Remark. The results of Theorem 1 and Theorem 2 can be applied to the notion of representable spaces. A space X is said to be *representable* [6, p. 263] if for every $x \in X$ and every open set U containing x there exists an open set $V \subset U$ containing x and such that, for every $y \in V$ there exists a homeomorphism of X onto X which carries x onto y and which leaves fixed every point in the complement of U. Theorem 1 implies that $M \times X$ is not representable if X is a nondegenerate continuum as in Theorem 1, and Theorem 2 implies that $M \times M$ is not representable.

4. Problems.

(1) Does there exist a number n (finite or countable) such that the Cartesian product M^n of n copies of the Menger universal curve is 2-homogeneous?

(2) Does there exist a non-degenerate continuum X such that $M \times X$ is 2-homogeneous?

(3) Suppose that $X_1, X_2, \ldots, X_n$ are 1-dimensional locally connected continua such that the product $X_1 \times X_2 \times \ldots \times X_n$ is 2-homogeneous. Is it true that every X_i is a simple closed curve?

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Topological contraction principle

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Abstract. A quite general fixed point theorem for functions in a quasi-uniform space and its converse, in the compact Hausdorff case, has been presented in this paper.

Let f be a function on a metric space $X = (X, d)$ into itself. It is called a *Banach contraction* if there exists $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $(x, y) \in X \times X$. In this case, according to the Banach contraction principle [1, p. 160], if X is complete, then f has a unique fixed point u and $\lim_n f^n(x) = u$ for all $x \in X$. The primary purpose of this paper is to establish a generalization, in quasi-uniform context, of the Banach contraction principle, and to show that it contains, among others, the results of Davis [6], Edelstein [7], Janos [10], Keeler–Meier [12], Knill [14], Naimpally [16], Reilly [20], Tan [22], Tarafdar [23] and Taylor [24]. The secondary purpose is to establish, in the compact Hausdorff case, the converse theorem. We note that in the non-compact case, because of the multiplicity of uniformities (or quasi-uniformities) defining the same topology, the notion of a converse theorem has no unique sense.

1. Contraction theorem. We begin with pertinent definitions specifying our context. A *quasi-uniformity* on a set X is a filter $\mathscr{U}$ on $X \times X$ satisfying the axioms of a uniformity, with the possible exception of the symmetry axiom. As in the case of a uniformity, it induces a topology $\tau_{\mathscr{U}}$ on X such that, for $x \in X$, the sets $U[x] = \{y \in X: (x, y) \in U\}$, $U \in \mathscr{U}$, form a $\tau_{\mathscr{U}}$-neighbourhood basis of x. This generalization of uniformity owes its importance to the fact that every topological space is quasi-uniformizable ([4, p. 171], [5, pp. 886–887], [17, p. 316]).

Henceforth in this section $X = (X, \mathscr{U})$ is a quasi-uniform space. According to Davis [5, p. 892], a filter $\mathscr{F}$ on X is "Cauchy" if, for every $U \in \mathscr{U}$, there exists $x = x(U) \in X$ such that $U[x] \in \mathscr{F}$. We define a *Cauchy sequence* in X to be a sequence $\{x_n\}_1^\infty$ in X whose corresponding Fréchet filter is Cauchy, that is, such that, for every $U \in \mathscr{U}$, there exists $x = x(U) \in X$ and a positive integer $n = n(U)$ such that $x_m \in U[x]$ for all $m \geqslant n$. Every convergent sequence in X is Cauchy; if, conversely, every Cauchy sequence in X converges, we say that X is *sequentially complete*. (For a uniform space the above definition of "Cauchy sequence" coincides with the usual definition. If, however, the usual definition were carried over the quasi-uniform context, there would exist non-Cauchy convergent sequences [13, p. 88]). Following Wilansky [25, p. 262], we say that a topological space is US if every convergent sequence has a unique limit. A US space is T_1 [25, p. 262] but may be non-Hausdorff [21, p. 777].

If A, B are binary relations on X their composition is defined by the formula

$$A\circ B = \{(x,y)\in X\times X\colon\ (x,z)\in B \text{ and } (z,y)\in A \text{ for some } z\in X\}.$$

We treat a function f on X into itself as a special binary relation on X. Its kth iterate f^k is defined inductively: f^0 is the identity, $f^k = f\circ f^{k-1}$ for $k\geqslant 1$. We note that

$$f\circ A\circ f^{-1} = \{(f(x),f(y))\colon\ (x,y)\in A\} \quad \text{for all } A\subseteq X\times X.$$

Let f be a function on X into itself. We say that f is *contractive*, or is a *contraction*, if, for every $U\in\mathcal{U}$, there exist $V = V(U)$, $W = W(U)\in\mathcal{U}$ such that $f\circ V\circ W\circ f^{-1}\subseteq V\subseteq U$. We say that it is *occasionally small* if, for every ordered pair $(x,y)\in X\times X$ and every $U\in\mathcal{U}$, there exists a positive integer $n = n((x,y),U)$ such that $(f^n(x),f^n(y))\in U$.

