a homeomorph of the Cantor set. Next observe that (iv) implies that

$$\bigcap \{ \pi_1(f(V_{\psi^m})): \{e_1, e_2, \ldots, e_n\} \in \mathcal{S}_k \} \neq \emptyset$$

for each $k \geq 1$.

It now follows that the set $\pi_1(f(C))$ contains exactly one point, say $x_0$. Finally, conditions (i) and (iv) imply that $\pi_1 \circ f$, restricted to $C$, is one-one. So $\mathcal{A}^m$ contains a homeomorph of the Cantor set.

We now turn to the proof of the theorem. Assume that $\mathcal{A}^n$ is countable for each $x \in X$. Then, by Lemma 9, there is $\lambda < \omega_1$ such that $Z_\lambda = \emptyset$. Now Lemma 7 with $\alpha = \lambda$ and $B = \emptyset$ yields a set $H \in \mathcal{G}_\alpha$ such that $A = f(C) \subseteq H$. This completes the proof of the theorem.

Discussions with H. Sarbadhikari and S. M. Srivastava are gratefully acknowledged.

References


Indian Statistical Institute
Coochbehar, India

Accepté par la Redaction le 19. 7. 1978

On pure semi-simple Grothendieck categories II

by

Daniel Simson (Toruń)

Abstract. Given a pure semi-simple Grothendieck category $\mathcal{A}$, we construct a new pure semi-simple functor category $I(\mathcal{A})$ such that $\text{gl.dim} \mathcal{A} = \text{gl.dim} I(\mathcal{A})$. The map $\mathcal{A} \to I(\mathcal{A})$ defines a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. Applications of this result are given.

Introduction. In [13] the notion of a pure semi-simple Grothendieck category is introduced as a "pure" counterpart of semi-simple categories (cf. [9]). We recall that a Grothendieck category is pure semi-simple if each of its objects is a direct sum of finitely presented objects. Pure semi-simple Grothendieck categories are investigated in [11]-[17].

In the present paper we give two constructions of new pure semi-simple Grothendieck categories from a given pure semi-simple one. Given a pure semi-simple Grothendieck category $\mathcal{A}$, a pure semi-simple functor category $I(\mathcal{A})$ is constructed in such a way that $\text{gl.dim} \mathcal{A} = \text{gl.dim} I(\mathcal{A})$ and the category of all noetherian injective objects in $\mathcal{A}$ are equivalent to the category of all finitely generated projective objects in $I(\mathcal{A})$. Further, given a skeletally small additive category $\mathcal{B}$ such that the functor category $\mathcal{B}$-Mod is locally coherent and $\mathcal{B}$-Mod is pure semi-simple, a pure semi-simple Grothendieck category $\mathcal{B}$ is constructed. The map $\mathcal{A} \to I(\mathcal{A})$ is the inverse (with respect to an equivalence) of the map $\mathcal{A}$-Mod $\to \mathcal{B}$, and conversely. These maps define a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. In Section 2 we illustrate our constructions by simple examples.

In Section 1 we recall from [12]-[14] some background material on functor categories and pure semi-simple Grothendieck categories. An extension of Theorem A in [3] is given.

Section 2 contains the constructions and main results mentioned above. As a consequence of our general considerations we get the following two corollaries. Any injective noetherian object of a pure semi-simple Grothendieck category has a right pure semi-simple endomorphism ring. If $\mathcal{B}$ is a skeletally small abelian category such that the category $\mathcal{B}$-Mod is perfect, then $\mathcal{B}$-Mod is locally noetherian.
§ 1. Notation and preliminaries. We start by recalling the terminology and notation from [9] and [12]-[14]. If \( \mathcal{A} \) is a locally finitely presented Grothendieck category, we denote by \( \text{fp}(\mathcal{A}) \) the full subcategory of \( \mathcal{A} \) consisting of all finitely presented objects. By an additive category we mean a category with an abelian group structure on each of its Hom sets.

Let \( \mathcal{C} \) be a skeletally small additive category. We denote by \( \mathcal{C} \text{-Mod} \) the category of all covariant additive functors from \( \mathcal{C} \) to the category \( \mathcal{A} \) of abelian groups. The objects of \( \mathcal{C} \text{-Mod} \) are called \( \mathcal{C} \text{-modules} \). Given an object \( X \in \mathcal{C} \), the \( \mathcal{C} \text{-module} (X, -) \) is denoted by \( H^X \) and the \( \mathcal{C} \text{-module} (-, X) \) is denoted by \( H_X \). For convenience we will write \( F_X \) instead of \( F \circ H_X \). The category of finitely generated projective \( \mathcal{C} \text{-modules} \) will be denoted by \( \text{pg}(\mathcal{C}) \).

