On saturated sets of Boolean rings and Ulam's problem on sets of measures

by

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Abstract. Our Theorem 5 implies the following corollary: Let F be a family of σ-fields on the real line R such that for every $A \in F$ all one-element subsets of R belong to A and $A \neq \mathfrak{P}(R)$. Then any of the conditions (i) $|F| < \omega$, (ii) $|F| < 2^{\omega}$ and $2^{\omega} = \omega_1$, (iii) $|F| < 2^{2^{\omega}}$ and Gödel's axiom of constructibility, implies $\bigcup F \neq \mathfrak{P}(R)$. Theorem 5 is a generalization of some results of Ulam, Alaoglu–Erdös, Jensen, Prikry, and Taylor connected with Ulam's problem on sets of measures.

A set F of Boolean subrings of a Boolean ring A is called *weakly ν-saturated* iff there is no set $C \subset A - \bigcup F$ such that $|C| = \nu$ and elements of C are pairwise disjoint. We prove the following lemma if $|F| \leqslant \nu \geqslant \omega$ and A is $(|F|^+ + \omega)$–complete, then F is weakly ν-saturated iff the set $\{I(B)\colon B \in F\}$ of ideals in A, where

$$I(B) = \{b \in B\colon \forall (a \in A)(a \subset b \to a \in B)\},$$

is weakly ν-saturated. The lemma (and its generalizations) applied to some results of Ulam, Alaoglu–Erdös, Prikry, and Taylor (see [2], [5], [7] and [4]) connected with Ulam's problem one sets of measures (see problem 81 of [3], or [9], or else the end of the present paper) lead to a generalization and strengthening of those results. The results of this paper were presented at the 6th Winter School on Abstract Analysis at Spindlerův Mlýn in Czechoslovakia in February 1978.

1. Notation and terminology. Throughout this paper, small Greak letters denote ordinals, $\varkappa$, μ always denoting infinite cardinals and λ, ν, δ any (finite or infinite) cardinals. By λ^+ we denote the cardinal successor of λ. $|S|$ denotes the cardinality of the set S and $\mathcal{P}(S)$ the power set of S, and we write

$$[S]^{\delta} = \{X \subset S\colon |X| = \delta\}, \quad [S]^{\geqslant \delta} = \{X \subset S\colon |X| \geqslant \delta\}, \quad \text{and} \quad [S]^{<\delta} = \{X \subset S\colon |X| < \delta\}.$$

For Boolean rings we will use the same terminology and notation as for rings of sets. The least element of A is denoted by 0. A Boolean ring B will be called *μ-complete* iff for every $X \subset B$ with $|X| < \mu$ the Boolean join of X exists (in our notation $\bigcup X$ exists). In this paper A will always denote a fixed Boolean ring.

If B is a Boolean subring of A (shortly, a subring of A), then we put

$$I(B) = \{b \in B:\ \forall (a \in A)(a \subset b \to a \in B)\}, \quad B^+ = A - B,$$

and

$$B(a) = \{c \in A:\ a \cap c \in B\} \quad \text{for every } a \in A.$$

Note that $I(B)$ is an ideal in A, $B(a)$ is a subring of A, and if, in addition, B is an ideal in A, then $B(a)$ is an ideal.

F will always denote a non-empty set of subrings of A.

For every F we put

$$I(F) = \{I(B):\ B \in F\}, \quad F^+ = A - \bigcup F, \quad \text{and} \quad F(a) = \{B(a):\ B \in F\}$$

for every $a \in A$.

Instead of $(I(B))^+$ and $(I(F))^+$ we will write $I^+(B)$ and $I^+(F)$, respectively. Since $I(F(a)) = (I(F))(a)$, we can write $I(F)(a)$.

For the case where F is a family of ideals in $A = \mathscr{P}(\varkappa)$ the above definitions were introduced by Taylor in [7].

If I is an ideal in A contained in $\bigcap I(F)$, then F will be called *ν-saturated with respect to I* (shortly, *ν-saturated w.r.t. I*) iff for every collection $\{x_\alpha:\ \alpha < \nu\} \subset F^+$ there exists a set $\{\alpha, \beta\} \in [\nu]^2$ such that $x_\alpha \cap x_\beta \notin I$.

