_n, $n = 1, 2, \ldots$, there exists an $x \in X_I$, such that $f_{i_n}(x) \in e_n$ for all n.

It is clear that a system $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a.s.m. if and only if $f_J(X_I) \cap e \neq \emptyset$ for every $\mathscr{C}_J$-atom e.

It is easy to construct an example of an a.s.m. system which is not s.m.

DEFINITION 4.7 [15]. A system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of countably generated measure spaces is *atomic sequentially almost maximal* (*a.s.a.m.*) if, for every $\varepsilon > 0$ and every $J = \{i_n\} \in \Xi$, there exists a descending sequence of sets $A_n \in \mathscr{C}_{i_n}$ with the following properties:

(a) $A_n \cap \mathscr{C}_{i_n}$ are quasi-countably generated;

(b) $\mu_{i_n}(X_{i_n} \setminus A_n) < \varepsilon$, $n = 1, \ldots$

(c) for each descending sequence of $\mathscr{C}_{i_n}$-atoms $e_n \subset A_n$ $n = 1, 2, \ldots$, there exists a point $x \in X_I$ such that $f_{i_n}(x) \in e_n$ for all n.

Remark 4.2. The above two properties of systems were introduced in [15] without names.

The following proposition can be proved exactly as Proposition 4.2.

PROPOSITION 4.2. *A sequentially convergent system* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *of countably generated measure spaces is a.s.a.m. if and only if for every* $J \in \Xi$ *there exists a set* $D \in \mathscr{C}_J$ *with the following properties:*

(α) $D \cap \mathscr{C}_J$ *is quasi-countably generated;*

(β) $\mu_J(X_J \setminus D) = 0$;

(γ) $f_J(X_I) \cap e \neq \emptyset$ *for every* $\mathscr{C}_J$*-atom* $e \subset D$.

In the special case of countably generated measure spaces Theorem 4.2 and partially Theorem 4.1 can be reformulated as follows:

THEOREM 4.4. *Let* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *be a sequentially convergent system of countably generated measure spaces.*

(i) *If the system is convergent and* μ_I *is perfect, then it satisfies the a.s.a.m. condition.*

(ii) *If the system is a.s.a.m. (and has perfect limits), then it is convergent (to a perfect measure space).*

Proof. Only the perfectness of μ_I in (ii) requires a proof. But this follows from Theorem 3.2 because the σ-algebras $D \cap \mathscr{C}_J$ and $f_J(X_I) \cap D \cap \mathscr{C}_J$ (D from the a.s.a.m. condition) are σ-isomorphic.

As a direct consequence of Theorem 4.4 we get also

THEOREM 4.5. *If a system* $(X_i, \mathscr{C}_i, f_{ij}, I)$ *of quasi-countably generated measurable spaces is a.s.m., then each sequentially convergent system* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *(with perfect limits) is convergent (to a perfect measure space).*

Proof. The system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is always a.s.a.m., and so we may apply Theorem 4.4 (ii).

From Theorem 4.2 and 4.4 we get

THEOREM 4.6. *Let* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *be a sequentially convergent measure system.*

(i) *If the system is convergent to a perfect measure space, then each system* $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$ *of countably generated measure spaces with* $D_i \subset \mathscr{C}_i$ *is a.s.a.m.*

(ii) *If all the systems* $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$ *of countably generated measure spaces with* $\mathscr{D}_i \subset \mathscr{C}_i$ *are a.s.a.m. (and have perfect limits), then* $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ *is convergent (to a perfect measure space).*

Proof. (i) The convergence of the initial system yields the convergence of an arbitrary $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$. In particular for countably generated measure spaces the a.s.a.m. condition is satisfied (Theorem 4.4 (i)).

(ii) It follows from the a.s.a.m. condition for all the systems $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$, that for each $J \in \Xi$ the set $f_J(X_I)$ is thick in $(X_J, \mathscr{C}_J, \mu_J)$, and so the existence of μ_I follows from Theorem 4.1.

The perfectness of μ_I is a consequence of Theorem 4.2 (iii) and Proposition 4.2.

Remark 4.3. We will show in § 7 that, in fact, every measure system satisfying the a.s.a.m. condition is sequentially convergent.

In view of the theorems proved in this section, we devote the rest of the paper to determining the conditions under which a given measure system is sequentially convergent.

5. Projective limits of separable perfect measure spaces. In this section the problem of the existence of a projective limit measure in the case of separable perfect measure spaces is considered.

It is proved that every system of separable and perfect measures ordered by a set with a countable cofinal subset has a perfect limit. Moreover, we give a necessary and sufficient condition for the existence of a perfect limit of an arbitrary system of separable measure spaces.

Our starting point is a theorem on product measures proved by C. Ryll-Nardzewski in [21]. We quote it without proof.

THEOREM 5.1. *Let $(X_n, \mathscr{C}_n, \mu_n)_{n=1}^{\infty}$ be a sequence of perfect probability spaces. Each finitely additive set function μ: $\prod_{n=1}^{\infty} \mathscr{C}_n \to [0, 1]$ such that*

$$\mu[(p^n)^{-1}(E)] = \mu_n(E)$$

whenever $E \in \mathscr{C}_n$, $n = 1, 2, \ldots$, is countably additive. The unique extension of μ to $\mathscr{C}^N$ is a perfect measure.