If \( \mathcal{C} \) is a full additive subcategory of the category \( \mathcal{C} \), we denote by \( \mathcal{C} \) the two-sided ideal in \( \mathcal{C} \) generated by \( \mathcal{C} \). It is clear that \( \mathcal{C} \) is the subfunctor of the two variable functor \( (-, ?): \mathcal{C} \times \mathcal{C} \rightarrow \text{Ab} \) defined as follows. Given two objects \( X \) and \( Y \) in \( \mathcal{C} \), the group \( \mathcal{C}(X, Y) \) is the subgroup of \( \mathcal{C}(X, Y) \) consisting of all finitely generated projective objects in \( \mathcal{C} \) which factor through objects in \( \mathcal{C} \).

We often use the following result proved in [12], § 6.

**Proposition 1.1.** If \( \mathcal{C} \) is a skeletally small additive category, then there exists an equivalence \( \text{fp}(\mathcal{C}) \simeq \text{fp}(\mathcal{C}) \).

We recall that \( \mathcal{C} \text{-Mod} \) is perfect (resp. semi-perfect) if every \( \mathcal{C} \text{-module} \) (resp. every finitely generated \( \mathcal{C} \text{-module} \) has a projective cover (cf. [11], [12]). The following lemma results from Lemma 1.1 in [14] and in fact is proved in [14], p. 293.

**Lemma 1.2.** Let \( \mathcal{C} \) be a skeletally small additive category and suppose that every object in \( \mathcal{C} \) is a finite direct sum of objects having local endomorphism rings. Then \( \mathcal{C} \text{-Mod} \) is semi-perfect.

For other basic notation and properties of pure semi-simple Grothendieck categories the reader is referred to [9] and [12]-[14]. We are now going to investigate pure semi-simple Grothendieck categories. The following result is an extension of a theorem of Auslander [3] from module categories over Artin algebras to arbitrary Grothendieck categories.

**Theorem 1.3.** Let \( \mathcal{A} \) be a locally finitely presented Grothendieck category. The following statements are equivalent:

1. \( \mathcal{A} \) is pure semi-simple.
2. \( \mathcal{A} \) is locally noetherian, and given any sequence
   \[
   X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_n \rightarrow \cdots
   \]
   of monomorphisms between indecomposable noetherian objects in \( \mathcal{A} \), there is an integer \( n \) such that \( f_i \) is an isomorphism for all \( i \geq n \).
3. \( \mathcal{A} \) is locally noetherian and every indecomposable object in \( \mathcal{A} \) is noetherian.

**Proof.** The implications (a) \( \Rightarrow \) (b) and (b) \( \Rightarrow \) (c) follow from Theorem 6.3 in [12]. The implications (b) \( \Rightarrow \) (a) and (c) \( \Rightarrow \) (a) may be proved by the method of Auslander [3].

For the convenience of the reader we sketch the proof. Let assume that \( \mathcal{A} \) is locally noetherian and pure semi-simple. It follows from Theorem 1.9 in [13] that the category \( \text{fp}(\mathcal{A}) \text{-Mod} \) is not semisimple, i.e. that there exists an \( \text{fp}(\mathcal{A}) \text{-module} \)
\[
F: \text{fp}(\mathcal{A}) \rightarrow \mathcal{A}
\]
which has no simple submodules, or equivalently, given an object \( A \) in \( \text{fp}(\mathcal{A}) \) and a nonzero element \( x \) in \( F(A) \) there is a morphism \( f: A \rightarrow A \)
in \( \text{fp}(\mathcal{A}) \) which is not a split monomorphism and such that \( f(x) = 0 \) (use arguments from the proof of Proposition 2.9 in [2]). Since each object in \( \text{fp}(\mathcal{A}) \) is noetherian, one can construct a sequence (see the proof of Theorem 1.5 in [3])
\[
M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_i \rightarrow M_i
\]
in \( \text{fp}(\mathcal{A}) \) such that \( M_i \) is indecomposable noetherian for each \( i = 1, 2, \ldots \), each \( f_i \) is a proper monomorphism and \( \text{lim}(M_i) \) is indecomposable and not noetherian. Then the required implications follow and the theorem is proved.

