[5] M. Auslander and I. Reiten, *Stable equivalence of dualizing R-varieties II: Hereditary dualizing R-varieties*, ibid. 17 (1975), pp. 93–121.
[6] — *Representation theory of Artin algebras IV: Invariants given by almost split sequences*, Comm. in Algebra 5 (1977), pp. 443–518.
[7] C. U. Jensen and D. Simson, *Purity and generalized chain conditions*, J. Pure Appl. Algebra 14 (1979), pp. 297–305.
[8] J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965), pp. 211–264.
[9] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), pp. 1–161.
[10] J. E. Roos, *Locally noetherian categories and generalized linearly compact rings. Applications*, Lecture Notes in Math. 92 (1969), pp. 197–277.
[11] D. Simson, *On colimits of injectives in Grothendieck categories*, Arkiv för Matematik 12 (1974), pp. 161–165.
[12] — *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. 96 (1977), pp. 91–116.
[13] D. Simson, *On pure semi-simple Grothendieck categories I*, Fund. Math. 100 (1978), pp. 211–222.
[14] — *Pure semi-simple categories and rings of finite representation type*, J. Algebra 48 (1977), pp. 290–296.
[15] — *Categories of representations of species*, J. Pure Appl. Algebra 14 (1979), pp. 101–114.
[16] — and A. Skowroński, *On the category of commutative connected graded Hopf algebras over a perfect field*, Fund. Math. 101 (1978), pp. 137–149.
[17] A. Skowroński, *On the category of abelian Hopf algebras over a nonperfect field*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), pp. 675–682.
[18] — *The category of abelian Hopf algebras*, Fund. Math. 107 (1980), pp. 167–193.

INSTITUTE OF MATHEMATICS
NICHOLAS COPERNICUS UNIVERSITY

Accepté par la Rédaction le 19. 7. 1978

Irreducible continua with degenerate end-tranches and arcwise accessibility in hyperspaces

by

J. Grispolakis and **E. D. Tymchatyn*** (Saskatoon, Sas.)

Abstract. In a 1960 paper G. W. Henderson proved that every hereditarily decomposable chainable continuum has a subcontinuum with a degenerate tranche. In this paper some other classes of hereditarily decomposable continua which also have this property are investigated. In particular it is proved that in a rational continuum of finite rim-type every point is a degenerate tranche of some continuum. An example of a hereditarily decomposable chainable continuum such that no subcontinuum has a cut-point is presented. Hence the degenerate tranches guaranteed by Henderson's construction are end-tranches. These results are used to answer several questions of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. In 1960, G. W. Henderson [2] proved that every hereditarily decomposable chainable continuum contains an irreducible subcontinuum with a degenerate end-tranche. In 1967, W. Mahavier asked whether Henderson's theorem is true for any hereditarily decomposable continuum. In Theorem 3.2, we give another class of continua with the property that they contain irreducible subcontinua with a degenerate end-tranche. In particular, we prove that hereditarily decomposable continua which contain subcontinua of finite rim-type at some point have this property. In 5.1 we give an example of a hereditarily decomposable chainable continuum with the property that no subcontinuum has a cut-point, and hence, no subcontinuum has a degenerate tranche other than an end-tranche.

In Section 4, we prove that the existence of irreducible continua with degenerate end-tranches implies the arcwise accessibility of points in hyperspaces (see Theorem 4.1), and we resolve several problems raised by Nadler in [5] and [6].

2. Preliminaries. Throughout this paper by a *continuum* we mean a connected, compact, metric space and by a *mapping* we mean a continuous function. A continuum X is said to be *chainable* (or *arc-like* or *snake-like*) provided for each $\varepsilon>0$ there exists a finite open cover $\{U_1, \ldots, U_n\}$ consisting of open sets with diameter less than ε and such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A mapping $f\colon X \to Y$

* The first author was supported by a University of Saskatchewan Postdoctoral Fellowship and the second author was supported partially by a NRC (Canada) grant No. A5616.

from a compact metric space X onto a metric space Y is said to be *monotone* provided for every connected subset K of Y $f^{-1}(K)$ is connected.

Let X be a continuum. Then the point $x \in X$ is said to be a *cut-point* of X provided that

$$X \setminus \{x\} = P \cup Q$$

where P and Q are two non-empty separated subsets of X.

The point $x \in X$ is said to be a *local separating point of* X provided there exists a neighbourhood U of x and points $p, q \in U \setminus \{x\}$ such that p and q lie in a component of U but p and q can be separated in $U \setminus \{x\}$.

Let $X_0, X_1, \ldots$ be a sequence of continua and let $f_i^{i+1}\colon X_{i+1} \to X_i$ be a mapping of X_{i+1} onto X_i for each $i \in \omega$. Then the *inverse limit* of the inverse system $\{X_i, f_i^{i+1}, \omega\}$ denoted by $\varprojlim\{X_i, f^{i+1}, \omega\}$ is the continuum

$$\{x \in \prod_{i=0}^{\infty} X_i |\ f_n^{n+1} \circ p_{n+1}(x) = p_n(x)\}$$

endowed with the subspace topology of the product $\prod_{i=0}^{\infty} X_i$ (here p_n denotes the projection of $\prod_{i=0}^{\infty} X_i$ onto X_n).

Let A be a subset of a space X. Then by $\mathrm{Cl}(A)$ we denote the closure of A in X and by $\mathrm{Bd}(A)$ we denote the boundary of A in X. A continuum is said to be *rational* provided it has a basis of open sets with countable boundaries. A continuum X is said to be *Suslinian* provided every collection of mutually disjoint non-degenerate subcontinua of X is countable. A continuum X is said to be *decomposable* provided that $X = P \cup Q$ where P and Q are proper subcontinua of X. Finally, a continuum X is said to be *hereditarily decomposable* provided every non-degenerate subcontinuum of X is decomposable. It is well-known that rational continua are Suslinian, and Suslinian continua are hereditarily decomposable.

Let A be a subset of a topological space X, and let A' denote the derived set of A. Let $A^{(0)} = A$ and by transfinite induction define $A^{(\alpha)}$ for each ordinal α, by $A^{(\alpha+1)} = (A^{(\alpha)})'$ and $A^{(\lambda)} = \bigcap \{A^{(\alpha)} |\ \alpha < \lambda\}$ for each limit ordinal λ. If C is a compact, countable subset of a metric space, then there exists a countable ordinal α such that $C^{(\alpha)} = \emptyset$. We call the smallest such ordinal α the *topological type* of C. If X is a continuum that has a basis at the point $p \in X$ consisting of open sets with countable boundaries, then we define the *rim-type of* X *at* p to be the smallest ordinal α such that X has a neighbourhood basis at p of open sets $\{U_i\}_{i \in \omega}$ such that the topological type of $\mathrm{Bd}(U_i)$ is at most α for each $i \in \omega$. Otherwise, define the rim-type of X at p to be Ω (i.e., the first uncountable ordinal).