1.1. THEOREM. *Let $X = (X,\mathcal{U})$ be a sequentially complete* US *quasi-uniform space and let f be a function on X into itself. If at least one iterate f^k is an occasionally small contraction, then f has a unique fixed point u. Moreover, for arbitrary $x_0\in X$, $\lim_n f^{kn}(x_0) = u$.*

Proof. It will be shown that $h = f^k$ has at most one fixed point. Let u, v be fixed points of h. Let $U\in\mathcal{U}$ be arbitrary. Since h is occasionally small, for some positive integer n, $(u,v) = (h^n(u),h^n(v))\in U$. Since X is T_1, it follows that $u = v$.

Let $x_0\in X$. It will be shown that $\{h^n(x_0)\}_1^\infty$ is Cauchy. Let $U\in\mathcal{U}$ be arbitrary. Since h is contractive, there exist V, $W\in\mathcal{U}$ such that $h\circ V\circ W\circ h^{-1}\subseteq V\subseteq U$. Then, since h is occasionally small, there exists a positive integer n such that

$$(h^n(x_0),h^{n+1}(x_0)) = (h^n(x_0),h^n(h(x_0)))\in W.$$

It will suffice to show that

$$h^m(x_0)\in V[h^{n+1}(x_0)] \quad \text{for all } m\geqslant n+1.$$

This being so for $m = n+1$, we suppose it for an arbitrary $m\geqslant n+1$. The inductive assumption $(h^{n+1}(x_0),h^m(x_0))\in V$, together with $(h^n(x_0),h^{n+1}(x_0))\in W$, implies $(h^n(x_0),h^m(x_0))\in V\circ W$. It follows that $(h^{n+1}(x_0),h^{m+1}(x_0))\in V$, so the induction is complete.

Since X is sequentially complete there exists $u = u(x_0)\in X$ such that $h^n(x_0)\to u$. Being contractive, h is, in particular, continuous, so $h(u) = \lim_n h(h^n(x_0)) = u$. Thus u is a fixed point of h, therefore the unique fixed point of h. In particular, u is independent of x_0. Since $h(f(u)) = f(h(u)) = f(u)$, $f(u)$ is a fixed point of h, therefore $f(u) = u$. But a fixed point of f is also a fixed point of h, so u is the unique fixed point of f.

In the remainder of this section we deduce from Theorem 1.1 a corollary generalizing several fixed point theorems in quasi-uniform context. We recall some terminology of the authors.

A *quasi-pseudo metric* on a set X is a non-negative real function p on $X\times X$ such that $p(x,x) = 0$ and $p(x,y)\leqslant p(x,z)+p(z,y)$. A real function f on a quasi-uniform space $X = (X,\mathcal{U})$ is *quasi-uniform lower semi-continuous* if, for $\varepsilon>0$, there exists $U\in\mathcal{U}$ such that $(x,y)\in U$ implies $f(y)>f(x)-\varepsilon$. It has been shown by Reilly [19, p. 318] that if $X = (X,\mathcal{U})$ is a quasi-uniform space, $\mathcal{U}$ is generated by the family π of all quasi-pseudo-metrics on X which are quasi-uniform lower semi-continuous on $(X\times X, \mathcal{U}\times\mathcal{U}^{-1})$, where $\mathcal{U}^{-1} = \{U^{-1}\colon\ U\in\mathcal{U}\}$. In this case, $\mathcal{U}$ has, as subbase, the sets of the form $U(p,\varepsilon) = \{(x,y)\in X\times X\colon\ p(x,y)<\varepsilon\}$, $p\in\pi$, $\varepsilon>0$. Let π be a family of quasi-pseudo-metrics on a quasi-uniform space $X = (X,\mathcal{U})$ which generates $\mathcal{U}$. A function f of X into itself is a *π-contraction* or *π-contractive* if, for each $p\in\pi$, there exists $c_p\in[0,1)$ such that $p(f(x),f(y))\leqslant c_p p(x,y)$ for all $(x,y)\in X\times X$. The following result generalizes the theorems of Janos [10, p. 69], Reilly [20, p. 361], Tan [22, p. 832] and Tarafdar [23, p. 212].

1.2. COROLLARY. *Let $X = (X,\mathcal{U})$ be a sequentially complete* US *quasi-uniform space and let π be a family of quasi-pseudo-metrics on X which generates $\mathcal{U}$. Then every π-contractive function f on X into itself has a unique fixed point $u\in X$. Moreover, for all $x\in X$, $f^n(x)\to u$ in the τ_u-topology.*

Proof. It will suffice to show that f is occasionally small and contractive. Let $U\in\mathcal{U}$. There exist $p_i\in\pi$, $\varepsilon_i>0$, $1\leqslant i\leqslant k$ such that $V = \bigcap_{i=1}^{k} U(p_i,\varepsilon_i)\subseteq U$. Since f is π-contractive, for $i = 1,2,\ldots,k$, there exists $c_{p_i}\in[0,1)$ such that

$$p_i(f(x),f(y))\leqslant c_{p_i}p_i(x,y) \quad \text{for all } (x,y)\in X\times X.$$