As follows from the above theorem and the examples presented in § 4, that the problem of perfectness of projective limits in the case of general systems is much more complicated than in the case of product spaces. It appears that even the countability of I and the surjectivity of f_{ij} do not guarantee the perfectness of projective limits.

EXAMPLE 5.1. Take a subset A of $[0, 1]$ such that $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$ and put $\mathscr{C} = \sigma(\mathscr{B}_{[0,1]} \cup \{A\})$. Since $\mathscr{C}$ is countably generated, there exists a sequence $\{A_k\}_{k=1}^{\infty}$ of sets generating $\mathscr{C}$.

Put $\mathscr{C}_n = \sigma(\{A_k\}_{k=1}^{n})$ and $X_n = [0, 1]$, $n = 1, \ldots$

If μ is an extension of λ onto $\mathscr{C}$, $f_{n,n+1}$: $X_{n+1} \to X_n$ is the identity injection, and $\mu_n = \mu | \mathscr{C}_n$, then $(X_n, \mathscr{C}_n, \mu_n, f_{n,n+1}, N)$ is a system of perfect measure spaces with the limit $([0, 1], \mathscr{C}, \mu)$, which is not perfect.

Let us observe that, setting $(Y_n, \mathscr{D}_n, \nu_n) = (X_n \times X_n, \sigma(\mathscr{C}_n \times \mathscr{B}_{X_n}), \mu_n \times \lambda)$ and $g_{n,n+1}(y) = y$ we get a system of perfect measure spaces with non-atomic measures ν_n and with a non-perfect limit $([0, 1] \times [0, 1], \sigma(\mathscr{C} \times \mathscr{B}_{[0,1]}), \mu \times \lambda)$.

It is our aim to show that in the case of separable perfect measures such pathological constructions are impossible.

We begin with a simple consequence of Corollary 2.1.

PROPOSITION 5.1. *Let $(X_i, \mathscr{C}_i, f_{ij}, I)$ be a measurable system of (quasi-) separable measurable spaces. If I has an at most countable cofinal subset, then $(X_I, \mathscr{C}_I)$ is (quasi-) separable. Moreover, if I is at most countable, then $X_I \in \mathscr{C}^I$.*

Proof. Assume that I is at most countable. Then, by Corollary 2.1, we have $X_{ij} \in \mathscr{C}^{ij}$ for every $i<j$. Since

$$X_I = \bigcap_{\{(i,j):\, i<j\}} (p^{ij})^{-1}(X_{ij})$$

and

$$(p^{ij})^{-1}(X_{ij}) \in \mathscr{C}^I,$$

we have $X_I \in \mathscr{C}^I$.

If I has an at most countable cofinal subset, then the (quasi-) separability of $(X_I, \mathscr{C}_I)$ is a consequence of Proposition 2.2 (iv) and the (quasi-) separability of $\mathscr{C}_I$ in the case of an at most countable I.

Now we are in a position to prove a theorem that is fundamental for all the subsequent results of this paper. The theorem generalizes Theorem V. 3.2 of Parthasarathy [18], proved for standard spaces $(X_i, \mathscr{C}_i)$ and surjective maps f_{ij}. The proof presented here is completely different from that of Parthasarathy.

THEOREM 5.2. *If I is a directed set possessing a countable cofinal subset, then each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of separable and perfect measure spaces is convergent to a separable perfect measure space $(X_I, \mathscr{C}_I, \mu_I)$.*

Proof. In virtue of Remark 3.2 we may assume that $I = N$, and a simple calculation shows also that without loss of generality we may assume the quasi-separability of all $(X_i, \mathscr{C}_i)$, $i \in N$.

Take the finitely additive set function $\tilde{\mu}_0$ on $\prod_{i \in I} \mathscr{C}_i$, whose existence was proved in Proposition 2.3.

In virtue of Theorem 5.1 there exists a unique extension of $\tilde{\mu}_0$ to a perfect probability measure $\tilde{\mu}$ on $\mathscr{C}^N$.

Since

$$\tilde{\mu}[(p^{1\ldots n})^{-1}(X_{1\ldots n})] = \mu_{1\ldots n}(X_{1\ldots n}) = 1,$$

$$X_I = \bigcap_{n=1}^{\infty} (p^{1\ldots n})^{-1}(X_{1\ldots n}) \in \mathscr{C}^N,$$

and $\tilde{\mu}$ is a measure, we have $\tilde{\mu}(X_I) = 1$, and the restriction of $\tilde{\mu}$ to X_I is a perfect measure. Since $\tilde{\mu} | X_I = \mu_I$ (see the commentary after Proposition 2.4) and $\mathscr{C}_I$ is quasi-separable (Proposition 5.1), μ_I is separable (Proposition 3.2) and this completes the proof.

As Example 4.1 shows, if I does not possess any countable cofinal subset, then Theorem 5.2 does not hold.