2.1. THEOREM ([4, p. 216]). *If X is an irreducible continuum such that each indecomposable subcontinuum of X is nowhere dense, then there exists a finest monotone mapping $\varphi\colon X \to [0, 1]$ of X onto the unit interval* [0, 1]. *The point-inverses under φ are nowhere dense subcontinua of X and are called the tranches of X.*

Throughout this paper the subcontinua $\varphi^{-1}(0)$ and $\varphi^{-1}(1)$ of X will be called the *end-tranches* of X.

The following theorem is proved by using methods that are almost identical with those used in the proof of Theorem 2.2 in [1]. We include the proof for the sake of completeness.

2.2. THEOREM. *Let X be a hereditarily decomposable irreducible continuum and let p be a point of a non-degenerate end-tranche T_0 of X. Suppose also that every non-degenerate subcontinuum of X containing p is of rim-type n at the point p for some positive integer n. Then there exists an irreducible hereditarily decomposable continuum Y and a mapping $f\colon Y \to X$ of Y onto X such that:*

(i) *f is one-to-one except at countably many points, where f is two-to-one, and $f^{-1}(p)$ is degenerate;*

(ii) *Y has an end-tranche S such that $f(S) = T_0$, Y is of rim-type n at $f^{-1}(p)$, and S is of rim-type $n-1$ at $f^{-1}(p)$;*

(iii) *$X \setminus T_0$ is homeomorphic to $f^{-1}(X \setminus T_0)$ under f.*

Proof. Let $\varphi\colon X \to [0, 1]$ be a finest monotone mapping of X onto [0, 1] (see Theorem 2.1) and suppose, without loss of generality, that $T_0 = \varphi^{-1}(1)$. Let $\mathcal{U} = \{U_1, U_2, \ldots\}$ be a nested countable neighbourhood basis at p of open sets with boundaries having topological type $\leqslant n$. Since X is metric, we may assume that the boundaries of the members of $\mathcal{U}$ are pairwise disjoint.

Let $D_m = [T_0 \cap \mathrm{Bd}(U_m)] \setminus [\mathrm{Bd}(U_m)]'$ for each $m \in \{1, 2, \ldots\}$. Then for each $m \in \{1, 2, \ldots\}$ D_m is countable. Let $D_m = \{x_{m,1}, x_{m,2}, \ldots\}$ for each $m \in \{1, 2, \ldots\}$.

By induction, we construct an inverse sequence of continua as follows: Let Y_1 be the compactification of $Y_0 \setminus D_1$ (where $Y_0 = X$) which is larger than Y_0 and such that if $f_1\colon Y_1 \to Y_0$ is the extension over Y_1 of the inclusion of $Y_0 \setminus D_1 \subset Y_1$ into Y_0, then $f_1^{-1}(x_{1,i}) = \{y_{1,i}, z_{1,i}\}$. A basic open neighbourhood of $y_{1,i}$ (respectively, $z_{1,i}$) is given by $f_1^{-1}[U \cap U_1] \cup \{y_{1,i}\}$ (respectively, $f_1^{-1}[U \setminus \mathrm{Cl}(U_1)] \cup \{z_{1,i}\}$), where U is a neighbourhood of $x_{1,i}$ in Y_0. Then Y_1 is an irreducible continuum of rim-type n at $f_1^{-1}(p)$ and $f_1^{-1}(T_0) = T_1$ is an end-tranche of Y_1. Notice that the topological type of $T_1 \cap \mathrm{Bd}[\mathrm{Cl}(f_1^{-1}(U_1))]$ is less than or equal to $n-1$ since f_1 maps $T_1 \cap \mathrm{Bd}[\mathrm{Cl}(f_1^{-1}(U_1))]$ homeomorphically onto a subset of $[\mathrm{Bd}(U_1)]'$. Also, f_1 maps $\mathrm{Bd}[\mathrm{Cl}(f_1^{-1}(U_1))]$ homeomorphically onto $\mathrm{Bd}(U_1) \setminus D_1$. We identify points and subsets of $Y_0 \setminus D_1$ with their preimages in Y_1.

Suppose that $m > 1$ and $Y_1, \ldots, Y_{m-1}$ are irreducible continua, $f_i\colon Y_i \to Y_{i-1}$ is a mapping of Y_i onto Y_{i-1}, the rim-type of Y_i at $f_i^{-1} \circ \ldots \circ f_1^{-1}(p)$ is n, f_i maps $Y_i \setminus f_i^{-1} \circ \ldots \circ f_1^{-1}(D_i)$ homeomorphically onto $Y_{i-1} \setminus f_{i-1}^{-1} \circ \ldots \circ f_1^{-1}(D_i)$, and the topological type of $T_i \cap \mathrm{Bd}[\mathrm{Cl}(f_i^{-1} \circ \ldots \circ f_1^{-1}(U_i))]$ is at most $n-1$, where $T_i = f_i^{-1}(T_{i-1})$ for each $i \in \{1, \ldots, m-1\}$. We identify points and subsets in $Y_{i-1} \setminus f_{i-1}^{-1} \circ \ldots \circ f_1^{-1}(D_i)$ with their preimages under f_i for each $i \in \{1, \ldots, m-1\}$. Let Y_m be the compactification of $Y_{m-1} \setminus D_m$ that is larger than Y_{m-1} and such that if $f_m\colon Y_m \to Y_{m-1}$ is the extension over Y_m of the inclusion of $Y_{m-1} \setminus D_m \subset Y_m$ into Y_{m-1}, then $f_m^{-1}(x_{m,i}) = \{y_{m,i}, z_{m,i}\}$. A basic open neighbourhood of $y_{m,i}$ (respectively, $z_{m,i}$) is given by $f_m^{-1}[U \cap U_m] \cup \{y_{m,i}\}$ (respectively, $f_m^{-1}[U \setminus \mathrm{Cl}(U_m)] \cup \{z_{m,i}\}$), where U is a neighbourhood of $x_{m,i}$ in Y_{m-1}. We identify points and subsets of $Y_{m-1} \setminus D_m$ with their preimages in Y_m under f_m. Then Y_m is an irreducible continuum of rim-type n at the point $f_m^{-1} \circ \ldots \circ f_1^{-1}(p)$, and $T_m = f_m^{-1}(T_{m-1})$ is an end-tranche of Y_m. Notice that the topological type of $T_m \cap \mathrm{Bd}\big[\mathrm{Cl}\big(f_m^{-1} \circ \ldots \circ f_1^{-1}(U_m)\big)\big]$ is less than or equal to $n-1$, since it is mapped by $f_1 \circ \ldots \circ f_m$ homeomorphically onto a subset of $[\mathrm{Bd}(U_m)]'$. Also, $f_1 \circ \ldots \circ f_m$ maps $\mathrm{Bd}\big[\mathrm{Cl}\big(f_m^{-1} \circ \ldots \circ f_1^{-1}(U_m)\big)\big]$ homeomorphically onto $\mathrm{Bd}(U_m) \setminus D_m$.