Let $\varepsilon = \min_{1\leqslant i\leqslant k}\varepsilon_i$. Let $(x_0,y_0)\in X\times X$. For $i = 1,2,\ldots,k$,

$$p_i(f^n(x_0),f^n(y_0))\leqslant c_{p_i}^n\, p_i(x_0,y_0)\downarrow 0 \quad (n\to\infty).$$

Hence there is a positive integer n such that

$$p_i(f^n(x_0),f^n(y_0))<\varepsilon \quad \text{for} \quad i = 1,2,\ldots,k.$$

Then

$$(f^n(x_0),f^n(y_0))\in\bigcap_{i=1}^{k} U(p_i,\varepsilon) = V = U,$$

proving that f is occasionally small. Now choose $\delta_i>0$ such that $\delta_i c_{p_i}<\varepsilon_i - c_{p_i}\varepsilon_i$, $1\leqslant i\leqslant k$. Let $W = \bigcap_{i=1}^{k} U(p_i,\delta_i)$. Let $(x,y)\in V\circ W$. There exists $z\in X$ such that $(x,z)\in W$ and $(z,y)\in V$. Let $i = 1,2,\ldots,k$ be arbitrary. Since $p_i(x,z)<\delta_i$, $p_i(z,y)<\varepsilon_i$ we have $p_i(f(x),f(z))\leqslant c_{p_i}\delta_i$, $p_i(f(z),f(y))\leqslant c_{p_i}\varepsilon_i$ so that

$$p_i(f(x),f(y))\leqslant c_{p_i}(\delta_i+\varepsilon_i)<\varepsilon_i,$$

that is, $(f(x),f(y))\in U(p_i,\varepsilon_i)$. Therefore $f\circ V\circ W\circ f^{-1}\subseteq V\subseteq U$, so f is contractive.

2. Well-chained spaces. A quasi uniform space $X = (X, \mathscr{U})$ is said to be *well-chained* if, for every ordered pair $(x, y) \in X\times X$ and every $U \in \mathscr{U}$ there exists a positive integer $n = n((x, y), U)$ such that $(x, y) \in U^n$ [6, p. 982]. For example, a connected uniform space is well-chained. It will be shown that a version of Theorem 1.1 for well-chained spaces contains the results in this context of Taylor, Knill and Davis.

2.1. LEMMA. *Let $(X, \mathscr{U})$ be a well-chained quasi-uniform space and let f be a function on X into itself. If f is contractive, then f is occasionally small.*

Proof. Let $W \in \mathscr{U}$. It will be shown inductively that $f\circ W\circ f^{-1}\subseteq W$ implies $f\circ W^n\circ f^{-1}\subseteq W^n$ for all $n = 1, 2, \ldots$ The implication being trivial for $n = 1$, we prove if for $n+1$, assuming it for n. Let $(x, y) \in W^{n+1}$. There exists $z \in X$ such that $(x, z) \in W^n$ and $(z, y) \in W$. Then $(f(z), f(y)) \in W$ and, by the induction hypothesis, $(f(x), f(z)) \in W^n$, so $(f(x), f(y)) \in W^{n+1}$. This shows that $f\circ W^{n+1}\circ f^{-1}\subseteq W^{n+1}$.

Let $U \in \mathscr{U}$. Since f is contractive there exists $V = V(U)$, $W_1 = W_1(U) \in \mathscr{U}$ such that $f\circ V\circ W_1\circ f^{-1}\subseteq V\subseteq U$. Then f being contractive, there exists $W = W(W_1) \in \mathscr{U}$ such that $f\circ W\circ f^{-1}\subseteq W\subseteq W_1$. Then $f\circ V\circ W\circ f^{-1}\subseteq V\subseteq U$. It will be shown inductively that $f^n\circ V\circ W^n\circ f^{-n}\subseteq V$ for all $n = 1, 2, \ldots$ This being true for $n = 1$, we prove it for $n+1$, assuming it for n. Note that, for $A, B\subseteq X\times X$, $A\circ B\subseteq A\circ f^{-1}\circ f\circ B$, hence, applying the result of the first paragraph and the induction hypothesis, we have

$$\begin{aligned} f^{n+1}\circ V\circ W^{n+1}\circ f^{-(n+1)} &= f^n\circ f\circ V\circ W\circ W^n\circ f^{-1}\circ f^{-n} \\ &\subseteq f^n\circ (f\circ V\circ W\circ f^{-1})\circ (f\circ W^n\circ f^{-1})\circ f^{-n} \\ &\subseteq f^n\circ V\circ W^n\circ f^{-n}\subseteq V. \end{aligned}$$

This established, let $(x, y) \in X\times X$. Since X is well chained, there exists a positive integer $n = n((x, y), W)$ such that $(x, y) \in W^n\subseteq V\circ W^n$, so $(f^n(x), f^n(y)) \in V\subseteq U$.