Remark 5.1. Using the proof of the above theorem, it is easy to construct examples of probability spaces $(X_n, \mathscr{C}_n, \mu_n)$ for which there are finitely additive set functions, defined on $\prod_{n=1}^{\infty}\mathscr{C}_n$ and generated by μ_n's, which cannot be extended to any measure on $\mathscr{C}^N$.

In fact, if $(X_n, \mathscr{C}_n, \mu_n, f_{n,n+1}, N)$ is a system of probability spaces without any limit, then $\tilde{\mu}_0$ has the desired properties.

In particular, taking a decreasing sequence of sets $Y_n \subset [0, 1]$ such that $\bigcap_{n=1}^{\infty} Y_n = \emptyset$ and Y_n, $n = 1, 2, \ldots$ are thick with respect to λ (see Halmos [6], p. 214 for a construction of such a sequence), we can construct a sequence of measures $\mu_n\colon \mathscr{C}_n \to [0, 1]$, $n = 1, 2, \ldots$, where $\mathscr{C}_n = \sigma(\mathscr{B}_{[0,1]} \cup \{Y_1\} \cup \ldots \cup \{Y_n\})$ (cf. Halmos [6], p. 71) which are extensions of λ and are such that $\mu_{n+1}|\mathscr{C}_n = \mu_n$ and, $\mu_n(Y_n) = 1$ for every n.

Of course, $\varprojlim \mu_n$ does not exist (we take $(X_n = [0, 1], \mathscr{C}_n, \mu_n, f_{n,n+1}$ — identity map, N)) and so the finitely additive set function $\tilde{\mu}_0\colon \prod_{n=1}^{\infty}\mathscr{C}_n \to [0, 1]$ is not countably additive.

The following theorem completely solves the problem of the existence and perfectness of $(X_I, \mathscr{C}_I, \mu_I)$ in the case of arbitrary system of perfect separable measure spaces.

THEOREM 5.3. *Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a system of perfect separable measure spaces. Then, a necessary and sufficient condition for the existence of the perfect space $(X_I, \mathscr{C}_I, \mu_I)$ is the sequential almost maximality of the system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$.*

Proof. In virtue of Theorem 5.2 $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a sequentially convergent system with perfect limits, and so the assertion follows from Theorem 4.4.

As a straightforward consequence of Theorems 5.3 we get the following result:

THEOREM 5.4. *If a system $(X_i, \mathscr{C}_i, f_{ij}, I)$ of quasi-separable measurable spaces is s.m., or I has a countable cofinal subset, then every system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is convergent to a perfect measure space.*

Now we shall show that the converse to Theorem 5.4 is false even for countable I.

EXAMPLE 5.2. We denote by W the rational numbers of the interval $(0, 1)$ and set $I = W$. Setting $X_w = (0, 1)\setminus W \cap (0, w)$, $w \in W$, $\mathscr{C}_w = \mathscr{B}_{X_w}$ and $f_{ww'}\colon X_{w'} \to X_w$ to be the identity injection whenever $w \leqslant w'$, we obtain a system of separable measurable spaces which is not s.m.

On the other hand, in virtue of Theorem 5.2, every system $(X_w, \mathscr{C}_w, \mu_w, f_{ww'}, W)$ has a perfect limit.

In the case of separable measures the compactness (Marczewski [12]) and perfectness of measures coincide (Ryll-Nardzewski [21]). Thus it is natural to ask about the compactness of a projective limit of separable compact measure spaces. Unfortunately, the answer is negative: there exists a system of separable measure spaces whose limit is perfect but not compact.

EXAMPLE 5.3. Denote by $\Gamma(\Lambda)$ the set of all non-limit (limit) ordinals less than ω_1. Then let $Y_\gamma = \{0, 1\}$, let ν_γ be the uniform probability measure defined on $\mathscr{D}_\gamma = 2^{Y_\gamma}$ and let $(Y^\gamma, \mathscr{D}^\gamma, \nu^\gamma)$ be the product space where ν^γ is the direct product of all ν_α, with $\alpha < \gamma < \omega_1$ and $\alpha, \gamma \in \Gamma$.

For each $\lambda \in \Lambda$ choose a sequence $\gamma_0 < \gamma_1 < \ldots$ cofinal with λ and put

$$M^\lambda = \{\{x_\gamma\}_{\gamma<\lambda}\colon x_{\gamma_n} = 0,\ n = 1, 2, \ldots\}$$

and

$$M_\delta^\lambda = M^\lambda \times \prod_{\lambda<\gamma<\delta} Y_\gamma, \qquad \delta \in \Gamma.$$

Clearly we have $\nu^\delta(M_\delta^\lambda) = 0$.

Put $X_\gamma = Y^\gamma \setminus \bigcup_{\lambda<\gamma} M_\gamma^\lambda$, $\mathscr{C}_\gamma = \mathscr{D}^\gamma \cap X_\gamma$ and $\mu_\gamma = \nu^\gamma | X_\gamma$. Moreover, let $f_{\alpha\gamma}$, $\alpha < \gamma$ and $\alpha, \gamma \in \Gamma$ be the canonical projections of X_γ into X_α. The σ-algebras $\mathscr{C}_\gamma$ are separable and the measures μ_γ are compact. It is easily seen that the system $(X_\gamma, \mathscr{C}_\gamma, \mu_\gamma, f_{\alpha\beta}, \Gamma)$ is s.a.m. and so — in virtue of Theorem 5.3 — the limit measure space $(X_\Gamma, \mathscr{C}_\Gamma, \mu_\Gamma)$ exists and is perfect. Moreover, $(X_\Gamma, \mathscr{C}_\Gamma, \mu_\Gamma)$ is σ-isomorphic to a measure space $(Z, \mathscr{D}^\Gamma \cap Z, \nu^\Gamma|Z)$ where $Z = Y^\Gamma \setminus \bigcup_{\lambda<\omega_1} M_{\omega_1}^\lambda$ and the σ-isomorphism is induced by a one-to-one map.