Consider the inverse system $\{Y_m, f_m, \omega\}$ and let $Y = \varprojlim\{Y_m, f_m, \omega\}$, and let $f\colon Y \to Y_0 = X$ be the mapping induced by the inverse limit. Then f is one-to-one except at the points of the set $\bigcup_{m=1}^{\infty} D_m$, where f is two-to-one, and f maps $f^{-1} \circ \varphi^{-1}([0,1)) = f^{-1}(X \setminus T_0)$ homeomorphically onto $\varphi^{-1}([0,1)) = X \setminus T_0$. We also have that $f^{-1} \circ \varphi^{-1}([0,1))$ is dense in Y, Y is irreducible, and

$$S = \varprojlim\{T_m, f_m | T_m, \omega\}$$

is an end-tranche of Y such that $f(S) = T_0$. A basic neighbourhood of p in Y is of the form $\mathrm{Int}[\mathrm{Cl}(f^{-1}(U_i))]$ for some $U_i \in \mathscr{U}$. Then

$$\mathrm{Bd}\big[\mathrm{Int}\big(\mathrm{Cl}(f^{-1}(U_i))\big)\big] = \mathrm{Bd}\big[\mathrm{Cl}\big(f^{-1}(U_i)\big)\big]$$

is homeomorphic to $\mathrm{Bd}\big[\mathrm{Cl}\big(f_i^{-1} \circ \ldots \circ f_1^{-1}(U_i)\big)\big]$ and the latter set has topological type less than or equal to n. Also, $S \cap \mathrm{Bd}\big[\mathrm{Cl}\big(f^{-1}(U_i)\big)\big]$ is homeomorphic to $\mathrm{Bd}\big[\mathrm{Cl}\big(f_i^{-1} \circ \ldots \circ f_1^{-1}(U_i)\big)\big] \cap S$. Hence, the rim-type of Y is at most n at $f^{-1}(p)$ and the rim-type of S is at most $n-1$ at $f^{-1}(p)$.

3. Irreducible continua with degenerate end-tranches. In this section we investigate the existence of irreducible subcontinua with degenerate end-tranches in continua with certain properties. We first prove the following:

3.1. Lemma. *Let X be a hereditarily decomposable irreducible continuum with a degenerate end-tranche $\{q\}$, and let $f\colon X \to Y$ be a mapping of X onto a hereditarily decomposable continuum Y such that f is one-to-one except at a countable subset D of $X \setminus \{q\}$, where f is two-to-one. Then Y contains an irreducible subcontinuum Z with $\{f(q)\}$ being an end-tranche of Z.*

Proof. Let $\varphi\colon X \to [0,1]$ be a finest monotone mapping of X onto $[0,1]$ (see Theorem 2.1). Since D is countable and f is one-to-one on $X \setminus D$, we may assume, without loss of generality, that $\varphi^{-1}(0) \subset X \setminus D$. Assume, also, that $\varphi(q) = 1$. We shall construct, now, an irreducible subcontinuum of Y with $\{f(q)\}$ as an end-tranche.

Let $t \in (0,1)$ and define an increasing sequence $\{t(n)\}_{n\in\omega}$ of numbers in $[t,1]$ as follows: Let $t(0) = t$ and suppose that $t(i)$ has been defined for each $i \leqslant n$. Then define $t(n+1) \in [t(n), 1)$ to be such that

$$f[\varphi^{-1}(t(n))] \cap f[\varphi^{-1}(t(n+1))] \neq \emptyset,$$

and such that

$$t(n+1) - t(n) = \max\{r - t(n) | f[\varphi^{-1}(t(n))] \cap f[\varphi^{-1}(r)] \neq \emptyset\}.$$

Let $t(\omega) = \lim_{n\to\infty} t(n)$. Similarly, let $\{(n)t\}_{n\in\omega}$ be a decreasing sequence in $(0, t]$ that is defined in a way analogous to that in which $\{t(n)\}_{n\in\omega}$ was defined. Let $(\omega)t = \lim_{n\to\infty} (n)t$. Consider a continuum $P(t)$ in $\bigcup_{n\leqslant\omega} f[\varphi^{-1}(\{t(n), (n)t\})]$ which is irreducible with respect to meeting both the continua $f[\varphi^{-1}(t(\omega))]$ and $f[\varphi^{-1}((\omega)t)]$.

If for some $t \in (0,1)$ $t(\omega) = 1$, then we claim that $\{f(q)\}$ is an end-tranche of $P(t)$. For this notice that

$$\{f(q)\} = \operatorname*{Lim}_{n\to\infty} f[\varphi^{-1}(t(n))],$$

and that each tranche of $P(t)$ which is not an end-tranche is either contained in

$$f[\varphi^{-1}(t(n))] \cup f[\varphi^{-1}(t(n+1))] \quad \text{or in} \quad f[\varphi^{-1}((n)t)] \cup f[\varphi^{-1}((n+1)t)]$$

for some n. Hence, the end-tranche of $P(t)$ which contains $f(q)$ is degenerate.

We suppose, therefore, that for each $t \in (0,1)$ $t(\omega) < 1$. Notice that if $s, t \in (0,1)$ and $t < s$, then either

$$P(s) = P(t) \quad \text{or} \quad P(s) \cap P(t) = \emptyset \quad \text{or}$$
$$P(s) \cap P(t) \subset f[\varphi^{-1}(t(\omega))] \cap f[\varphi^{-1}((\omega)s)].$$

Let $t_1 \in (0,1)$ such that $t_1(\omega) - (\omega)t_1 \geqslant t(\omega) - (\omega)t$ for each $t \in (0,1)$. If $P(t_1) \cap f[\varphi^{-1}(0)] \neq \emptyset$, let $t_2 = t_1$. If $P(t_1) \cap f[\varphi^{-1}(0)] = \emptyset$, then let $t_2 \in (0, (\omega)t_1)$ such that

$$t_2(\omega) - (\omega)t_2 = \max\{t(\omega) - (\omega)t | \ t \in (0, (\omega)t_1)\}.$$

Let, also, $t_3 \in (t_1(\omega), 1)$ such that

$$t_3(\omega) - (\omega)t_3 = \max\{t(\omega) - (\omega)t | \ t \in (t_1(\omega), 1)\}.$$

Inductively, by a Cantor-type construction, we define a sequence $t_1, t_2, \ldots$ of points in $(0,1)$ such that $t_i < t_j$ implies that

$$P(t_i) \cap P(t_j) \subset f[\varphi^{-1}((\omega)t_j)] \cap f[\varphi^{-1}(t_i(\omega))],$$

and such that if $x, y \in X \setminus \bigcup_{i=1}^{\infty} \varphi^{-1}([(\omega)t_i, t_i(\omega)])$ and $f(x) = f(y)$, then $\varphi(x) = \varphi(y)$ (i.e. x and y belong to the same tranche of X).