Theorem 1.1, with Lemma 2.1, yields the following result:

2.2. THEOREM. *Let $X = (X, \mathscr{U})$ be a sequentially complete* US *well-chained quasi-uniform space and let f be a function on X into itself. If at least one iterate f^k is contractive, then f has a unique fixed point u. Moreover, for arbitrary $x_0 \in X$, $\lim_n f^{kn}(x_0) = u$.*

Let $X = (X, \mathscr{U})$ be a uniform space and let $\mathscr{B}$ be a basis for $\mathscr{U}$. According to Taylor [24, p. 165], a function f on X into itself is a *$\mathscr{B}$-contraction* if, for every $U \in \mathscr{U}$ there exists $W = W(U) \in \mathscr{B}$ such that $f\circ U\circ W\circ f^{-1}\subseteq U$. It is clear that a $\mathscr{B}$-contraction is a contraction. Conversely if f is a contraction,

$$\mathscr{B} = \{V \in \mathscr{U}\colon f\circ V\circ W\circ f^{-1}\subseteq V \text{ for some } W \in \mathscr{U}\}$$

is a basis for $\mathscr{U}$ and f is a $\mathscr{B}$-contraction. Consequently Theorem 2.2 generalizes the Basic Lemma of Taylor [24, p. 166].

Let $X = (X, \mathscr{U})$ be a uniform space. According to Knill [14, p. 450], a function f on X into itself is a *uniform contraction* if, for every $U \in \mathscr{U}$, there exist $V = V(U)$, $W = W(U) \in \mathscr{U}$ such that $f\circ V\circ f^{-1}\circ W\subseteq V\subseteq U$. In this case there exists $W_1 = W_1(W) \in \mathscr{U}$ such that $f\circ W_1\circ f^{-1}\subseteq W_1\subseteq W$, then $f\circ V\circ W_1\circ f^{-1}\subseteq (f\circ V\circ f^{-1})\circ (f\circ W_1\circ f^{-1})\subseteq f\circ V\circ f^{-1}\circ W\subseteq V\subseteq U$, so f is a contraction. Consequently Theorem 2.2 generalizes the Uniform contraction principle of Knill [14, p. 451].

Let $X = (X, \mathscr{U})$ be a quasi-uniform space and let f be a function on X into itself. Let r, s be positive integers. According to Davis [6, p. 982], f is an *(r/s)-map* if for every $U \in \mathscr{U}$, $f\circ U^s\circ f^{-1}\subseteq U^r$; and f is an *eventual contraction* if some iterate f^k is an (r/s)-map with $r<s$. In this case we may choose $U_1 \in \mathscr{U}$ such that $U_1^r\subseteq U$ and then $f^k\circ U_1^r\circ U_1^{s-r}\circ f^{-k} = f^k\circ U_1^s\circ f^{-k}\subseteq U_1^r\subseteq U$, so f^k is a contraction. Consequently, Theorem 2.2 generalizes Theorem 2 of Davis [6, p. 984] (with a correction, because, to assure the uniqueness of the sequential limit, his condition T_0 should be replaced by US).

3. Metric spaces. In this section $X = (X, d)$ is a metric space.

Let $\varepsilon > 0$, $\lambda \in [0, 1)$. According to Edelstein [7, pp. 7–8] a function f on X into itself is an *(ε, λ)-uniformly local contraction* if $d(f(x), f(y))\leqslant \lambda d(x, y)$ whenever $d(x, y)<\varepsilon$. According to Edelstein [8, p. 78], X is *ε-chainable* if, for every $(x, y) \in X\times X$, there exists a finite sequence $\{z_i\}_0^n$ in X such that $x = z_0$, $z_n = y$ and $d(z_{i-1}, z_i)<\varepsilon$ for $1\leqslant i\leqslant n$. To prove that the fixed-point theorem of Edelstein [7, p. 8] is a corollary of Theorem 1.1, we will show that an (ε, λ)-uniformly local contraction f on an ε-chainable metric space X is occasionally small and contractive, relatively to the uniformity $\mathscr{U}_d$ having as basis the sets

$$U_\varepsilon = \{(x, y) \in X\times X\colon d(x, y)<\varepsilon\}, \quad \varepsilon>0.$$

Let $U \in \mathscr{U}_d$. Choose a positive integer $n\geqslant 2$ such that $V = U_{\varepsilon/n}\subseteq U$. Let

$$0<\delta<\min\left\{\frac{\varepsilon-\lambda\varepsilon}{n\lambda}, \frac{n-1}{n}\varepsilon\right\}$$

and set $W = U_\delta$. It is easy to verify that $f\circ V\circ W\circ f^{-1}\subseteq V\subseteq U$, so f is contractive. Now let $(x, y) \in X\times X$ choose a finite sequence $\{z_i\}_0^n$ in X such that $x = z_0$, $z_n = y$ and $d(z_{i-1}, z_i)<\varepsilon$ for $1\leqslant i\leqslant n$. Then $d(f(z_{i-1}), f(z_i))\leqslant \lambda d(z_{i-1}, z_i)$ so

$$d(f^m(z_{i-1}), f^m(z_i))<\lambda^m\varepsilon \quad (1\leqslant i\leqslant n),$$

therefore

$$d(f^m(x), f^m(y))\leqslant \sum_{i=1}^n d(f^m(z_{i-1}) f^m(z_i))<\lambda^m\varepsilon n \downarrow 0 \quad (m\to\infty).$$

