check that X is a Suslinian continuum, which is not rational. Let K be a non-degenerate subcontinuum of X. Then, by the construction of X, we infer that K contains a homeomorphic copy of X. Thus, K is not rational.

Added in proof. Prof. L. G. Oversteegen has pointed out to the authors that Example 5.1 of this paper has the same properties as the example on pp. 50–53 of E. S. Thomas, Jr. *Monotone decompositions of irreducible continua*, Dissertationes Math. 50 (1966), pp. 1–13.

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On the 2-homogeneity of Cartesian products

by

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Dedicated to the memory of Ralph Bennett

Abstract. The Cartesian product of the circle S^1 and the Menger universal curve M is not 2-homogeneous. This solves two problems: one of R. Bennett and one of G. S. Ungar. Some generalizations of this result are given.

1. Introduction. A space X is *n-homogeneous* (see [8], [4], [7]) if for every pair A, B of n-element subsets of X there exists a homeomorphism of X onto X which maps A onto B. A space is *homogeneous* if it is 1-homogeneous. A space X is *countable dense homegeneous* (Bennett, [3]) if for any pair A, B of countable dense subsets of X there exists a homeomorphism of X onto X which maps A onto B. Connected manifolds without boundary are the simplest and the most natural examples of spaces which satisfy all of these homogeneity conditions.

R. D. Anderson proved in [1] that the Menger universal curve M is n-homogeneous for every n. Using another result of R. D. Anderson [2] concerning the homogeneity of curves, R. Bennett [3] showed that M is countably dense homogeneous. Looking for higher dimensional countable dense homogeneous continua which are not manifolds, R. Bennett asked: "Is the property of being countable dense homogeneous preserved in Cartesian products?"

Investigating the n-homogeneous spaces, G. S. Ungar [7] proved that every 2-homogeneous metric continuum is locally connected, which solved a problem of C. E. Burgess [4]. Consequently, in a private conversation, Ungar asked if there exists a homogeneous locally connected metric continuum which is not 2-homogeneous.

In this paper we prove that the product of the circle S^1 and the universal curve M is not countable dense homogeneous, or even 2-homogeneous. In fact, every homeomorphism h of $S^1 \times M$ onto $S^1 \times M$ preserves the circular fibers, i.e. for every point $a \in M$ there exists a point $b \in M$ such that $h(S^1 \times \{a\}) = S^1 \times \{b\}$. This solves both Ungar's and Bennett's problems.

2. Terminology and notation. By a *space* we will understand a compact metric space. A *continuum* is a connected space. A *map* is a continuous function. A map is *inessential*, if it is homotopic to a constant map, otherwise it is *essential*. Given two spaces X and Y with distance functions d_1 and d_2, respectively, by their *Cartesian product* (or simply *product*) we mean the set $X \times Y$ of pairs (x, y) with $x \in X$, $y \in Y$ furnished with the distance function d defined by $d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1^2(x_1, x_2) + d_2^2(y_1, y_2)}$. The Menger universal curve will be denoted by M. The unit circle in the plane will be denoted by S. A *simple closed curve* is a set homeomorphic to S. A *loop* in a space X is a map of S into X. A space X is *weakly locally simply connected* if for every point x in X there exists an open set U containing x such that every loop in X whose image lies in U is inessential (in X).

3. Homeomorphisms of $M \times X$.

LEMMA 1 (M. L. Curtis and M. K. Fort, Jr. [5, p. 141]). *If X is a 1-dimensional space and if $f\colon S \to X$ is an inessential loop in X, then f is inessential in $f(S)$.*

LEMMA 2. *If X is a 1-dimensional continuum and if f_1 and f_2 are two essential loops in X such that $f_1(S) \cap f_2(S) = \emptyset$, then f_1 and f_2 are not homotopic.*

Proof. Assume that f_1 and f_2 are homotopic. Then consider the quotient space $X/f_2(S)$ and apply Lemma 1.

THEOREM 1. *Let X be a pathwise connected and weakly locally simply connected continuum. If h is a homeomorphism of $M \times X$ onto $M \times X$, then for every point $a \in M$ there exists a point $b \in M$ such that $h(\{a\} \times X) = \{b\} \times X$.*

Proof. Given a point $a \in M$, pick some $x_1 \in X$ and let b be the first coordinate of $h(a, x_1)$. First, we will prove that $h(\{a\} \times X) \subset \{b\} \times X$. Suppose on the contrary, that, for some $x_2 \in X$, the first coordinate of $h(a, x_2)$ is $c \neq b$. Let $p_M\colon M \times X \to M$ and $p_X\colon M \times X \to X$ be the projection maps. Since X is weakly locally simply connected, there exists a positive number δ such that every loop in X whose image is of diameter smaller than δ, is inessential (in X). Let ε be a positive number such that, for every set $A \subset M \times X$, the inequality $\operatorname{diam} A < \varepsilon$ implies

$$\operatorname{diam} h(A) < \min\{\tfrac{1}{2}\operatorname{dist}(b, c), \delta\}.$$