Since $\nu^\Gamma|Z$ is not compact (Musiał [16]), it follows that μ_Γ is not compact either.

6. Systems of quasi-countably generated measurable spaces. In this section we consider the problem of the existence of a projective limit measure in the case of perfect measures defined on quasi-countably generated measurable spaces. Using the results of § 5, we obtain a necessary and sufficient condition for the existence of perfect limits of all the systems of measures defined on an established system of quasi-countably generated measurable spaces and ordered by a set with a countable cofinal subset, and a sufficient condition in the case of an arbitrary ordering set.

The results obtained by Parthasarathy in [19] in the case of standard spaces are particular cases of our Corollaries 6.1 and 6.2. However, the proofs given here are quite different from that contained in [19].

Let $(X_i, \mathscr{C}_i, f_{ij}, I)$ be a system of quasi-countably generated measurable spaces $(X_i, \mathscr{C}_i)$, $i \in I$. We write $\overline{X}_i$ for the set of all $\mathscr{C}_i$-atoms, and define $h_i\colon X_i \to \overline{X}_i$ as the map $x \to e_x^i$, where e_x^i is the $\mathscr{C}_i$-atom containing $x \in X_i$.

We define $\overline{\mathscr{C}}_i$ as the collection of all the subsets $A \subset \overline{X}_i$ such that $h_i^{-1}(A) \in \mathscr{C}_i$. Clearly, $\overline{\mathscr{C}}_i$ are quasi-separable and the σ-algebras $\mathscr{C}_i$ and $\overline{\mathscr{C}}_i$ are σ-isomorphic.

$(\overline{X}_i, \overline{\mathscr{C}}_i)$ is called the *measurable space associated with* $(X_i, \mathscr{C}_i)$. The map $A \to h_i^{-1}(A)$ is a σ-isomorphism between $\overline{\mathscr{C}}_i$ and $\mathscr{C}_i$.

Define the map $\overline{f}_{ij}\colon \overline{X}_j \to \overline{X}_i$ by the formula

$$\overline{f}_{ij}(e_x^j) = e_{f_{ij}(x)}^i, \quad \text{if} \quad i \leqslant j.$$

It is easy to see that the definition of $\bar{f}_{ij}$ is correct and the equality

$$h_i \circ f_{ij} = \bar{f}_{ij} \circ h_j$$

holds.

We write $(\bar{X}_I, \bar{\mathscr{C}}_I)$ for the space $\varprojlim(\bar{X}_i, \bar{\mathscr{C}}_i, \bar{f}_{ij}, I)$ with maps $\bar{f}_i\colon \bar{X}_I \to \bar{X}_i$ given by $\bar{f}_i(\{\bar{x}_j\}_{j\in I}) = \bar{x}_i$.

$(\bar{X}_i, \bar{\mathscr{C}}_i, \bar{f}_{ij}, I)$ is said to be *associated with* $(X_i, \mathscr{C}_i, f_{ij}, I)$. $(\prod_{i\in I} \bar{X}_i, \sigma(\prod_{i\in I} \bar{\mathscr{C}}_i))$ is denoted by $(\bar{X}^I, \bar{\mathscr{C}}^I)$.

Assume further that (perfect) measures μ_i are defined on $(X_i, \mathscr{C}_i)$, $i \in I$, in such a way that the family $(X_i, \mathscr{C}_i, \mu_i)_{i\in I}$ is a system with respect to the maps f_{ij}, $i \leqslant j$.

Using the σ-isomorphism h_i, we can define on $(\bar{X}_i, \bar{\mathscr{C}}_i)$ a (perfect) measure $\bar{\mu}_i$ by the formula

$$\bar{\mu}_i(A) = \mu_i[h_i^{-1}(A)] \quad \text{for} \quad A \in \bar{\mathscr{C}}_i.$$

From the equality $h_i \circ f_{ij} = \bar{f}_{ij} \circ h_j$ it follows that $(\bar{X}_i, \bar{\mathscr{C}}_i, \bar{\mu}_i, \bar{f}_{ij}, I)$ is a measure system with quasi-separable spaces $(\bar{X}_i, \bar{\mathscr{C}}_i)$, $i \in I$.

As above, we say that the measures $\bar{\mu}_i$ and the system $(\bar{X}_i, \bar{\mathscr{C}}_i, \bar{\mu}_i, \bar{f}_{ij}, I)$ are associated with μ_i and $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$, respectively.