Let $\delta>0$ be such that $U_\delta\subseteq U$ and choose m so that $\lambda^m\varepsilon n<\delta$. Then $(f^m(x), f^m(y)) \in U_\delta\subseteq U$, proving that f is occasionally small.

Recall that a function f on X into itself is a *strict contraction* if $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. According to Keeler and Meier [12, p. 326] it is a *weakly uniformly strict contraction* if, for $\varepsilon>0$, there exists $\delta = \delta(\varepsilon)>0$ such that

$\varepsilon \leqslant d(x, y) < \varepsilon+\delta$ implies $d(f(x), f(y)) < \varepsilon$. In this case, supposing $x_0 \neq y_0$, set $\varepsilon = d(x_0, y_0)$, and choose $\delta > 0$ such that $\varepsilon \leqslant d(x, y) < \delta+\varepsilon$ implies $d(f(x), f(y)) < \varepsilon$. Then, in particular $d(f(x_0), f(y_0)) < \varepsilon = d(x_0, y_0)$. Thus, a weakly uniformly strict contraction is, in particular, a strict contraction.

3.1. Lemma. *A function f on X into itself is a weakly uniformly strict contraction if and only if, for $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $d(x, y) < \varepsilon+\delta$ implies $d(f(x), f(y)) < \varepsilon$.*

Proof. It suffices to show that the condition is necessary. Suppose that f is a weakly uniformly strict contraction not satisfying the condition. Then, for some $\varepsilon_0 > 0$, there exists for every $\delta > 0$ a point (x_δ, y_δ) of $X \times X$ such that $d(x_\delta, y_\delta) < \varepsilon_0+\delta$ and $d(f(x_\delta), f(y_\delta)) \geqslant \varepsilon_0$. Then, in particular, $x_\delta \neq y_\delta$. Let $\delta_0 > 0$ be such that $\varepsilon_0 \leqslant d(x, y) < \varepsilon_0+\delta_0$ implies $d(f(x), f(y)) < \varepsilon_0$. Since f is a strict contraction

$$\varepsilon_0+\delta_0 > d(x_{\delta_0}, y_{\delta_0}) > d(f(x_{\delta_0}), f(y_{\delta_0})) \geqslant \varepsilon_0 .$$

therefore $d(f(x_{\delta_0}), f(y_{\delta_0})) < \varepsilon_0$, which is a contradiction.

Following Geraghty [9, p. 811], we consider the class T of all functions $\alpha\colon (0, \infty) \to [0, 1)$ such that, if $\{t_n\}_1^\infty$ is a decreasing sequence in $(0, \infty)$ and $\alpha(t_n) \to 1$, then $t_n \downarrow 0$.

3.2. Lemma. *Let $\alpha \in T$. For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\sup\{\alpha(t)\colon \varepsilon \leqslant t < \varepsilon+\delta\} < 1$.*

Proof. Assuming the lemma to be false, there exists $\varepsilon_0 > 0$ such that, for all $\delta > 0$, $\sup\{\alpha(t)\colon \varepsilon_0 \leqslant t < \varepsilon_0+\delta\} = 1$. We will deduce a contradiction by proving inductively the existence of a decreasing sequence $\{t_n\}_1^\infty$ in (ε_0, ∞) such that

$$1 > \alpha(t_n) > \frac{2^{n-1}-1+\alpha(\varepsilon_0)}{2^{n-1}} = 1-\frac{1-\alpha(\varepsilon_0)}{2^{n-1}} .$$

Since $\sup\{\alpha(t)\colon \varepsilon_0 \leqslant t < \varepsilon_0+1\} = 1$ and $\alpha(\varepsilon_0) < 1$, there exists $t_1 \in (\varepsilon_0, \varepsilon_0+1)$ such that $1 > \alpha(t_1) > \alpha(\varepsilon_0)$. Thus the first term is established so we may suppose the first n terms established: $t_1, t_2, \ldots, t_n$. Since

$$\sup\{\alpha(t)\colon \varepsilon_0 \leqslant t < t_n\} = 1 \quad \text{and} \quad 1 > \frac{2^n-1+\alpha(\varepsilon_0)}{2^n},$$

there exists $t_{n+1} \in [\varepsilon_0, t_n)$ such that

$$1 > \alpha(t_{n+1}) > \frac{2^{(n+1)-1}-1+\alpha(\varepsilon_0)}{2^{(n+1)-1}} .$$