We will say that F is *ν-saturated* (*weakly ν-saturated*) iff F is ν-saturated w.r.t. $I = \bigcap I(F)$ ($I = \{0\}$ respectively).

For the case where F consists only of ideals in $A = \mathscr{P}(\varkappa)$, the definition of ν-saturatedness of F was given by Taylor in [7].

It is easy to see that if I is $(\nu+\omega)$-complete, F is ν-saturated w.r.t. I iff F is weakly ν-saturated.

Other definitions will be given at the beginning of Section 3.

2. Lemmas on saturated sets of Boolean rings. Let F be a set of subrings of a Boolean ring A and let I be an ideal in A such that $I \subset \bigcap I(F)$. It is evident that if F is not ν-saturated w.r.t. I, then $I(F)$ is not ν-saturated w.r.t. I either. Observe that if $|F| = 1$, then F is not ν-saturated w.r.t. I iff $I(F)$ is not ν-saturated w.r.t. I. Our Lemmas 1 (iii) 1 (iv), and 2 generalize this observation to the case $|F| \leqslant \nu$ (mainly when $\nu \geqslant \omega$). In Section 3 we will nead only Lemma 2(ii).

LEMMA 1. *Let A be a $(\lambda^+ + \omega)$-complete Boolean ring, and let F be a set of subrings of A such that $|F| \leqslant \lambda$. Then:*

(i) *If $Q \subset I^+(F)$ satisfies $|Q| = \lambda \geqslant \omega$ and elements of Q are pairwise disjoint, then there exists an $a \in F^+$ such that $a \subset \bigcup Q$.*

(ii) *Assume $\lambda < \omega$ and let $Q \subset I^+(F)$ be such that $|Q| \geqslant \frac{1}{2}(\lambda^2 + 3\lambda - 2)$ and elements of Q are pairwise disjoint. Then there exist a $Q' \subset Q$ with $|Q'| = \lambda$ and an $a \in F^+$ with $a \subset \bigcup Q'$.*

(iii) *Assume $\omega \leqslant \nu \geqslant \lambda$. If $I(F)$ is not weakly ν-saturated, then F is not weakly ν-saturated.*

(iv) *Assume $\lambda < \omega$ and $\delta < \omega$. If $I(F)$ is not $(\frac{1}{2}(\lambda^2 + 3\lambda - 2) + \delta \cdot \lambda)$-saturated, then F is not $(\delta \cdot \lambda + 1)$-saturated.*

Proof of (i). Let $\{Q_\alpha:\ \alpha < \lambda\}$ be a pairwise disjoint family contained in $[Q]^\lambda$ and let $\{B_\alpha:\ \alpha < \lambda\}$ be an enumeration of F (we repeat elements of F if necessary). By hypothesis we have in particular $Q_\alpha \subset I^+(B_\alpha)$ for every $\alpha < \lambda$.

For every $q \in Q_\alpha$ $(\alpha < \lambda)$ choose one element $r \subset q$, $r \in B_\alpha^+$ and denote by R_α the family formed by the chosen elements r.

It is easy to see that in order to finish the proof of (i) it is enough to define a sequence $\{s_\alpha:\ \alpha < \lambda\}$ such that $s_\alpha \in R_\alpha$ for every $\alpha < \lambda$ and $\bigcup \{s_\alpha:\ \alpha < \lambda\} \in F^+$. It is also easy to see that, for every $\alpha < \lambda$, $|R_\alpha| = \lambda$, $R_\alpha \subset B_\alpha^+$ and the elements of R_α are pairwise disjoint;

$$\bigcup R_\alpha \cap \bigcup R_\beta = 0 \quad \text{for every } \alpha, \beta < \lambda \text{ with } \alpha \neq \beta.$$

Now we define auxiliary partial functions f and g.

For every $\alpha < \lambda$, every $b \in A$ such that $b \subset \bigcup \bigcup \{R_\beta:\ \beta < \alpha\}$, and every $S \in [R_\alpha]^{\geqslant 2}$, choose one $b' \in S$ such that $b \cup b' \in B_\alpha^+$ and denote this b' by $f(\alpha, b, S)$.