Let L be a simple closed curve in M such that $\operatorname{diam}(L \cup \{a\}) < \varepsilon$. Since $\dim M = 1$, L represents an essential loop $f\colon S \to M$. Define loops g_1 and g_2 in $M \times X$ by $g_i(s) = (f(s), x_i)$ for $s \in S$, $i = 1, 2$. Since X is pathwise connected, g_1 and g_2 are homotopic and so are the loops $p_M h g_1$ and $p_M h g_2$ in M. However, because of our choice of ε, we get $p_M h g_1(S) \cap p_M h g_2(S) = \emptyset$, and, by Lemma 2, $p_M h g_1$ is inessential. The loop $p_X h g_1$ is inessential too, since $\operatorname{diam} p_X h g_1 < \delta$. Therefore $h g_1$ is an inessential loop, which implies that the loop $p_M h^{-1} h g_1 = f$ is inessential, and we have a contradiction. Thus $h(\{a\} \times X) \subset \{b\} \times X$. Now applying this result to h^{-1}, we get $h^{-1}(\{b\} \times X) \subset \{a\} \times X$, which concludes the proof.

COROLLARY 1. *If a non-degenerate space X is as in Theorem 1 (for instance, if $X = S$), then $M \times X$ is not 2-homogeneous.*

Proof. Pick two distinct points a, b in M, and two distinct points x_1, x_2 in X. By Theorem 1, there is no homeomorphism of $M \times X$ onto $M \times X$ that maps the set $\{(a, x_1), (a, x_2)\}$ onto the set $\{(a, x_1), (b, x_2)\}$.

COROLLARY 2. *If a non-degenerate space X is as in Theorem 1 (for instance, if $X = S$) then $M \times X$ is not countable dense homogeneous.*

Proof. Let A_1 and A_2 be countable dense subsets of M and of X, respectively. The set $A = A_1 \times A_2$ is a countable dense subset of $M \times X$. Notice that for every point $a \in M$, the set $(\{a\} \times X) \cap A$ is either empty of infinite. Now, pick a point $a_0 \in M \setminus A_1$ and an arbitrary point $x_0 \in X$, and let $B = A \cup \{(a_0, x_0)\}$. By Theorem 1, there exists no homeomorphism of $M \times X$ onto $M \times X$ which maps A onto B.

THEOREM 2. *If h is a homeomorphism of $M \times M$ onto $M \times M$, then one of the following holds true:*

(i) *there are homeomorphisms h_1 and h_2 of M onto M such that for every point (a, b) in $M \times M$, $h(a, b) = (h_1(a), h_2(b))$ or*

(ii) *there are homeomorphisms h_1 and h_2 of M onto M such that for every point (a, b) in $M \times M$, $h(a, b) = (h_2(b), h_1(a))$.*

Proof. Let us call a *fiber* every subset of $M \times M$ of the form $\{a\} \times M$ or $M \times \{b\}$ (a *vertical* fiber or a *horizontal* fiber, respectively). The idea of the proof is as follows: First we prove that h maps every vertical fiber into a fiber. By symmetry, it will follow that h maps every fiber into a fiber, and so does h^{-1}. Therefore h maps every fiber onto a fiber. Then we notice that if some vertical fiber gets mapped by h onto a vertical fiber, then h maps every vertical fiber onto a vertical fiber and every horizontal fiber onto a horizontal fiber and therefore (i) holds true. On the other hand, if some vertical fiber gets mapped by h onto a horizontal fiber, then every vertical fiber gets mapped by h onto a horizontal fiber and every horizontal fiber gets mapped by h onto a vertical fiber, and then (ii) holds true.

Our problem is thus reduced to proving that h maps every vertical fiber into some fiber. The proof is very similar to the proof of Theorem 1.