If $t_{n+1} = \varepsilon_0$ then

$$\alpha(\varepsilon_0) > \frac{2^n-1+\alpha(\varepsilon_0)}{2^n},$$

that is, $\alpha(\varepsilon_0) > 1$, a contradiction. Thus $t_{n+1} \in (\varepsilon_0, t_n)$, and the induction is complete

3.3. Lemma. *Let f be a function on X into itself. If there exists $\alpha \in T$ such that $d(f(x), f(y)) \leqslant \alpha(d(x, y))\, d(x, y)$ whenever $x \neq y$, then f is a weakly uniformly strict contraction.*

Proof. Let $\varepsilon > 0$. By Lemma 3.2 there exists $\eta = \eta(\varepsilon) > 0$ such that

$$\sup\{\alpha(t)\colon \varepsilon \leqslant t < \varepsilon+\eta\} = c(\varepsilon) < 1 .$$

Let $\delta > 0$ be such that $\delta < \eta$ and $\delta c(\varepsilon) < \varepsilon-\varepsilon c(\varepsilon)$. If $\varepsilon \leqslant d(x, y) < \varepsilon+\delta$ then

$$d(f(x), f(y)) \leqslant \alpha(d(x, y))\, d(x, y) < c(\varepsilon)(\varepsilon+\delta) < \varepsilon .$$

3.4. Lemma. *If a function f on X into itself is a weakly uniformly strict contraction, then f is occasionally small and contractive relatively to $\mathscr{U}_d$.*

Proof. Let $(x, y) \in X \times X$. Write $c_n = d(f^n(x), f^n(y))$, $n = 1, 2, \ldots$ Since f is a strict contraction $c_n \downarrow \varepsilon \geqslant 0$. To prove that f is occasionally small, we will show that $\varepsilon = 0$. Suppose, on the contrary, that $\varepsilon > 0$. Let $\delta = \delta(\varepsilon) > 0$ be such that $\varepsilon \leqslant d(x, y) < \varepsilon+\delta$ implies $d(f(x), f(y)) < \varepsilon$. There exists a positive integer m such that $c_m < \varepsilon+\delta$. Then we have

$$\varepsilon \leqslant c_{m+1} = d(f^{m+1}(x), f^{m+1}(y)) = d(f(f^m(x)), f(f^m(y))) < \varepsilon ,$$

a contradiction.

Let $U \in \mathscr{U}_d$. Then $U_\varepsilon \subseteq U$ for some $\varepsilon > 0$. By Lemma 3.1 there exists $\delta = \delta(\varepsilon) > 0$ such that $d(x, y) < \varepsilon+\delta$ implies $d(f(x), f(y)) < \varepsilon$. Since

$$U_\varepsilon \circ U_\delta \subseteq U_{\varepsilon+\delta}, \qquad f \circ U_\varepsilon \circ U_\delta \circ f^{-1} \subseteq f \circ U_{\varepsilon+\delta} \circ f^{-1} \subseteq U_\varepsilon \subseteq U .$$

This proves that f is a contraction on $X = (X, \mathscr{U}_d)$ into itself.

By Lemma 3.4, Theorem 1.1 contains the Keeler–Meier fixed point theorem [12, p. 326], which in turn, by Lemma 3.3, contains the theorem of Geraghty [9, p. 811] and its corollaries 3.1 (Rakotch), 3.2, 3.3 (Boyd–Wong), 3.4 (Browder). We note also that the Keeler–Meier fixed point theorem, apart from generalizing the Banach Contraction Principle, contains the fixed-point Theorem 3.1 of Edelstein [8, p. 75].

The Banach contraction principle has been generalized by means of the following generalization of the notion of metric space: Let $I = (I, \leqslant)$ be a partially ordered set and let $R = \prod_{i \in I} R_i$, where each R_i is a copy of the real line. Let each R_i be endowed with the usual topology and let R be assigned the product topology. Let R be partially ordered by the lexicographic formula: $(r_i) \leqslant (r_i')$ if, whenever $r_j > r_j'$ for some j, there exists $k < j$ such that $r_k < r_k'$ and $r_m \leqslant r_m'$ for all $m \leqslant k$. Defining $(r_i)+(r_i') = (r_i+r_i')$ and $\lambda(r_i) = (\lambda r_i)$ where λ is real, R is a real partially ordered vector space. A *generalized metric* on a set X is a function $\varrho\colon X \times X \to R = \prod_{i \in I} R_i$ satisfying the usual axioms for a metric (where 0 is the neutral element of R) [11, p. 936]. The subsets of R of the form

$$N = N(\varepsilon, i_1, \ldots, i_n) = \{(r_i)\colon |r_{i_j}| < \varepsilon \text{ for } j = 1, \ldots, n\}, \quad \varepsilon > 0 ,$$

$\{i_1, \ldots, i_n\}$ being a finite sequence in I, constitute a basis of the neighbourhood filter of 0 in R; and the sets of the form $U(N) = \{(x, y) \in X \times X\colon \varrho(x, y) \in N\}$ constitute a basis for a Hausdorff uniformity $\mathcal{U}$ on X, which we will say is *induced* by ϱ. The uniformity of a Hausdorff uniform space $X = (X, \mathcal{U})$ is induced by a generalized metric ϱ on X [11, p. 937].