DEFINITION 6.1. A system $(X_i, \mathscr{C}_i, f_{ij}, I)$ of quasi-countably generated spaces is *atomically maximal* (*a.m.*) if for each descending family $\{e_i\}_{i\in I}$, where e_i is a $\mathscr{C}_i$-atom for every $i \in I$, there exists an $x \in X_I$ such that $f_i(x) \in e_i$, for every $i \in I$.

The following two results present some basic relations between $(X_I, \mathscr{C}_I)$ and $(\bar{X}_I, \bar{\mathscr{C}}_I)$, and between $(X_I, \mathscr{C}_I, \mu_I)$ and $(\bar{X}_I, \bar{\mathscr{C}}_I, \bar{\mu}_I)$, where $\bar{\mu}_I = \varprojlim \bar{\mu}_i$.

PROPOSITION 6.1. *If $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a system of quasi-countably generated spaces, then there exists a map $h\colon X_I \to \bar{X}_I$ such that $h^{-1}\colon A \to h^{-1}(A)$ is a σ-homomorphism of $\bar{\mathscr{C}}_I$ onto $\mathscr{C}_I$ and for each $i \in I$ the equality $h_i \circ f_i = \bar{f}_i \circ h$ holds. If the system is a.m., then h^{-1} is a $\bar{\sigma}$-isomorphism of $\bar{\mathscr{C}}_I$ onto $\mathscr{C}_I$.*

Proof. We define $h\colon X_I \to \bar{X}_I$ by putting $h(\{x_i\}_{i\in I}) = \{h_i(x_i)\}_{i\in I}$.

THEOREM 6.1. *Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a measure system. Then:*

(i) *If the system is convergent, then the associated system is also convergent and*

$$(*) \qquad \bar{\mu}_I(A) = \mu_I[h^{-1}(A)]$$

for all $A \in \bar{\mathscr{C}}_I$. The perfectness of μ_I yields the perfectness of $\bar{\mu}_I$ and $\bar{\mu}_I | h(X_I)$.

(ii) *If the system $(X_i, \mathscr{C}_i, f_{ij}, I)$ is an a.m. system of quasi-countably generated spaces, then the convergence of $(\bar{X}_i, \bar{\mathscr{C}}_i, \bar{\mu}_i, \bar{f}_{ij}, I)$ yields the convergence of $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$, the spaces $(X_I, \mathscr{C}_I, \mu_I)$ and $(\bar{X}_I, \bar{\mathscr{C}}_I, \bar{\mu}_I)$ are σ-isomorphic and $(*)$ holds. If $\bar{\mu}_I$ is perfect, then μ_I is perfect as well.*

Proof. The assertions follows easily from Proposition 6.1, Theorem 3.1 and Theorem 3.2.

The following theorem is, in a sense, the main result of this section:

THEOREM 6.2. *Let $(X_i, \mathscr{C}_i, f_{ij}, I)$ be a system of quasi-countably generated spaces. If I contains a countable cofinal subset, then:*

(i) *If each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is convergent, then $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a.m.;*

(ii) *If $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a.m., then each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is convergent to a perfect measure space.*

Proof. (i) Suppose that there exists a descending family of $\mathscr{C}_i$-atoms e_i, with the empty intersection (i.e. $\bigcap_{i\in I} f^{-1}(e_i) = \varnothing$) and each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is convergent.

We define measures μ_i as follows: μ_i is defined on $\mathscr{C}_i$ and $\mu_i(C) = 1$ or 0, according as $e_i \subset C$ or $C \cap e_i = \varnothing$. The inclusion $e_j \subset f_{ij}^{-1}(e_i)$ implies that $\mu_j f_{ij}^{-1}(C) = \mu_i(C)$ if $C \in \mathscr{C}_i$, and so $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a convergent system of perfect measure spaces.

On the other hand, we have $\mu_I[f_{i_n}^{-1}(e_{i_n})] = 1$ for all n, while $\bigcap_{n=1}^{\infty} f_{i_n}^{-1}(e_{i_n}) = \varnothing$ and $f_{i_1}^{-1}(e_{i_1}) \supset f_{i_2}^{-1}(e_{i_2}) \supset \ldots$ (the sequence $\{i_n\}$ is chosen to be cofinal with I), and this contradicts the countable additivity of μ_I.

(ii) Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a system of perfect measure spaces where $(X_i, \mathscr{C}_i, f_{ij}, I)$ is a.m.

In virtue of Theorem 5.4 the associated system is convergent to a perfect measure space, and so the assertion is a consequence of Theorem 6.1.

As a special case we obtain a generalization of Theorem 4.1 from [19].

COROLLARY 6.1. *Let $(X, \mathscr{C})$ be a countably generated measurable space and let $\mathscr{C}_n$, $n = 1, \ldots$ be countably generated σ-algebras such that*

$$\mathscr{C}_1 \subset \mathscr{C}_2 \subset \ldots \quad \textit{and} \quad \mathscr{C} = \sigma(\bigcup_n \mathscr{C}_n).$$

If for each consistent sequence $\{\mu_n\}$ of perfect measures $\mu_n\colon \mathscr{C}_n \to [0, \infty]$, $n = 1, 2, \ldots$ (i.e. $\mu_{n+1}|\mathscr{C}_n = \mu_n$) there exists a measure on $\mathscr{C}$ which is their common extension, then each decreasing sequence of $\mathscr{C}_n$-atoms e_n has a non-empty intersection $\bigcap_{n=1}^{\infty} e_n$. And conversely, if each decreasing sequence of $\mathscr{C}_n$-atoms e_n has a non-empty intersection, then each consistent sequence of perfect measures μ_n on $\mathscr{C}_n$, $n = 1, \ldots$ has a common perfect extension on the whole of $\mathscr{C}$.