Observe that $f(\alpha, b, S)$ exists. Indeed, otherwise there would exist $\alpha < \lambda$ and $b \in A$ such that $b \subset \bigcup \bigcup \{R_\beta:\ \beta < \alpha\}$, and there would exist $b_1, b_2 \in R_\alpha$ with $b_1 \neq b_2$, $b \cup b_1 \in B_\alpha$ and $b \cup b_2 \in B_\alpha$. Since b_1, b_2, b are pairwise disjoint, we would have $b_1 = (b \cup b_1) - (b \cup b_2)$ and hence $b_1 \in B_\alpha$, which contradicts $b_1 \in R_\alpha \subset B_\alpha^+$.

Fix $\alpha < \lambda$ and $b \in A$. If there exists a sequence $\{r_\beta:\ \alpha < \beta < \lambda\}$ such that

$$(*) \qquad r_\beta \in R_\beta \quad \text{for every } \alpha < \beta < \lambda \quad \text{and} \quad b \cup \bigcup \{r_\beta:\ \alpha < \beta < \lambda\} \in B_\alpha,$$

then choose such a sequence and denote it by $\{r_\beta(\alpha, b):\ \alpha < \beta < \lambda\}$ (we put, by definition, $r_\beta(\alpha, b) = r_\beta$ for every $\alpha < \beta < \lambda$). Now for every (α, b, β) such that $\alpha < \lambda$, $b \in A$ and $\alpha < \beta < \lambda$ we define $g(\alpha, b, \beta)$ as either a one-element subset of R_β or the empty set as follows. Let (α, b, β) be as above. If, for α, b there exists a sequence $\{r_\beta:\ \alpha < \beta < \lambda\}$ with the property $(*)$, then we put $g(\alpha, b, \beta) = \{r_\beta(\alpha, b)\}$, $r_\beta(\alpha, b)$ being defined previously in $(*)$. If for α, b there is no sequence with property $(*)$, then we put $g(\alpha, b, \beta) = \emptyset$. Now we define by transfinite induction a sequence $\{s_\alpha:\ \alpha < \lambda\}$ such that, for every $\alpha < \lambda$, $s_\alpha \in R_\alpha$ and

$$(**) \qquad s_\alpha = f\big(\alpha, \bigcup \{s_\beta:\ \beta < \alpha\}, R_\alpha - \bigcup \{g(\beta, \bigcup \{s_\gamma:\ \gamma \leqslant \beta\}, \alpha):\ \beta < \alpha\}\big).$$

Put $a = \bigcup \{s_\alpha:\ \alpha < \lambda\}$. It is clear that, in order to finish the proof of (i), it is enough to prove $a \in F^+$. Suppose $a \notin F^+$. Then there exists an $\alpha_0 < \lambda$ such that $a \in B_{\alpha_0}$. We have also $a = \bigcup \{s_\beta:\ \beta \leqslant \alpha_0\} \cup \bigcup \{s_\beta:\ \alpha_0 < \beta < \lambda\}$. Hence, by the definition of g applied for $b = \bigcup \{s_\beta:\ \beta \leqslant \alpha_0\}$ and $\alpha = \alpha_0$, we have $c \in B_{\alpha_0}$, where

$$c = \{s_\beta:\ \beta \leqslant \alpha_0\} \cup \bigcup \bigcup \{g(\alpha_0, \bigcup \{s_\gamma:\ \gamma \leqslant \alpha_0\}, \beta):\ \alpha_0 < \beta < \lambda\}.$$

By $(**)$ and by the definition of f, we have

$$s_\beta \in R_\beta - \bigcup \{g(\alpha, \bigcup \{s_\gamma:\ \gamma \leqslant \alpha\}, \beta):\ \alpha < \beta\} \quad \text{for every } \beta < \lambda.$$

Hence

$$s_\beta \notin g(\alpha_0, \bigcup \{s_\gamma:\ \gamma \leqslant \alpha_0\}, \beta) \quad \text{for every } \beta > \alpha_0.$$

Since also $g(\alpha_0, \bigcup\{s_\gamma: \gamma\leqslant\alpha_0, \beta)$ is a one-element subset of R_β for every $\beta>\alpha_0$ and elements of R_β are pairwise disjoint for every $\beta<\lambda$, we have $a\cap c=\bigcup\{s_\beta: \beta\leqslant\alpha_0\}$. Hence $\bigcup\{s_\beta: \beta\leqslant\alpha_0\}\in B_{\alpha_0}$, because $a, c\in B_{\alpha_0}$. But by (**), $\bigcup\{s_\beta: \beta\leqslant\alpha_0\}\in B^+_{\alpha_0}$, and hence we have a contradiction.