Let $X = (X, \mathcal{U})$ be a Hausdorff space, where the uniformity $\mathcal{U}$ is induced by a generalized metric ϱ. Let $\lambda \in [0, 1)$. According to Naimpally [16, p. 479] a function f of X into itself is *λ-globally contractive* if $\varrho(f(x), f(y)) \leqslant \lambda\varrho(x, y)$ for all $(x, y) \in X \times X$.

3.5. Lemma. *In the context of the preceeding paragraph, let f be a function of $X = (X, \mathcal{U})$ into itself. If f is λ-globally contractive, then f is an occasionally small contraction.*

Proof. Since $\varrho(f^n(x), f^n(y)) \leqslant \lambda^n \varrho(f(x), f(y))$, f is occasionally small. Let $U \in \mathcal{U}$. There exists a neighbourhood $N = N(\varepsilon, i_1, \ldots, i_n)$ of 0 in R such that $U(N) \subseteq U$. Let $N'(\delta, i_1, \ldots, i_n)$ where $\delta > 0$ and $\lambda\delta < \varepsilon - \lambda\varepsilon$. Let $(x, y) \in U(N) \circ U(N')$. There exists $z \in X$ such that $(x, z) \in U(N')$ and $(z, y) \in U(N)$. Then $\varrho(x, z) \in N'$, $\varrho(z, y) \in N$ so $\varrho(x, y) \in N'' = N''(\varepsilon + \delta, i_1, \ldots, i_n)$. Hence

$$\varrho(f(x), f(y)) \leqslant \lambda\varrho(x, y) \in N''' = N'''(\lambda(\varepsilon+\delta), i_1, \ldots, i_n) \subseteq N.$$

It follows that $\varrho(f(x), f(y)) \in N$ so that $(f(x), f(y)) \in U(N)$. This shows that $f \circ U(N) \circ U(N') \circ f^{-1} \subseteq U(N) \subseteq U$.

By Lemma 3.5, Theorem 1.1 contains Theorem 2.12 of Naimpally [16, p. 479].

4. Converse theorem for a compact Hausdorf space. If X is a compact Hausdorff space, it is a uniform space $X = (X, \mathcal{U})$, with unique uniformity $\mathcal{U}$.

4.1. Theorem. *Let X be a compact Hausdorff space. For a continuous function f on X into itself, the following statements are equivalent:*

(i) *f is an occasionally small contraction.*

(ii) *f has a unique fixed point $u \in X$ and $\lim_n f^n(x) = u$ for all $x \in X$.*

(iii) *There exists $u \in X$ such that $\bigcap_{n=0}^{\infty} f^n(X) = \{u\}$.*

(iv) *The filter basis $\mathcal{B}_f = \{f^n(X)\colon n = 0, 1, 2, \ldots\}$ converges.*

Proof. (i)⇒(ii). Theorem 1.1.

(ii)⇒(iv). Using the argument of Meyers [15, p. 74], we will show that $\mathcal{B}_f \to u$. Let V be an open neighbourhood of u. For $x \in X$ let $n(x)$ be the first positive integer such that $f^n(x) \in V$ for all $n \geqslant n(x)$. The conclusion will follow if we show that $\sup_{x \in X} n(x) < \infty$. Supposing the contrary, there is a sequence $\{x_k\}_1^\infty$ in X such that $n(x_k) > k$ and $x_k \to y$ for some $y \in X$. Since $f^{n(y)}(y) \in V$ and $f^{n(y)}$ is continuous, there is a neighbourhood U of y such that $f^{n(y)}(z) \in V$ for all $z \in U$. Thus $z \in U$ implies $n(z) \leqslant n(y)$. But $x_k \in U$ eventually, which is a contradiction.

(iv)⇒(iii). Suppose that $\mathcal{B}_f \to u$. Since X is compact, $\bigcap_{n=0}^{\infty} f^n(X) \neq \emptyset$. Let $v \in \bigcap_{n=0}^{\infty} f^n(X)$ and suppose $v \neq u$. Let U, V be disjoint neighbourhoods of u, v, respectively. For some positive integer n, $f^n(X) \subseteq U$, so $f^n(X) \cap V = \emptyset$, a contradiction.

(iv)⇒(i). Suppose that $\mathcal{B}_f \to u$. This implies, in particular, that $f^n(x) \to u$ for all $x \in X$. Let $(x, y) \in X \times X$ and let $U \in \mathcal{U}$. Choose a symmetric member W of $\mathcal{U}$ such that $W^2 \subseteq U$. There exists a positive integer n such that $f^n(x), f^n(y) \in W[u]$, so $(f^n(x), f^n(y)) \in W^2 \subseteq U$. This shows that f is occasionally small.