As is shown by the examples given below, in general there is no correlation between the atomic maximality condition and the atomic sequential maximality condition of a measurable system.

EXAMPLE 6.1. Let $I = X$ be the set of all the ordinals less than ω_1. Put $X_\gamma = X$ for every $\gamma < \omega_1$ and let $\mathscr{C}_\gamma$ be the smallest σ-algebra on X_γ containing all the ordinals less than γ. If $f_{\alpha\beta}\colon X_\beta \to X_\alpha$, $\alpha \leqslant \beta$, is the identity map $x \to x$, then $(X_\gamma, \mathscr{C}_\gamma, f_{\alpha\beta}, I)$ is an a.s.m. system, but it is not a.m.

EXAMPLE 6.2. Let I be the collection of all at most countable subsets of the interval $(0, 1]$ ordered by inclusion. Setting $X_i = [0, 1] \setminus i$, $\mathscr{C}_i = \mathscr{B}_{X_i}$ and $f_{ij}\colon X_j \to X_i$ to be the identity injection, we get a system which is a.m. but not a.s.m. (see Example 4.1).

As can be observed, this example shows that the second part of Theorem 6.2 does not hold for an arbitrary directed set I. However, we can prove the following:

THEOREM 6.3. *If $(X_i, \mathscr{C}_i, f_{ij}, I)$ is an a.s.m. system of quasi-countably generated spaces, then each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is convergent to a perfect measure space.*

Proof. If follows that for each $J \in \Xi$ the system $(X_i, \mathscr{C}_i, f_{ij}, J)$ is a.m., and hence, in virtue of Theorem 6.2 (ii), each system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ of perfect measure spaces is sequentially convergent and has perfect limits.

Thus, the assertion is a direct consequence of Theorem 4.5.

COROLLARY 6.2. *Let $(X, \mathscr{C})$ be a measurable space, I — a directed set and $\{\mathscr{C}_i\}_{i \in I}$ — a collection of countably generated σ-algebras $\mathscr{C}_i \subset \mathscr{C}$ with the following properties:*

(i) *if $i \leqslant j$, then $\mathscr{C}_i \subset \mathscr{C}_j$;*

(ii) $\mathscr{C} = \sigma(\bigcup_{i \in I} \mathscr{C}_i)$;

(iii) *for each sequence $i_1 < i_2 < \ldots$ and each decreasing sequence of $\mathscr{C}_{i_n}$-atoms e_n, we have $\bigcap_{n=1}^{\infty} e_n \neq \emptyset$.*

Then, for each consistent family of perfect measures μ_i on $\mathscr{C}_i$, $i \in I$, there exists a perfect measure μ on $\mathscr{C}$ which is their common extension.

Remark 6.2. In virtue of Theorem 6.3 if $(X_\gamma, \mathscr{C}_\gamma, f_{\alpha\beta}, I)$ is the system considered in Example 6.1, then each system $(X_\gamma, \mathscr{C}_\gamma, \mu_\gamma, f_{\alpha\beta}, I)$ has a limit. Since $(X_\gamma, \mathscr{C}_\gamma, f_{\alpha\beta}, I)$ is not a.m., it follows that the first part of Theorem 6.2 does not hold in the case of an arbitrary I.

Remark 6.3. In the case of systems of quasi-separable measurable spaces the sequential maximality condition and the atomic sequential maximality condition are equivalent. Thus, Example 5.2 shows that the converse to Theorem 6.3 does not hold, even in the case of systems possessing a countable cofinal subset.

7. Systems of countably generated measure spaces. In this section, we give a necessary and sufficient condition for the existence and perfectness of a projective limit of an arbitrary system of countably generated measure spaces.

THEOREM 7.1. *If $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a system of perfect countably generated measure spaces, then a necessary and sufficient condition for the existence and perfectness of $(X_I, \mathscr{C}_I, \mu_I)$ is the atomic sequential almost maximality of the system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$.*

Proof. In virtue of Theorem 4.4 it is sufficient to prove the assertion in the

case of $I = N$. Without loss of generality we may assume that the spaces $(X_n, \mathscr{C}_n)$, $n = 1, 2, \ldots$ are quasi-countably generated.

Necessity. Assume that the system is convergent to a perfect measure space $(X_N, \mathscr{C}_N, \mu_N)$.

In virtue of Theorem 6.1 (i) the associated system has the limit $(\overline{X}_N, \overline{\mathscr{C}}_N, \overline{\mu}_N)$ and this measure space is perfect.