Let P be a subcontinuum of the continuum

$$K = f\Big[X \setminus \bigcup_{i=1}^{\infty} \varphi^{-1}\big(((\omega)t_i, t_i(\omega))\big)\Big] \cup \bigcup_{i=1}^{\infty} P(t_i)$$

irreducible between $f(q)$ and $f[\varphi^{-1}(0)]$. Then, by the construction, we deduce that each tranche of P is contained in the union of the images of at most two tranches of X. Hence, $\{f(q)\}$ is an end-tranche of P.

3.2. THEOREM. *For any hereditarily decomposable continuum X and every point p of X, which belongs to a subcontinuum K of X with finite rim-type at p, there exists an irreducible subcontinuum of X with $\{p\}$ as an end-tranche.*

Proof. Let Z be an irreducible subcontinuum of K with one end-tranche T at the point p and such that Z has finite rim-type at p. The proof is by induction on the rim-type of the continuum Z at the point p.

If the rim-type of Z at p is equal to one, then Z has a basis of open sets at p with finite boundaries. If T were a non-degenerate tranche, then there would be an open neighbourhood U of p in Z such that for any open set V with $p \in V \subset U$ the boundary of V is infinite. This implies that $T = \{p\}$.

Suppose that the theorem is true for any hereditarily decomposable continuum X and any point p of X with the property that there exists a subcontinuum K of X containing p and having rim-type at most $n-1$ at p.

Let X_0 be a hereditarily decomposable continuum and p_0 a point of X_0 for which the theorem fails, and Z_0 an irreducible subcontinuum of X_0 with a non-degenerate end-tranche T_0 containing the point p_0, and such that Z_0 is of rim-type n at p_0. Then every non-degenerate subcontinuum of Z_0 which contains p_0 is of rim-type n at p_0. By Theorem 2.2, there exists an irreducible hereditarily decomposable continuum Y and a mapping $f\colon Y \to Z_0$ of Y onto Z_0 which satisfies the following properties:

(i) f is one-to-one except at countably many points where f is two-to-one and $f^{-1}(p_0)$ is degenerate;

(ii) Y has an end-tranche S such that $f(S) = T_0$, Y is of rim-type n at $f^{-1}(p_0)$, and S is of rim-type $n-1$ at $f^{-1}(p_0)$;

(iii) $Z_0 \setminus T_0$ is homeomorphic to $f^{-1}(Z_0 \setminus T_0)$ under f. By the inductive hypothesis, if $\{q_0\} = f^{-1}(p_0)$, since $q_0 \in S$ and S is of rim-type $n-1$ at q_0, we have that $\{q_0\}$ is an end-tranche of some irreducible subcontinuum J of Y. Then $f|J\colon J \to f(J)$ is a mapping which satisfies the hypothesis of Lemma 3.1, and hence, $f(J)$ contains an irreducible subcontinuum with $\{p_0\}$ being an end-tranche. With this contradiction the proof of the theorem is complete.

3.3. THEOREM. *Let X be a Suslinian continuum and let p be a point of X. Then $\{p\}$ is an end-tranche of some irreducible subcontinuum of X if and only if p belongs to a subcontinuum of X with finite rim-type at the point p.*

Proof. Let p belong to a subcontinuum of X with finite rim-type at p. Then, by Theorem 3.2, $\{p\}$ is an end-tranche of some irreducible subcontinuum of X. Conversely, let $\{p\}$ be an end-tranche of some irreducible subcontinuum J of X. Since J in Suslinian, the set of degenerate tranches of J is a dense subset of J. Now it is clear that we can form a countable neighbourhood basis $\{U_i\}_{i \in \omega}$ of p in J so

that the boundary of U_i is degenerate (in fact, it is one of the degenerate tranches of J) for each $i \in \omega$. Thus, J is of finite rim-type at the point p, and the theorem is proved.

3.4. COROLLARY. *Let X be a rational continuum of finite rim-type. Then every point of X is an end-tranche of some irreducible subcontinuum of X.*

4. Arcwise accessibility in hyperspaces. For any compact metric space we denote by 2^X (respectively, $C(X)$) the space of all non-empty, compact subsets (respectively, subcontinua) of X with the topology induced by the Hausdorff metric (for the definitions see [5] or [6, Ch. 0]). Let $C_2(X)$ denote the subspace of 2^X consisting of those compact subsets of X which have at most two components.

Let X be a continuum, and let $x \in X$. Then, $\{x\}$ is said to be *arcwise accessible from* $2^X \setminus C(X)$ (resp., $C_2(X) \setminus C(X)$) provided there exists an arc A in 2^X (resp., $C_2(X)$) with one end-point at x and such that $A \cap C(X) = \{x\}$. For a study of arcwise accessibility see [6, Chapter XII] and [5].

The following theorem gives a useful sufficient condition for a point to be arcwise accessible from $C_2(X) \setminus C(X)$.

4.1. THEOREM. *Let X be a continuum such that every indecomposable subcontinuum of X is nowhere dense and let p be a point of x such that $\{p\}$ is an end-tranche of an irreducible subcontinuum of X. Then p is arcwise accessible from $C_2(X) \setminus C(X)$.*

Proof. Let J be an irreducible subcontinuum of X such that $\{p\}$ is an end-tranche of J. By Theorem 2.1, there exists a finest monotone mapping $\varphi\colon J \to [0, 1]$ of J onto $[0, 1]$ such that $\varphi^{-1}(0) = \{p\}$.

Consider the points $A_n = \{p\} \cup \varphi^{-1}([1/2^n, 1/2^{n-1}])$ and $B_n = \{p\} \cup \varphi^{-1}(1/2^n)$ for $n \in \{1, 2, \ldots\}$ in $C_2(J) \setminus C(J)$. Let K_n be an order arc in $C_2(J) \setminus C(J)$ from A_n to B_n and let L_n be an order arc in $C_2(J) \setminus C(J)$ from B_n to A_{n+1} for each $n \in \{1, 2, \ldots\}$. Then $K_n, L_n \subset C_2(J) \setminus C(J)$ for each $n \in \{1, 2, \ldots\}$. Let

$$L = \mathrm{Cl}\Big[\bigcup_{n=1}^{\infty} (K_n \cup L_n)\Big].$$

Then we claim that

$$L = \bigcup_{n=1}^{\infty} (K_n \cup L_n) \cup \{p\}.$$

For this notice that $\operatorname{Lim}_{n\to\infty} K_n = \operatorname{Lim}_{n\to\infty} L_n = \{p\}$ and that if $C_n \in K_n$ and $D_n \in L_n$ are arbitrary points, then the sequence $C_1, D_1, C_2, D_2, \ldots$ converges to $\{p\}$. Hence, L is an arc in $C_2(J)$ such that $L \cap C(J) = \{p\}$.