By the argument of Knill in the second part of the proof of his Theorem 2.7 [14, pp. 453–454] — where the connectedness of X is not applied — for every $U \in \mathcal{U}$ there exists a closed member V of $\mathcal{U}$ such that $f \circ V \circ f^{-1} \subseteq \mathring{V} \subseteq U$, where $\mathring{V}$ denotes the interior of V in $X \times X$. Since V is closed, $V = \bigcap \{W \circ V \circ W\colon W$ is a symmetric member of $\mathcal{U}\}$. By the compactness of X, the sets of the form $W \circ V \circ W$, where W is a symmetric member of $\mathcal{U}$, constitute a basis for the neighbourhood filter of V. Then, since $f^{-1} \circ \mathring{V} \circ f$ is open and $V \subseteq f^{-1} \circ \mathring{V} \circ f$, there is a symmetric member W of $\mathcal{U}$ such that $W \circ V \circ W \subseteq f^{-1} \circ \mathring{V} \circ f$, so, in particular, $f \circ V \circ W \circ f^{-1} \subseteq \mathring{V} \subseteq V \subseteq U$. This shows that f is contractive.

(iii)⇒(ii). Since $\{u\} = \bigcap_{n=0}^{\infty} f^{n+1}(X) \supseteq f\left(\bigcap_{n=0}^{\infty} f^n(X)\right) = \{f(u)\}$, $f(u) = u$. Let $x \in X$. Since X is compact, the set $L(x)$ of cluster points of $\{f^n(x)\}_1^\infty$ is non-empty. Suppose the existence of $v \in L(x)$ such that $v \neq u$. Then there is a positive integer n_0 such that $v \notin f^n(X)$ for all $n \geqslant n_0$. Since $f^{n_0}(X)$ is closed, $U[v] \cap f^{n_0}(X) = \emptyset$ for some $U \in \mathcal{U}$. On the other hand, since $v \in L(x)$, $f^n(x) \in U[v]$ for an infinity of the indices n, so we have a contradiction, proving that $L(x) = \{u\}$.

Suppose that $\{f^n(x)\}_1^\infty$ does not converge to u. Then, since X is compact, some subsequence of $\{f^n(x)\}_1^\infty$ converges to some point $w \neq u$. But then $w \in L(x)$, a contradiction. Therefore $f^n(x) \to u$. Finally, let z be any fixed point of f. We have $f^n(z) = z$ for all $n = 1, 2, \ldots$ But since $f^n(z) \to u$, $z = u$.

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On normality and countable paracompactness

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Abstract. The purpose of this paper is to present certain construction techniques developed by the author in the study of the relationship between normality and countable paracompactness. Based on these techniques, the following results are obtained: (1) (MA+¬CH) There exists a countably paracompact, separable, non-normal Moore space. (2) (MA+¬CH) There exists a countably paracompact, screenable, non-normal Moore space. (3) There exists a pseudo-normal, separable Moore space which is not countably paracompact. (4) There exists a perfect, continuous map from a screenable Moore space onto a non-screenable Moore space. (5) (V = L) Each normal, first countable T_2-space in which each subset is an F_σ-set is σ-discrete. These results were obtained by the author in the period 1974–75, and all have been previously announced in the literature and presented at various mathematical meetings (see [R_5], [R_6], [R_7], [WFR], and [Ru_2]).

Motivation. It was established in 1951 by Dowker [D] that in perfect spaces (i.e. spaces in which closed sets are G_δ-sets), normality implies countable paracompactness. The extreme usefullness of this result led quite naturally to the search for (i) a Dowker space, a normal T_2-space which is not countably paracompact, and (ii) a countably paracompact, perfect, non-normal T_3-space. In particular, the relationships between normality, countable paracompactness, and pseudo-normality in Moore spaces have been of considerable interest ([Y], [Z], [P], [R_3], [T_2], and [K], for examples). Dowker spaces were finally constructed by M. E. Rudin in [Ru_1]. More recently, Wage in [W_1] developed an elegant construction technique which produced an example of a countably paracompact, perfect, non-normal T_3-space, and, under (Ma+¬CH), an example of a countably paracompact, non-normal Moore space. Furthermore, in [F_2], Fleissner showed that, under (CH), each coun-countably paracompact, separable Moore space is metrizable. In this paper, the author presents a basic construction technique that was derived from the consideration of Wage's in [W_1], but which is remarkable for its simplicity. Results (1)–(4) above are based on this technique. Note that (1) together with Fleissner's result from [F_2] establishes the consistency and independence w.r.t. ZFC of the existence of a countably paracompact, separable, non-normal Moore space. Result (2) answers a question often raised by F. D. Tall and contradicts an incorrect argument to the contrary in [G]. Results (3) and (4) answer questions raised by Proctor in [P] and Hodel in [Ho], respectively. Result (5) is based on Fleissner's theorem from [F_1] that, under (V = L), each normal T_2-space of character $\leqslant c$ is collectionwise Haus-