Moreover, $\overline{\mu}_N | h(X_N)$ is perfect as well. In view of Theorem 3.3 there exists a set $D \in \overline{\mathscr{C}}_N$ such that

$$D \subset h(X_N) \quad \text{and} \quad \overline{\mu}_N(\overline{X}_N \setminus D) = 0 .$$

If $\varepsilon > 0$ is arbitrary, then from the elementary properties of outer measures follows the existence of a descending sequence $\{A_n\}$ of sets $A_n \in \mathscr{C}_n$ such that

$$\overline{X}_N \setminus D \subset \bigcup_{n=1}^{\infty} \overline{f}_n^{-1}(\overline{X}_n \setminus \overline{A}_n) \in \overline{\mathscr{C}}_N$$

and

$$\overline{\mu}_N \overline{f}_n^{-1}(\overline{X}_n \setminus \overline{A}_n) < \varepsilon , \quad n = 1, 2, \ldots ,$$

where $\overline{A}_n = h_n(A_n)$.

It is easy to see that the sequence $\{A_n\}$ satisfies the conditions formulated in Definition 4.7 and so the system $(X_n, \mathscr{C}_n, \mu_n, f_{n,n+1}, N)$ is a.s.a.m.

Sufficiency. Assume that the system is a.s.a.m. and take for every $k = 1, 2, \ldots$ a descending sequence of sets $A_n^k \in \mathscr{C}_n$ with the following properties:

(a) $\mu_n(X_n \setminus A_n^k) < 1/k$, $n = 1, 2, \ldots$

(b) if $e_n \subset A_n^k$ is a $\mathscr{C}_n$-atom and, $e_{n+1} \subset f_{n,n+1}^{-1}(e_n)$, then $\bigcap_{n=1}^{\infty} f_n^{-1}(e_n) \neq \emptyset$.

Thus we have

$$\bigcap_{n=1}^{\infty} \overline{f}_n^{-1}(\overline{A}_n^k) \subset h(X_N)$$

and hence

$$\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{f}_n^{-1}(\overline{A}_n^k) \subset h(X_N) .$$

Since

$$\bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{f}_n^{-1}(\overline{A}_n^k) \in \overline{\mathscr{C}}_N$$

and

$$\overline{\mu}_N[\overline{X}_N \setminus \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \overline{f}_n^{-1}(\overline{A}_n^k)] = 0$$

(the existence and perfectness of $\overline{\mu}_N$ follow from Theorem 5.2 since $\overline{\mathscr{C}}_n$, $n = 1, 2, \ldots$ are quasi-separable), the restriction of $\overline{\mu}_N$ to $(h(X_N), h(X_N) \cap \overline{\mathscr{C}}_N)$ — say $\hat{\mu}$ — is a perfect measure.

Since the spaces $(X_N, \mathscr{C}_N)$ and $(h(X_N), \overline{\mathscr{C}}_N \cap h(X_N))$ are σ-isomorphic, it follows from Theorem 3.2 that a measure $\mu\colon \mathscr{C}_N \to [0, 1]$, given by

$$\mu h^{-1}(A) = \hat{\mu}(A)$$

whenever $A \in \overline{\mathscr{C}}_N \cap h(X_N)$, is perfect. It is easily seen that $\mu = \mu_N$. This completes the proof of the theorem.

Observe that in general it is impossible to find the sets A_n such that $\mu_n(A_n) = 0$ for all n.

EXAMPLE 7.1. Put $X_n = (0, 1]$ and let $\mathscr{C}_n$ be the algebra generated by the intervals

$$\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right], \quad k = 0, 1, \ldots, 2^n - 1 .$$

Put

$$\mu_n\left\{\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right\} = \frac{1}{2^n}, \quad \text{if} \quad k = 0, 1, \ldots, 2^n - 1$$

and

$$f_{n,n+1}(x) = x, \quad x \in (0, 1] .$$

It is easily seen that $(X_N, \mathscr{C}_N, \mu_N) = ((0, 1], \mathscr{B}_{(0,1]}, \lambda)$.

In view of Theorem 7.1 the system is a.s.a.m. Suppose there exist sets $A_n \in \mathscr{C}_n$, $n = 1, 2, \ldots$ satisfying the conditions of Definition 4.7 and such that $\mu_n(A_n) = 0$ for all n. Then, since each measure μ_n is purely atomic, we have $A_n = X_n$ for every n. But for $e_n = (0, 1/2^n]$, $n = 1, 2, \ldots$ we have $\bigcap_{n=1}^{\infty} e_n = \emptyset$, and this contradicts the atomic sequential almost maximality of the measure system.

Let us also observe that the above example shows that Theorem 7.1 cannot be deduced from the theorems proved by Choksi [3] and Motivier [13].

Indeed, if φ_n is a compact system approximating $\mathscr{C}_n$ with respect to μ_n, then we have $\mathscr{C}_n \subset \varphi_n$ (because μ_n is purely atomic and positive except for the empty set).

It is clear that a condition $f_{n,n+1}(\varphi_{n+1}) \subset \varphi_n$ (Choksi [3], Theorem 3.3) does not hold.

In a similar way we can show that the compactness of a family

$$\mathscr{K}_n = \bigcup_{m \geqslant n} f_{n,m}(\varphi_m)$$

(Metivier [13], Theorem 2.3) cannot hold.