Proof of (ii). Let $Q_0, Q_1, \ldots, Q_{\lambda-1}$ be a pairwise disjoint family of subsets of Q such that $|Q_0|=1$, $|Q_1|=3$, $|Q_2|=4, \ldots, |Q_{\lambda-2}|=\lambda$ and $|Q_{\lambda-1}|=\lambda+1$ (by hypothesis such a family exists). The rest of the proof is similar to that of (i). We put $Q'=\{q\in Q: \exists(\alpha<\lambda)(s_\alpha\subset q)$.

Proof of (iii). Since $\nu\cdot\lambda=\nu$, (iii) easily follows from (i) and (ii).

Proof of (iv). (iv) easily follows from (ii).

LEMMA 2. *Let A be a $(\lambda^+ +\omega)$-complete Boolean ring, let F be a set of subrings of A such that $|F|\leqslant\lambda$ and let I be an ideal in A such that $I\subset\bigcap I(F)$. Then*:

(i) *Assume $\omega\leqslant\nu\geqslant\lambda$ and I λ^+-complete. If $I(F)$ is not ν-saturated w.r.t. I, then F is not ν-saturated w.r.t. I.*

(ii) *Assume $\omega\leqslant\nu\geqslant\lambda$, I $(\lambda+\omega)$-complete and for every $a\in I^+(F)$, $I(F)(a)$ not ν-saturated w.r.t. I. Then $F(a)$ is not ν-saturated w.r.t. I for any $a\in I^+(F)$.*

Proof of (i). Since $I(F)$ is not ν-saturated w.r.t. I, there exists a set $Q\subset I^+(F)$ such that $|Q|=\nu$ and $p\cap q\in I$ for every $p, q\in Q$ with $p\neq q$. Let $\{Q_\alpha: \alpha<\nu\}$ be a pairwise disjoint family contained in $\mathscr{P}(Q)$ such that $|Q_\alpha|=\lambda\cdot\omega$ for every $\alpha<\lambda$. By the $(\lambda+\omega)$-completeness of I and by $I\subset\bigcap I(F)$, for every $\alpha<\nu$ there exists a $Q'_\alpha\subset I^+(F)$ such that the elements of Q'_α are pairwise disjoint and for every $p\in Q'_\alpha$ there exists exactly one $q\in Q_\alpha$ such that $p\subset q$. Now we distinguish two cases: $\lambda<\omega$ and $\lambda\geqslant\omega$.

Case $\lambda<\omega$. By Lemma 1 (ii) for each $\alpha<\nu$ there exists a $Q''_\alpha\subset Q'_\alpha$ such that $|Q''_\alpha|<\omega$ and there exists an $a_\alpha\in F^+$ with $a_\alpha\subset\bigcup Q''_\alpha$. Consider the set $\{a_\alpha: \alpha<\nu\}$. Since each a_α is contained in a finite union of elements of Q_α we have $a_\alpha\cap a_\beta\in I$ for every $\alpha, \beta<\nu$ with $\alpha\neq\beta$. The existence of the set $\{a_\alpha: \alpha<\nu\}$ with the above properties proves that F is not ν-saturated w.r.t. I.

Case $\lambda\geqslant\omega$. Here we apply Lemma 1 (i) and the λ^+-completeness of I. The proof is similar to the proof of the case $\lambda<\omega$ and we omit the details.