4.2. COROLLARY. *Let X be a hereditarily decomposable continuum and let p be a point of X which belongs to a subcontinuum of X with finite rim-type at p. Then p is arcwise accessible from $C_2(X) \setminus C(X)$.*

Proof. This follows directly from Theorem 3.2 and Theorem 4.1.

4.3. Corollary. *Let X be a rational continuum of finite rim-type. Then every point of X is arcwise accessible from $C_2(X)\setminus C(X)$.*

Corollary 4.3 provides a partial solution to a problem of Nadler (see [5, (6.5)]). The following example shows that the converses of Theorem 4.1, Corollary 4.2 and Corollary 4.3 are not true.

4.4. Example. This is a rational continuum X such that every point is arcwise accessible from $C_2(X)\setminus C(X)$, although there are many points which do not belong to subcontinua of X which are of finite rim-type at these points.

Let $J_0, J_1, J_2, \ldots$ be a null sequence (i.e., the limit of the diameters of $J_0, J_1, J_2, \ldots$ is zero) of the Janiszewski chainable rational continuum J (see [3] and [5, (3.12)] for a complete description of this continuum). It is proved in [5, (3.12)] that J has countably infinitely many points, say $p_1, p_2, \ldots$ which are not arcwise accessible from $2^X\setminus C(X)$. Let $X_0 = J_0$. We form a continuum $X_1 = \bigcup_{i=0}^{\infty} J_i$ such that $J_0 \cap J_j = \{p_j\}$ for each $j>0$. We suppose that p_j is not arcwise accessible in $2^{J_j}\setminus (J_j)$ for each $j>0$. We assume, also, that if $0<i<j$, then $J_i \cap J_j = \emptyset$. Since X_1 is the union of a sequence of rational continua, it follows that X_1 is a rational continuum. Let $f_1\colon X_1 \to X_0$ be the function defined by

$$f_1(x) = \begin{cases} x, & \text{if} \quad x \in J_0\,, \\ p_i, & \text{if} \quad x \in J_i\,. \end{cases}$$

It is easy to check, using the fact that the sequence $J_1, J_2, \ldots$ is null, that f_1 is a mapping of X_1 onto X_0.

Let, now, $p_{i,1}, p_{i,2}, \ldots$ be the points of $J_i\setminus\{p_i\}$ which are not arcwise accessible from $2^{X_1}\setminus C(X_1)$ for each $i \in \{1, 2, \ldots\}$. Let, also, $\{J_{i,j}|\ i,j \in \{1, 2, \ldots\}\}$ be a null sequence of mutually disjoint homeomorphic copies of J. We construct a continuum

$$X_2 = X_1 \cup \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} J_{i,j}$$

so that $X_1 \cap J_{i,i} = \{p_{i,j}\}$ and the point $p_{i,j}$ is not arcwise accessible from $2^{J_{i,j}}\setminus C(J_{i,j})$. Since the sequence $\{J_{i,j}|\ i,j \in \{1, 2, \ldots\}\}$ is null, it is clear that X_2 is a rational continuum. Let $f_2\colon X_2 \to X_1$ be a function defined by

$$f_2(x) = \begin{cases} x, & \text{if} \quad x \in X_1\,, \\ p_{i,j} & \text{if} \quad x \in J_{i,j}\,. \end{cases}$$

It is easy to check that f_2 is a mapping of X_2 onto X_1.

Inductively, we construct a sequence $X_0, X_1, X_2, \ldots$ of rational continua and mappings $f_{i+1}\colon X_{i+1} \to X_i$ for each $i \in \omega$. Let $X = \varprojlim\{X_i, f_i, \omega\}$. It is easy to check that X satisfies the hypothesis of Lemma 3.15 in [5], and hence, X is a rational continuum.

We claim that every point of X is arcwise accessible from $C_2(X)\setminus C(X)$. For this notice that if $p_{i_1,i_2,\ldots,i_n}$ is a point of $J_{i_1,\ldots,i_{n-1}} \subset X_{n-1}$ which is not arcwise accessible from $C_2(X_{n-1})\setminus C(X_{n-1})$, then it becomes a cut-point of the continuum X_n by attaching to X_{n-1} the continuum $J_{i_1,\ldots,i_n}$ at the point $p_{i_1,\ldots,i_n}$. Thus, it is easy to check that if p is a point of X, then either p is a cut-point of X or $\{p\}$ is the end-tranche of some irreducible subcontinuum of X, and hence, by [5, (3.9)] and Theorem 4.1, p is arcwise accessible from $C_2(X)\setminus C(X)$.

By the construction, now, it follows that there exist countably many points in X which do not belong to non-degenerate subcontinua of finite rim-type at these points.

In [5, (6.1) and (6.6)], Nadler asks whether for any hereditarily decomposable continuum X there must be a point which is arcwise accessible from $2^X\setminus C(X)$. Theorems 4.5 and 4.7 give partial answers to this question.

4.5. Theorem. *Let X be a Suslinian continuum. Then the set D of points of X which are arcwise accessible from $C_2(X)\setminus C(X)$ is dense in X. Moreover, the set $X\setminus D$ contains no (non-degenerate) continuum.*

Proof. Let J be an irreducible subcontinuum of X. Then J is a Suslinian continuum, and hence, J is hereditarily decomposable. By Theorem 2.1, J admits a finest monotone mapping $\varphi\colon J \to [0, 1]$ onto the unit interval $[0, 1]$. It follows from the definition of a Suslinian continuum that only countably many of the tranches of J are non-degenerate. Let $t \in (0, 1)$ be a point such that $\varphi^{-1}(t)$ is degenerate. Then $A = \varphi^{-1}([0, t])$ is an irreducible subcontinuum of J with a degenerate end-tranche $\varphi^{-1}(t)$. Let $\varphi^{-1}(t) = \{p\}$. By Theorem 4.1, $\{p\}$ is arcwise accessible from $C_2(X)\setminus C(X)$. It is easy to prove now that the set D of these points which are arcwise accessible form a dense set in X, and that $X\setminus D$ contains no continuum.

Theorem 4.5 generalizes Theorem 3.11 in [5].

In [2] Henderson proved (without stating) the following result:

4.6. Theorem (Henderson [2]). *Let X be a hereditarily decomposable chainable continuum. Then X contains an irreducible subcontinuum with a degenerate end-tranche.*

Using Theorem 4.6 and Theorem 4.1 one obtains the following result:

4.7. Theorem. *Let X be a chainable hereditarily decomposable continuum. Then the set of points of X which are arcwise accessible from $C_2(X)\setminus C(X)$ is dense in X.*

W. S. Mahavier has asked whether Theorem 4.6 is true for every hereditarily decomposable continuum. A positive answer to this question combined with Theorem 4.1 would give a positive answer to Nadler's question [5, (6.1)].