As a corollary we get the following result:

COROLLARY 7.1. *A measure space $(X, \mathscr{C}, \mu)$ is perfect if and only if there exists a family $\mathscr{C}_i$, $i \in I$ of finite subalgebras of $\mathscr{C}$, directed by inclusion and such that $(X, \mathscr{C}_i, \mu|\mathscr{C}_i, f_{ij}, I)$ forms an a.s.a.m. system with f_{ij} being the identity mappings.*

8. Systems of arbitrary perfect measure spaces. As a direct consequence of Theorem 7.1 we get the following necessary and sufficient condition for the existence and perfectness of a projective limit of an arbitrary system of measure spaces.

THEOREM 8.1. *If $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ is a system of perfect measure spaces then a necessary and sufficient condition for the existence and perfectness of $(X_I, \mathscr{C}_I, \mu_I)$ is the atomic sequential almost maximality of each system $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$ of countably generated measure spaces with $\mathscr{D}_i \subset \mathscr{C}_i$, $i \in I$.*

As an example of the application of the above theorem we give a proof of the following result of Topsøe:

THEOREM 8.2 (Topsøe [25], Theorem 3). *Let $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$ be a sequentially almost maximal measure system and let for each $i \in I$ $\mathscr{K}_i$ be a family of subsets of X_i approximating μ_i. Assume moreover that $f_{ij}(\mathscr{K}_j) \subset \mathscr{K}_i$ for every $i < j$ and that one of the following conditions is satisfied:*

(1) For every $i \in I$ and every chain of non-empty sets in $\mathscr{K}_i$, the interesection of the sets in the chain is a non-empty member of $\mathscr{K}_i$ (a family $\mathscr{F}$ is a chain if for all $A, B \in \mathscr{F}$ either $A \subset B$ or $B \subset A$);

(2) For each $i \in I$, $x \in X_i$, $j \geqslant i$ and each decreasing sequence $\{K_n\}_{n=1}^{\infty}$ of sets in $\mathscr{K}_j$ with $\bigcap_{n=1}^{\infty} K_n \cap f_{ij}^{-1})x) = \emptyset$ there exists an n such that $K_n \cap f_{ij}^{-1}(x) = \emptyset$.

Then $\varprojlim \mu_i$ exists and is perfect.

Proof. In virtue of Theorem 4.3 it is sufficient to prove the sequential convergence of the system $(X_i, \mathscr{C}_i, \mu_i, f_{ij}, I)$.

So take a sequence $J = \{i_n\}_{n=1}^{\infty} \in \Xi$. We shall prove the atomic sequential almost maximality of each system $(X_{i_n}, \mathscr{D}_{i_n}, \mu_{i_n}|\mathscr{D}_{i_n}, f_{i_n i_{n+1}}, N)$ with countably generated $\mathscr{D}_{i_n} \subset \mathscr{C}_{i_n}$. In virtue of Theorem 8.1 this will guarantee the convergence of $(X_{i_n}, \mathscr{C}_{i_n}, \mu_{i_n}, f_{i_n i_{n+1}}, N)$ to a perfect measure space.

In order to simplify the notations we shall write n instead of i_n. In virtue of a result of Vinokurov (cf. Musiał [16], Lemma) there exists a sequence of countably generated σ-algebras, $\mathscr{F}_n \subset \mathscr{C}_n$, such that $\mathscr{D}_n \subset \mathscr{F}_n$, $(X_n, \mathscr{F}_n, \mu_n|\mathscr{F}_n, f_{n,n+1}, N)$ is a measure system, and almost all $\mathscr{F}_n$-atoms belong to $\mathscr{K}_n$.

Take $B_n \in \mathscr{F}_n$ such that $\mu_n(B_n) = 0$ and all $\mathscr{F}_n$-atoms contained in $X_n \setminus B_n$ are in $\mathscr{K}_n$. Without loss of generality we may assume that $\{B_n\}_{n=1}^{\infty}$ is an ascending sequence. Then let $\{e_n\}_{n=1}^{\infty}$ be a descending sequence of $\mathscr{F}_n$-atoms $e_n \subset X_n \setminus B_n$.

Put now for each n $e_n^1 = e_n$ and suppose that for a non-limit ordinal γ sets, e_n^α, $\alpha \leqslant \gamma$, $n \in N$ are already constructed. Then we put for each n

$$e_n^{\gamma+1} = \bigcap_{m \geqslant n} f_{nm}(e_m^\gamma) .$$

If y is a limit ordinal then we put

$$e_n^\gamma = \bigcap_{\alpha < \gamma} e_n^\alpha \quad \text{for each} \quad n \in N .$$

It follows from (1) that if $E_n = \bigcap_{\gamma} e_n^\gamma$, then $E_n \neq \emptyset$ and $f_{n,n+1}(E_{n+1}) = E_n$, $n \in N$.

This shows that there exists $x \in X_J$ such that $f_{nJ}(x) \in e_n$ for every $n \in N$. Taking this result into account and applying Theorem 7.1 to the system $(X_i, \mathscr{D}_i, \mu_i|\mathscr{D}_i, f_{ij}, I)$ we easily get the a.s.a.m. of this system.

In a similar way the second part of the theorem can be proved.

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