Proof of (ii). Let $a\in I^+(F)$. By hypothesis there exists a family $Q\subset(I(F)(a))^+$ such that $|Q|=\nu$ and $p\cap q\in I$ for every $p, q\in Q$ with $p\neq q$. Choose one such family Q, which satisfies additionally $q\subset a$ for every $q\in Q$. We have in particular $Q\subset I^+(F)$. By hypothesis, for every $q\in Q$, $I(F)(q)$ is not ν-saturated w.r.t. I. Hence by Lemmas 1 (i) and 1 (ii) applied to $F(q)$ (recall that I is $(\lambda+\omega)$-complete) it is easy to see that for every $q\in Q$ there exists a $p\in(F(q))^+$. For every $q\in Q$ choose one such p and denote it by $p(q)$. Put $Q_1=\{q\cap p(q): q\in Q\}$. Evidently $q\cap p(q)\in(F(q))^+$ for every $q\in Q$. Hence, because $q\subset a$ for every $q\in Q$, we have $Q_1\subset(F(a))^+$. Evidently $(q\cap p(q))\cap(q'\cap p(q'))\in I$ for every $q, q'\in Q$ such that $q\neq q'$. The existence of the family Q_1 with the above properties proves that $F(a)$ is not ν-saturated.

Remark. It can easily be checked that in all the above definitions and lemmas instead of subrings of the Boolean ring A we can consider substructures of A closed only under subtraction and finite intersections. Lemmas 1 and 2 so generalized remain true with the same proofs.

3. Applications. In this section we apply our lemmas to some new results of Taylor's and an old one of Ulam's in order to obtain generalizations and strengthenings of them.

Consider $A=\mathscr{P}(\varkappa)$ as the complete Boolean algebra with the usual operations. Instead of saying that B is a subring of $\mathscr{P}(\varkappa)$ we will say that B is a ring on $\varkappa$. A subset B of $\mathscr{P}(\varkappa)$ will be called *non-trivial* iff $[\varkappa]^{<\omega}\subset B$ and $B\neq\mathscr{P}(\varkappa)$. The following two definitions are central for the considerations of this section.

If Q is a set of non-trivial rings on $\varkappa$, then the symbol

$$\text{“}\langle\varkappa: \lambda, \mu\rangle\xrightarrow{Q}\nu\text{”}$$

denotes the following assertion.

If $F\subset Q$, $|F|\leqslant\lambda$ and $I(B)$ is μ-complete for every $B\in F$, then F is not ν-saturated.

If Q is a set of non-trivial rings on $\varkappa$ and I is an ideal on $\varkappa$ (we do not exclude $I=\{\emptyset\}$), then the symbol

$$\text{“}\langle\varkappa: \lambda, \mu\rangle\xrightarrow{Q}\langle\nu, I\rangle\text{”}$$

denotes the following assertion.

If $F\subset Q$, $|F|\leqslant\lambda$, $I\subset\bigcap I(F)$ and $I(B)$ is μ-complete for every $B\in F$, then F is not ν-saturated w.r.t. I.

For the case where Q is a set of non-trivial ideals on $\varkappa$ the notation $\langle\varkappa: \lambda, \mu\rangle\xrightarrow{Q}\nu$ was introduced by Taylor in [7] (he required also $[\varkappa]^{<\varkappa}\subset B$ for every $B\in Q$ but for our purpose we do not need that). If Q is the set of all non-trivial ideals on $\varkappa$, then instead of $\langle\varkappa: \lambda, \mu\rangle\xrightarrow{Q}\nu$ and $\langle\varkappa: \lambda, \mu\rangle\xrightarrow{Q}\langle\nu, I\rangle$ we will write $\langle\varkappa: \lambda, \mu\rangle\to\nu$ and $\langle\varkappa: \lambda, \mu\rangle\to\langle\nu, I\rangle$, respectively (i.e. we suppress the superscript Q). For a fixed cardinal $\varkappa$ by R we denote the family of all non-trivial rings on $\varkappa$ (in paper [7] by Taylor, R has a different meaning).