Suppose that some two points from one vertical fiber, say (a, b) and (a, c) in $M \times M$ are carried by h onto two points that do not belong to the same fiber, say, $h(a, b) = (a^1, b^1)$ and $h(a, c) = (a^2, c^1)$ where $a^1 \neq a^2$ and $b^1 \neq c^1$. Let p_1 and p_2 be the projections of $M \times M$ onto the first and the second coordinate spaces, respectively. Let ε be a positive number such that for every subset A of $M \times M$, the inequality $\operatorname{diam} A < \varepsilon$ implies $\operatorname{diam} h(A) < \frac{1}{2}\min\{\operatorname{dist}(a^1, a^2), \operatorname{dist}(b^1, c^1)\}$. Let L be a simple closed surve in M such that $\operatorname{diam}(L \cup \{a\}) < \varepsilon$. Obviously, L defines an essential loop $f\colon S \to M$. Now, let g_b and g_c be loops in $M \times M$ defined by $g_b(s) = (f(s), b)$ and $g_c(s) = (f(s), c)$ for $s \in S$. Notice that g_b and g_c are homotopic to each other and essential. Because of the choice of ε, we get $p_i h g_b(S) \cap p_i h g_c(S) = \emptyset$ for $i = 1, 2$ which implies (see Lemma 2) that both $p_1 h g_b$ and $p_2 h g_b$ are inessential. Therefore $h g_b$ is inessential and so is g_b, which is a contradiction.

COROLLARY 3. *$M \times M$ is not 2-homogeneous.*

COROLLARY 4. *$M \times M$ is not countable dense homogeneous.*

The two corollaries above are proved in exactly the same fashion as Corollaries 1 and 2.

Remark. The results of Theorem 1 and Theorem 2 can be applied to the notion of representable spaces. A space X is said to be *representable* [6, p. 263] if for every $x \in X$ and every open set U containing x there exists an open set $V \subset U$ containing x and such that, for every $y \in V$ there exists a homeomorphism of X onto X which carries x onto y and which leaves fixed every point in the complement of U. Theorem 1 implies that $M \times X$ is not representable if X is a nondegenerate continuum as in Theorem 1, and Theorem 2 implies that $M \times M$ is not representable.

4. Problems.

(1) Does there exist a number n (finite or countable) such that the Cartesian product M^n of n copies of the Menger universal curve is 2-homogeneous?

(2) Does there exist a non-degenerate continuum X such that $M \times X$ is 2-homogeneous?

(3) Suppose that $X_1, X_2, \ldots, X_n$ are 1-dimensional locally connected continua such that the product $X_1 \times X_2 \times \ldots \times X_n$ is 2-homogeneous. Is it true that every X_i is a simple closed curve?

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Topological contraction principle

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Abstract. A quite general fixed point theorem for functions in a quasi-uniform space and its converse, in the compact Hausdorff case, has been presented in this paper.

Let f be a function on a metric space $X = (X, d)$ into itself. It is called a *Banach contraction* if there exists $\lambda \in [0, 1)$ such that $d(f(x), f(y)) \leqslant \lambda d(x, y)$ for all $(x, y) \in X \times X$. In this case, according to the Banach contraction principle [1, p. 160], if X is complete, then f has a unique fixed point u and $\lim_n f^n(x) = u$ for all $x \in X$. The primary purpose of this paper is to establish a generalization, in quasi-uniform context, of the Banach contraction principle, and to show that it contains, among others, the results of Davis [6], Edelstein [7], Janos [10], Keeler–Meier [12], Knill [14], Naimpally [16], Reilly [20], Tan [22], Tarafdar [23] and Taylor [24]. The secondary purpose is to establish, in the compact Hausdorff case, the converse theorem. We note that in the non-compact case, because of the multiplicity of uniformities (or quasi-uniformities) defining the same topology, the notion of a converse theorem has no unique sense.

1. Contraction theorem. We begin with pertinent definitions specifying our context. A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ satisfying the axioms of a uniformity, with the possible exception of the symmetry axiom. As in the case of a uniformity, it induces a topology $\tau_{\mathcal{U}}$ on X such that, for $x \in X$, the sets $U[x] = \{y \in X: (x, y) \in U\}$, $U \in \mathcal{U}$, form a $\tau_{\mathcal{U}}$-neighbourhood basis of x. This generalization of uniformity owes its importance to the fact that every topological space is quasi-uniformizable ([4, p. 171], [5, pp. 886–887], [17, p. 316]).

Henceforth in this section $X = (X, \mathcal{U})$ is a quasi-uniform space. According to Davis [5, p. 892], a filter $\mathcal{F}$ on X is "Cauchy" if, for every $U \in \mathcal{U}$, there exists $x = x(U) \in X$ such that $U[x] \in \mathcal{F}$. We define a *Cauchy sequence* in X to be a sequence $\{x_n\}_1^\infty$ in X whose corresponding Fréchet filter is Cauchy, that is, such that, for every $U \in \mathcal{U}$, there exists $x = x(U) \in X$ and a positive integer $n = n(U)$ such that $x_m \in U[x]$ for all $m \geqslant n$. Every convergent sequence in X is Cauchy; if, conversely, every Cauchy sequence in X converges, we say that X is *sequentially complete*. (For a uniform space the above definition of "Cauchy sequence" coincides with the usual definition. If, however, the usual definition were carried over the quasi-