Finally, we answer in the affirmative Question (11.18) in [6] by using Theorem (4.4) in [5] and simple facts about the structure of hereditarily decomposable irreducible continua.

4.8. Theorem (Nadler, [5, (4.4)]). *Let E be a decomposable non-degenerate proper subcontinuum of a continuum X. Then the following are equivalent:*

(1) *If Y is a subcontinuum of X such that*

$$Y \cap E \neq \emptyset \neq (X \setminus E),$$

then $Y \supset E$;

(2) *$C(X) \setminus \{E\}$ is not arcwise connected.*

The following theorem gives an affirmative answer to Question (11.18) in [6]:

4.9. THEOREM. *Let X be a hereditarily decomposable irreducible continuum which is not an arc. Then there exists a subcontinuum K of X and a subcontinuum E of K such that $C(K) \setminus \{E\}$ is not arcwise connected.*

Proof. Let $\varphi\colon X \to [0, 1]$ be a finest monotone mapping of X onto $[0, 1]$ (see Theorem 2.1). Since X is not an arc, there exists a point $t \in [0, 1]$ such that $\varphi^{-1}(t)$ is nondegenerate. Assume, without loss of generality, that $t \in (0, 1]$, and that $E = \mathrm{Bd}[\varphi^{-1}([0, t))]$ is a nowhere dense non-degenerate subcontinuum of $K = \mathrm{Cl}[\varphi^{-1}([0, t))]$. Let Y be a subcontinuum of K such that

$$Y \cap E \neq \emptyset \neq Y \cap (K \setminus E).$$

Then, $\varphi(Y) = [t_1, t]$ for some $0 \leqslant t_1 < t$. Then by the irreducibility of K between any point of $\varphi^{-1}(0)$ and any point of E, we infer that

$$\mathrm{Cl}[\varphi^{-1}((t_1, t))] \subset Y,$$

and hence, $E \subset Y$. By Theorem 4.8, $C(K) \setminus \{E\}$ is not arcwise connected.

5. Examples. In this section we present two constructions. Example 5.1 is a hereditarily decomposable chainable continuum such that no subcontinuum has a cut-point, (compare with 4.6) and hence no non-degenerate subcontinuum is Suslinian. Example 5.2 is a Suslinian continuum such that no non-degenerate subcontinuum is rational. These two examples resolve in the affirmative Problems (8.2) and (8.3) in [5], respectively.

5.1. EXAMPLE. Let C be the Cantor ternary set in $[0, 1]$. By an *adjacent segment to C* we mean the closure of a component of $[0, 1] \setminus C$. Let N denote the set of all positive integers.

Let $Y_0 = C \times [0, 1]$ and let $\sim_0$ be an equivalence relation on Y_0 defined by $(c_0, t) \sim_0 (c_0', t')$ if and only if either $(c_0, t) = (c_0', t')$ or c_0 and c_0' are the end-points of some adjacent segment to C and

$$t = t' = \begin{cases} 0, & \text{if } c_0 = p/3^{2n} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p, \\ 1, & \text{if } c_0 = p/3^{2n-1} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p. \end{cases}$$

Let $X_0 = Y_0/\sim_0$ and let $\pi_0\colon Y_0 \to X_0$ be the natural projection of Y_0 onto X_0. Then a simple chaining argument can be used to prove that X_0 is a hereditarily decomposable chainable continuum.

Let $Y_1 = C \times X_0$ and let $\sim_1$ be an equivalence relation on Y_1 defined by $(c_1, \pi_0(c_0, t)) \sim_1 (c_1', \pi_0(c_0', t'))$ if and only if either $(c_1, \pi_0(c_0, t)) = (c_1', \pi_0(c_0' t'))$ or c_1 and c_1' are the end-points of some adjacent segment to C, $t = t'$ and

$$c_0 = c_0' = \begin{cases} 0, & \text{if } c_1 = p/3^{2n} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p, \\ 1, & \text{if } c_1 = p/3^{2n-1} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p. \end{cases}$$

Let $X_1 = Y_1/\sim_1$ and let $\pi_1\colon Y_1 \to X_1$ be the natural mapping of Y_1 onto X_1. Then we can show that X_1 is a hereditarily decomposable chainable continuum. Define a mapping $\varphi_1\colon Y_1 \to Y_0$ by

$$\varphi_1(c_1, \pi_0(c_0, t)) = (c_1, \psi_0 \circ \pi_0(c_0, t))$$

for any point $(c_1, \pi_0(c_0, t))$ of Y_1, where $\psi_0\colon X_0 \to [0, 1]$ is a finest monotone mapping of X_0 onto $[0, 1]$ (see Theorem 2.1). Then φ_1 is monotone. It is clear now that there exists a unique mapping $f_0^1\colon X_1 \to X_0$ such that $\pi_0 \circ \varphi_1 = f_0^1 \circ \pi_1$, and that f_0^1 is monotone. Let also $\psi_1\colon X_1 \to [0, 1]$ be the finest monotone mapping of X_1 onto $[0, 1]$ such that $h \circ \psi_1 = \psi_0 \circ f_0^1$, where $h\colon [0, 1] \to [0, 1]$ is the identity mapping.

Let $Y_2 = C \times X_1$ and let $\sim_2$ be an equivalence relation on Y_2 defined by

$$(c_2, \pi_1(c_1, \pi_0(c_0, t))) \sim_2 (c_2', \pi_1(c_1', \pi_0(c_0', t')))$$

if and only if either

$$(c_2, \pi_1(c_1, \pi_0(c_0, t))) = (c_2', \pi_1(c_1', \pi_0(c_0', t')))$$

or c_2 and c_2' are the end-points of some adjacent segment to C, $\pi_0(c_0, t) = \pi_0(c_0', t')$ and

$$c_1 = c_1' = \begin{cases} 0, & \text{if } c_2 = p/3^{2n} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p, \\ 1, & \text{if } c_2 = p/3^{2n-1} \text{ for some } p,\ n \in N \text{ with } 3 \nmid p. \end{cases}$$

Let $X_2 = Y_2/\sim_2$ and let $\pi_2\colon Y_2 \to X_2$ be the natural mapping of Y_2 onto X_2. Then we can show that X_2 is a hereditarily decomposable chainable continuum. Define a mapping $\varphi_2\colon Y_2 \to Y_1$ by

$$\varphi_2(c_2, \pi_1(c_1, \pi_0(c_0, t))) = (c_2, \pi_0(c_1, \psi_0 \circ \pi_0(c_0, t)))$$

for any point $(c_2, \pi_1(c_1, \pi_0(c_0, t)))$ of Y_2. Then φ_2 is monotone. Let $f_1^2\colon X_2 \to X_1$ be the unique mapping of X_2 onto X_1 such that $\pi_1 \circ \varphi_2 = f_1^2 \circ \pi_2$. It is clear again that f_1^2 is monotone. Let also $\psi_2\colon X_2 \to [0, 1]$ be the finest monotone mapping of X_2 onto $[0, 1]$ such that $h \circ \psi_2 = \psi_1 \circ f_1^2$.