We have the following theorem:

THEOREM 3. *Assume $\lambda\leqslant\nu\geqslant\omega$.*

a) *If I is a $(\lambda+\omega)$-complete ideal on $\varkappa$, then*

$$\langle\varkappa: \lambda, \mu\rangle\xrightarrow{R}\langle\nu, I\rangle \quad \textit{iff} \quad \langle\varkappa: \lambda, \mu\rangle\to\langle\nu, I\rangle.$$

a') *If $\lambda\leqslant\mu$, then*

$$\langle\varkappa: \lambda, \mu\rangle\xrightarrow{R}\nu \quad \textit{iff} \quad \langle\varkappa: \lambda, \mu\rangle\to\nu.$$

Proof of a). The "only if" part is trivial. We only prove "if". Let I be a $(\lambda+\omega)$-complete ideal on $\varkappa$ and let F be a family of non-trivial μ-complete rings on $\varkappa$ such that $I\subset\bigcap F$, $|F|\leqslant\lambda$. We have to prove that F is not ν-saturated w.r.t. I. For every

$a\in I^+(F)$, $I(F)(a)$ is a family of non-trivial μ-complete ideals on $\varkappa$, $I\subset\bigcap I(F)(a)$ and $|I(F)(a)|\leqslant\lambda$. By hypothesis $I(F)(a)$ is not ν-saturated w.r.t. I for any $a\in I^+(F)$. By Lemma 2 (ii), $F(a)$ is not ν-saturated w.r.t. I for any $a\in I^+(F)$ and hence in particular F is not ν-saturated w.r.t. I, since $\varkappa\in I^+(F)$ and $F(\varkappa)=F$.

Proof of a′). Since "only if" is trivial, we only prove "if". Let F be a family of non-trivial μ-complete rings on $\varkappa$ such that $|F|\leqslant\lambda$. We have to prove that F is not ν-saturated. By assumption, $I(F)(a)$ is not ν-saturated w.r.t. $\bigcap I(F)(a)$ for any $a\in I^+(F)$. Put $I=\bigcap I(F)$. Since $\bigcap I(F)(a)=I(a)$ for every $a\in I^+(F)$, it is easy to see that $I(F)(a)$ is not ν-saturated w.r.t. I for any $a\in I^+(F)$. Hence, by Lemma 2 (ii), $F(a)$ is not ν-saturated w.r.t. I for any $a\in I^+(F)$. Therefore F is not ν-saturated w.r.t. I, since $\varkappa\in I^+(F)$ and $F(\varkappa)=F$. So F is not ν-saturated. Note that the most interesting part of a′), when $\lambda\geqslant\omega$, also follows directly from a).

With the help of Theorem 3 we will generalize the following result of Taylor (Theorem 2.2. and Theorem 4.4 of [7]). (We formulate it in a little more general form, which easily follows from the original one.)

THEOREM 4 (Taylor). a) *Assume $\nu\geqslant\lambda^++\omega$, $\mu\geqslant\lambda^++\omega$, $\lambda<\varkappa$ and I is a $(\lambda^++\omega)$-complete ideal on $\varkappa$. Then we have*

$$\langle\varkappa:\lambda,\mu\rangle\to\langle\nu,I\rangle \quad \textit{iff} \quad \langle\varkappa:1,\mu\rangle\to\langle\nu,I\rangle.$$

b) $\langle\omega_1:\omega_1,\omega_1\rangle\to\omega_2$ *iff* $\langle\omega_1:1,\omega_1\rangle\to\langle\omega_2,[\omega_1]^{<\omega_1}\rangle$.

Instead of Theorem 4 a) Taylor has formulated only $(\varkappa:\lambda,\lambda^+)\to\lambda^+$ iff $(\varkappa:1,\lambda^+)\to\lambda^+$ for every $\lambda<\varkappa$ (in fact Taylor has assummed also $\lambda\geqslant\omega$ but his proof also works for $\lambda<\omega$). His proof gives even $(\varkappa:\lambda,\mu)\to\lambda^+$ iff $(\varkappa:1,\mu)\to\lambda^+$ for every $\lambda<\varkappa$ and $\mu\geqslant\lambda^++\omega$. Now we show how a) follows from the above mentioned result. Let $F=\{B_\alpha:\ \alpha<\lambda\}$ be a family of ideals on $\varkappa$ which we have to consider to prove "if" (the only non-trivial part of a)). By the above mentioned generalized version of Taylor's result there exists in particular a pairwise disjoint family $\{X_\alpha:\alpha<\lambda\}$ such that $X_\alpha\in B_\alpha^+$ for every $\alpha<\lambda$. Let $\{X_\alpha^\beta:\ \beta<\nu\}\subset\mathscr{P}(X_\alpha)$ be a family such that $X_\alpha^\beta\cap X_\alpha^{\beta'}\in I$ for every $\beta,\beta'<\nu$ with $\beta\neq\beta'$. Put $Y_\beta=\bigcup\{X_\alpha^\beta:\ \alpha<\nu\}$. The existence of the family $\{Y_\beta:\ \beta<\nu\}$, as can easily be seen, proves "if" part of a).

Recall that Theorem 4 of Taylor is a strengthening and a generalization of some results of Ulam, Alaoglu–Erdös (see [2]), Jensen (see [1]), Prikry (see [5]) and the present author (see [4]). An exhaustive explanation of the relations between the above mentioned results and Theorem 4 can be found in Taylor's paper [7].

By Theorem 4 and Theorem 3 we immediately obtain the following generalization and strengthening of Theorem 4.

THEOREM 5. a) *Assume that I is a $(\lambda^++\omega)$-complete ideal on $\varkappa$ and $\nu\geqslant\lambda^++\omega$, $\mu\geqslant\lambda^++\omega$, $\lambda<\varkappa$. Then*

$$\langle\varkappa:\lambda,\mu\rangle\overset{R}{\to}\langle\nu,I\rangle \quad \textit{iff} \quad \langle\varkappa:1,\mu\rangle\to\langle\nu,I\rangle.$$

b) $\langle\omega_1:\omega_1,\omega_1\rangle\overset{R}{\to}\omega_1$ *iff* $\langle\omega_1:1,\omega_1\rangle\to\langle\omega_2,[\omega_1]^{<\omega_1}\rangle$.

Remark that if in Theorem 5b) we replace R by $R_0\subset R$, where R_0 is a collection of all fields on $\varkappa$ satisfying a certain natural chain conditions, then the particular case of Theorem 5b) thus obtained becomes a well known result of Taylor, which follows directly from Theorem 4 (see Corollary 4.13 of Taylor's paper [7]).

If in Theorem 5a) we put $\lambda=\omega$ and replace R by $R_0\subset R$, where R_0 is the family of all σ-fields on $\varkappa$ on which it is possible to define non-trivial real-valued measures, then the particular case of Theorem 5a) thus obtained can be greatly strengthen (see Prikry's paper [5] and paper [4] by present author).

Remark. For a given $\varkappa$ define

$$R_1=\{A\subset P(\varkappa):\ A \text{ is non-trivial and } \forall(a\in A)\forall(b\in A)\ (a\cap b\in A \text{ and } a-b\in A\}.$$

Then $R\subset R_1$ and Theorems 3 and 5 remain true if we replace R by R_1 (cf. the Remark at the end of Section 2).

To see for which $\varkappa$, λ, μ and ν Theorem 5 works, recall the following well-known facts. Ulam in [8] has proved that

$$\langle\varkappa^+:1,\varkappa^+\rangle\to\varkappa^+ \quad \text{and} \quad \langle 2^\varkappa:1,\varkappa^+\rangle\to\omega$$

hold for every $\varkappa$. It is easy to see that the first of the above mentioned results of Ulam implies that $\langle\varkappa:1,\mu\rangle\to\mu$ holds for every $\varkappa$ which is less than the first weakly inaccessible cardinal and for every $\omega_1\leqslant\mu\leqslant\varkappa$. Also $\langle\varkappa:1,\omega_1\rangle\to\omega$ holds for every $\varkappa$ which is less than the first strongly inaccessible cardinal (see [8]). By well-known results of Tarski and Solovay the relations hold if $\varkappa$ is even larger. It is also well known that Gödel's axiom of constructibility (V = L) implies the transversals hypothesis (TH) and TH implies $\langle\varkappa^+:1,\varkappa^+\rangle\to\langle\varkappa^{++},[\varkappa^+]^{<\varkappa^+}\rangle$ (see [5] and [6]). Recall that V = L implies GCH. An exhaustive discussion and exact references for V = L, TH and other related axioms can be found in [5], [6] and [7].