Inductively, we define an inverse system $\{X_i, f_i^{i+1}, \omega\}$ of hereditarily decomposable chainable continua $X_0, X_1, \ldots$ and monotone mappings $f_i^{i+1}\colon X_{i+1} \to X_i$ for each $i \in \omega$. Let $X = \varprojlim\{X_i, f_i^{i+1}, \omega\}$, $f_n\colon X \to X_n$ be the mapping induced by the inverse limit, and let $\psi\colon X \to [0, 1]$ be the mapping of X onto $[0, 1]$ induced by the inverse limit of the mappings ψ_i. Let $n \in N$. Then we have that $\psi_n \circ f_n = h \circ \psi$. It can be proved, also, that f_n is monotone, and, hence, ψ is a monotone finest mapping of X onto $[0, 1]$. Since X is the inverse limit of chainable continua with onto bonding mappings, X is a chainable continuum. It is clear, also, that for each $\varepsilon > 0$ there exists a positive integer $n(\varepsilon)$ such that the mapping $f_{n(\varepsilon)}$ is an ε-mapping (i.e. $f_{n(\varepsilon)}^{-1}(x)$ has diameter less than ε for each $x \in X_{n(\varepsilon)}$). Hence, since $X_{n(\varepsilon)}$ is hereditarily decomposable and f_n is monotone, we deduce that X is a hereditarily decomposable continuum.

Next, we prove that no subcontinuum of X has a cut-point. For this let K be a non-degenerate subcontinuum of X and suppose, on the contrary, that there exists a point $x \in X$ such that $K \setminus \{x\} = P \cup Q$, where P and Q are two disjoint non-empty open subsets of K. Then there exists a positive integer n such that $f_n(K)$ is a non-degenerate subcontinuum of X_n, and such that $f_n(x)$ is a cut-point of $f_n(K)$. Let

$$K' \setminus \{x'\} = P' \cup Q'$$

where $K' = f_n(K)$, $x' = f_n(x)$, $P' = f_n(P) \setminus \{x'\}$ and $Q' = f_n(Q) \setminus \{x'\}$. Then P' and Q' are two disjoint open subsets of K'. Consider now the continuum $f_{n+2}(K)$. Then it is easy to check that $f_{n+2}(K)$ is a non-degenerate subcontinuum of X_{n+2} such that $f_{n+2}(x)$ is a cut-point of $f_{n+2}(K)$. Notice that the point $f_n(x) = x'$ in X_n is such that

$$A = (f_{n+1}^{n+2})^{-1} \circ (f_n^{n+1})^{-1}(x')$$

i.e. the preimage of x' under the mapping $f_n^{n+1} \circ f_{n+1}^{n+2})$ is a homeomorphic copy of X_0, and that the continuum $f_{n+2}(K)$, being irreducible, contains A. It is easy to see, now, that A does not contain any cut-point of the continuum $f_{n+2}(K)$, which contradicts the assumption that the point $f_{n+2}(x)$ belongs to A and is a cut-point of $f_{n+2}(K)$.

It is easy to show, now, that X has no non-degenerate subcontinuum which is Suslinian. For this let K be a non-degenerate subcontinuum of X. Then K is chainable, and hence, irreducible. Let $\varphi\colon K \to [0, 1]$ be a finest monotone mapping of K onto $[0, 1]$ (see Theorem 2.1). Then, as above, K does not have any cut-point. Thus, $\varphi^{-1}(t)$ is non-degenerate for each $t \in [0, 1]$, which implies that K is a non-Suslinian continuum.

We should notice that the continuum X of Example 5.1 is a hereditarily decomposable continuum with the property that no non-degenerate subcontinuum of X is rational. Therefore, Example 5.1 already provides with an affirmative answer Problem (6.3) in [5]. The purpose of presenting Example 5.2 is to show that there exists an even "better" continuum (namely, Suslinian) which contains no non-degenerate rational subcontinuum.

5.2. Example. In [7, Example 1], the second author constructed a Suslinian but not rational continuum which is the confluent image of a rational continuum. Using the same ideas we shall construct our continuum.

Let S be the Sierpiński triangular curve (see [4, p. 276]) defined as follows: Let T be the equilateral triangle in the plane with vertices $(0, 0)$, $(1, 1)$ and $(\sqrt{2}, 0)$. Partition T into four congruent triangles T_0, T_1, T_2, T_3. Let T_0, T_1, T_2 be the triangles which have a vertex in common with T. The triangles T_0, T_1 and T_2 are numbered clockwise and T_0 is the left-most triangle of the three. Let v_0, v_1, v_2 be the vertices

of T_3 where v_0 is the left-most vertex of the three and the numbering is clockwise. In a similar way partition each of the triangles T_i for $i = 0, 1, 2$ into four congruent triangles $T_{i,0}$, $T_{i,1}$, $T_{i,2}$, $T_{i,3}$, where $T_{i,3}$ is the triangle which has no vertices in common with T_i. Let $v_{i,0}$, $v_{i,1}$ and $v_{i,2}$ be the vertices of $T_{i,3}$. The vertices $v_{i,0}$, $v_{i,1}$, $v_{i,2}$ and the triangles $T_{i,0}$, $T_{i,1}$, $T_{i,2}$ are numbered clockwise starting with the left-most one. Continuing inductively in this manner, let

$$S = \mathrm{Cl}\big[\bigcup_D \mathrm{Bd}\,(T_{\alpha_1,\ldots,\alpha_k})\big]$$

where $D = \{(\alpha_1, \ldots, \alpha_k) \mid k \in \{1, 2, \ldots\}$ and $\alpha_1, \ldots, \alpha_k \in \{0, 1, 2\}\}$. The local separating points of S are the vertices $v_{\alpha_1,\ldots,\alpha_k}$, where $(\alpha_1, \ldots, \alpha_k) \in D$.

Our example is obtained from the Sierpiński curve S as an inverse limit by successively exploding the local separating points to homeomorphic copies of S.

Let $X_0 = S$. Let $X_1 = [S \setminus \{v_0, v_1, v_2\}] \cup \bigcup_{i=0}^{2} S_i$ be the compactification of $S \setminus \{v_0, v_1, v_2\}$ with remainder consisting of three homeomorphic copies S_0, S_1 and S_2 of S in such a way that for each $i \in \{0, 1, 2\}$ if p is a local separating point of S_i, then p is not a local separating point of X_1. Moreover, X_1 is such that there exists a mapping $f_0^1\colon X_1 \to X_0$ which carries $S \setminus \{v_0, v_1, v_2\}$ homeomorphically onto itself in X_0 and such that for each $i \in \{0, 1, 2\}$ $f_0^1(S_i) = \{v_i\}$. We identify the points of $S \setminus \{v_0, v_1, v_2\}$ with their preimages in X_1. Let

$$X_2 = \Big[X_1 \setminus \Big(\big(\bigcup_{i=0}^{2} \{v_{i,0}, v_{i,1}, v_{i,2}\}\big) \cup \big(\bigcup_{j=0}^{2} \{v_0^j, v_1^j, v_2^j\}\big)\Big)\Big] \cup \bigcup_{i=0}^{2} \bigcup_{k=0}^{2} S_{i,k} \cup \bigcup_{j=0}^{2} \bigcup_{n=0}^{2} S_j^n$$

be a continuum, where v_0^j, v_1^j, v_2^j are those local separating points of S_j which are the homeomorphic images of v_0, v_1, v_2, respectively, and $S_{i,k}$ and S_j^n homeomorphic copies of S. The continuum X_2 is constructed as a compactification of

$$X_1 \setminus \Big(\big(\bigcup_{i=0}^{2} \{v_{i,0}, v_{i,1}, v_{i,2}\}\big) \cup \big(\bigcup_{j=0}^{2} \{v_0^j, v_1^j, v_2^j\}\big)\Big)$$

with remainder consisting of eighteen homeomorphic copies $S_{0,0}, \ldots, S_{2,2}, S_0^0, \ldots, S_2^2$ of S in such a way that for each $i, k \in \{0, 1, 2\}$ if p is a local separating point of either $S_{i,k}$ or S_i^k, then p is not a local separating point of X_2. Moreover, X_2 is such that there exists a mapping $f_1^2\colon X_2 \to X_1$ which carries

$$X_1 \setminus \Big(\big(\bigcup_{i=0}^{2} \{v_{i,0}, v_{i,1}, v_{i,2}\}\big) \cup \big(\bigcup_{j=0}^{2} \{v_0^j, v_1^j, v_2^j\}\big)\Big)$$

homeomorphically onto itself in X_1 and such that

$$f_1^2(S_{i,k}) = \{v_{i,k}\} \quad \text{and} \quad f_1^2(S_j^n) = \{v_j^n\} \quad \text{for each } i, k, j, n \in \{0, 1, 2\}.$$

By continuing inductively in this manner, we construct an inverse system $\{X_i, f_i^{i+1}, \omega\}$ of continua and mappings $f_i^{i+1}\colon X_{i+1} \to X_i$ for each $i \in \omega$. Let $X = \varprojlim \{X_i, f_i^{i+1}, \omega\}$. Then in a way similar to that in [7, Example 1] we can check that X is a Suslinian continuum, which is not rational. Let K be a non-degenerate subcontinuum of X. Then, by the construction of X, we infer that K contains a homeomorphic copy of X. Thus, K is not rational.

Added in proof. Prof. L. G. Oversteegen has pointed out to the authors that Example 5.1 of this paper has the same properties as the example on pp. 50–53 of E. S. Thomas, Jr. *Monotone decompositions of irreducible continua*, Dissertationes Math. 50 (1966), pp. 1–13.

References

[1] J. Grispolakis and E. D. Tymchatyn, *On the existence of arcs in rational curves*, Fund. Math., 108 (1980), pp. 23–26.
[2] G. W. Henderson, *Proof that every compact decomposable continuum which is topologically equivalent to each of its non-degenerate subcontinua is an arc*, Ann. of Math. 72 (1960), pp. 421–428.
[3] Z. Janiszewski, *Über die Begriffe "Linie" und "Fläche"*, Proc. Cambridge Internat. Congr. Math. 2 (1912), pp. 126–128.
[4] K. Kuratowski, *Topology II*, New York–London–Warszawa 1968.
[5] S. B. Nadler, Jr., *Arcwise accessibility in hyperspaces*, Dissertationes Math. 138 (1976), pp. 1–33.
[6] — *Hyperspaces of Sets*, New York 1978.
[7] E. D. Tymchatyn, *Some rational continua*, to appear.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SASKATCHEWAN
Saskatoon, Saskatchewan
Canada

Current address of J. Grispolakis
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CRETE
Iraklion, Crete,
Greece

Accepté par la Rédaction le 14. 8. 1978

On the 2-homogeneity of Cartesian products

by

K. Kuperberg, W. Kuperberg, and **W. R. R. Transue** (Auburn, Ala.)

Dedicated to the memory of Ralph Bennett

Abstract. The Cartesian product of the circle S^1 and the Menger universal curve M is not 2-homogeneous. This solves two problems: one of R. Bennett and one of G. S. Ungar. Some generalizations of this result are given.

1. Introduction. A space X is *n-homogeneous* (see [8], [4], [7]) if for every pair A, B of n-element subsets of X there exists a homeomorphism of X onto X which maps A onto B. A space is *homogeneous* if it is 1-homogeneous. A space X is *countable dense homegeneous* (Bennett, [3]) if for any pair A, B of countable dense subsets of X there exists a homeomorphism of X onto X which maps A onto B. Connected manifolds without boundary are the simplest and the most natural examples of spaces which satisfy all of these homogeneity conditions.

R. D. Anderson proved in [1] that the Menger universal curve M is n-homogeneous for every n. Using another result of R. D. Anderson [2] concerning the homogeneity of curves, R. Bennett [3] showed that M is countably dense homogeneous. Looking for higher dimensional countable dense homogeneous continua which are not manifolds, R. Bennett asked: "Is the property of being countable dense homogeneous preserved in Cartesian products?"

Investigating the n-homogeneous spaces, G. S. Ungar [7] proved that every 2-homogeneous metric continuum is locally connected, which solved a problem of C. E. Burgess [4]. Consequently, in a private conversation, Ungar asked if there exists a homogeneous locally connected metric continuum which is not 2-homogeneous.

In this paper we prove that the product of the circle S^1 and the universal curve M is not countable dense homogeneous, or even 2-homogeneous. In fact, every homeomorphism h of $S^1 \times M$ onto $S^1 \times M$ preserves the circular fibers, i.e. for every point $a \in M$ there exists a point $b \in M$ such that $h(S^1 \times \{a\}) = S^1 \times \{b\}$. This solves both Ungar's and Bennett's problems.

2. Terminology and notation. By a *space* we will understand a compact metric space. A *continuum* is a connected space. A *map* is a continuous function. A map is *inessential*, if it is homotopic to a constant map, otherwise it is *essential*. Given