The above facts and Theorem 5 imply in particular the following

COROLLARY 6. a) $\langle 2^\varkappa:\lambda,\varkappa^+\rangle\overset{R}{\to}\omega$ *for every $\lambda<\omega$ and every $\varkappa$*; $\langle\varkappa:\lambda,\omega_1\rangle\overset{R}{\to}\omega$ *for every $\lambda<\omega$ and every $\varkappa$ which is less than the first strongly innaccessible cardinal*;

b) $\langle\varkappa^+:\varkappa,\varkappa^+\rangle\overset{R}{\to}\varkappa^+$ *for every $\varkappa$*; $\langle\varkappa:\lambda,\mu\rangle\overset{R}{\to}\mu$ *for every $\varkappa$ which is less than the first weakly inaccessible cardinal, every $\lambda<\varkappa$ and every $\mu\geqslant\lambda^++\omega_1$*;

c) $\langle\varkappa^+:\varkappa,\varkappa^+\rangle\overset{R}{\to}\varkappa^{++}$ *if we assume* V = L *or merely* TH *for* $\varkappa^+$;

$\langle\omega_1:\omega_1,\omega_1\rangle\overset{R}{\to}\omega_2$ *if we assume* V = L *or merely* TH *for* ω_1.

Observe that if we omit the letter R in Corollary 6 then the particular case of Corollary 6 thus obtained becomes a well-known result. It is in fact a theorem of Taylor (parts b) and c)), and of Ulam (part a)). For details see the comments after Theorem 4.

At the end of our paper we would like to recall (in our terminology) the problem of Ulam stated in [3] as Problem 81. It is the following question: Does $\langle\omega_1:\omega_1,\omega_1\rangle\to 2$ hold? The authors of [3] say that they do not know what happens if we consider a stronger relation, e.g. $(\omega_1:\omega_1,\omega_1)\overset{M}{\to}2$, where M is the family of all non-trivial ω_1-complete fields on ω_1 on which it is possible to define a non-trivial real-valued measure (cf. [8]). Prikry has proved that even $\langle\omega_1:\omega_1,\omega_1\rangle\overset{M}{\to}\omega_1$, assuming the transversal hypothesis for ω_1. Taylor has strengthened Prikry's result to the form $\langle\omega_1:\omega_1,\omega_1\rangle\overset{M}{\to}\omega_2$, assuming only $\langle\omega_1:1,\omega_1\rangle\to\langle\omega_2,[\omega_1]^{<\omega_1}\rangle$. In the present paper we generalize Taylor's result to the form $\langle\omega_1:\omega_1,\omega_1\rangle\overset{R}{\to}\omega_2$ under the same assumption as in Taylor's result (see Theorem 5 and the comments after it). By Theorem 3, without any additional assumption on ZFC, we have in particular:

$$\langle\omega_1:\omega_1,\omega_1\rangle\to\omega_1 \quad\text{iff}\quad \langle\omega_1:\omega_1,\omega_1\rangle\overset{M}{\to}\omega_1 \quad\text{iff}\quad \langle\omega_1:\omega_1,\omega_1\rangle\overset{R}{\to}\omega_1,$$

and

$$\langle 2^\omega:\omega,\omega_1\rangle\to 2 \quad\text{iff}\quad \langle 2^\omega:\omega,\omega_1\rangle\overset{M}{\to}2 \quad\text{iff}\quad \langle 2^\omega:\omega,\omega_1\rangle\overset{R}{\to}2,$$

where M is a family of all non-trivial ω_1-complete fields on ω_1 (and 2^ω, respectively) on which it is possible to define a non-trivial real-valued measure (observe that

$$\langle 2^\omega:\omega,\omega_1\rangle\to 2 \quad\text{iff}\quad \langle 2^\omega:\omega,\omega_1\rangle\to\omega).$$

The above two statements together with Corollary 6c) are a contribution to the comments of the authors of [3] on the problem of Ulam concerning sets of measures.

The following question seems to be open in ZFC:

Does $\langle\omega_1:\omega_1,\omega_1\rangle\to 2$ imply $\langle\omega_1:\omega_1,\omega_1\rangle\overset{R}{\to}2$?

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