Let us replace in condition (b) of Theorem 1.3 the word “monomorphisms” by the word “epimorphisms” and denote the new condition by (b'). It follows from Theorem 2.4 in [14] that (b) and (b') are equivalent under the additional assumption that \( \mathcal{A} \) has only a finite number of isomorphism classes of simple objects. Now we give an example showing that (b) and (b') are not equivalent for \( \mathcal{A} \) locally finite.

**Example 1.4.** Let \( K \) be a perfect field of finite characteristic \( p > 0 \) and denote by \( \mathcal{A}_p \), the category of all bicommutative graded connected primitive generated Hopf \( K \)-algebras which are generated by elements of degrees \( 2p^i, i = 0, 1, 2, \ldots \) (see [8], [16], [17]). Let \( \mathcal{A}_p \) be the full subcategory of \( \mathcal{A}_p \) consisting of all directed unions of finite dimensional Hopf algebras in \( \mathcal{A}_p \). It is clear that \( \mathcal{A}_p \) is hereditary and by Theorem 5.3 in [16] it is pure semi-simple. It follows that \( \mathcal{A}_p \) is also hereditary and pure semi-simple. Denote by \( \mathcal{A}_p \) the category Lex \( \mathcal{A}_p \) of all left exact covariant additive functors from \( \text{fp}(\mathcal{A}_p) \) to \( \mathcal{A}_p \). Since \( \text{fp}(\mathcal{A}_p) \) is fp\( \mathcal{A}_p \), then by Theorem 6.3 in [12] condition (b') is satisfied for \( \mathcal{A} = \mathcal{A}_p \) whereas (b) does not hold (see proof of Corollary 5.5 in [16]).

§ 2. Categories \( I(\mathcal{C}) \) and \( \mathcal{C} \). Throughout this section \( \mathcal{A} \) will denote a Grothendieck category, and by \( I(\mathcal{A}) \) we denote the full subcategory of \( \mathcal{A} \) consisting of finite direct sums of indecomposable injective objects. Let us consider the category
\[
I(\mathcal{A}) = I(\mathcal{A}) \text{-Mod}.
\]
We are going to prove that \( I(\mathcal{A}) \) is pure semi-simple if and only if \( \mathcal{A} \) is such. Before proving our main results, we shall need some preliminary facts. We start with the following useful proposition.

**Proposition 2.1.** Suppose that \( \mathcal{C} \) is a skeletally small additive category such that every finitely generated projective \( \mathcal{C} \text{-module} \) is a direct sum of indecomposable modules. Let \( \mathcal{C} \) be a full additive subcategory of \( \mathcal{C} \). Then \( \mathcal{C} \text{-Mod} \) is perfect if and only if \( \mathcal{C} \text{-Mod} \) and \( \mathcal{C}(\mathcal{C}-\text{-Mod}) \) are perfect.

**Proof.** It follows from Proposition 3.1 in [1] that \( \text{pr}_{\mathcal{C} \text{-Mod}} \) can be considered as a full subcategory of \( \mathcal{C} \text{-Mod} \). Since there are natural equivalences \( \mathcal{C} \text{-Mod} \simeq \mathcal{C}(\mathcal{C}-\text{-Mod}) \).
and \( \mathcal{C}_0\text{-Mod} = \text{pr}_0^\mathcal{C}\text{-Mod} \), we have reduced the proposition to the case where \( \mathcal{C} \) and \( \mathcal{C}_0 \) are closed under direct sums. Now, using the same type of arguments as in the proof of Proposition 2.1 in [14], we get the desired result.

**Corollary 2.2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be locally finitely presented Grothendieck categories such that each object in \( \text{fp}(\mathcal{A}) \) and also each object in \( \text{fp}(\mathcal{B}) \) are finite direct sums of indecomposable objects. Suppose that \( \mathcal{C} \) is a full additive subcategory of \( \text{fp}(\mathcal{A}) \) and \( \mathcal{C} \) is a full additive subcategory of \( \text{fp}(\mathcal{B}) \) such that there are natural equivalences \( \mathcal{C} \cong \mathcal{C}' \) and \( \text{fp}(\mathcal{A})(\mathcal{C}) \cong \text{fp}(\mathcal{B})(\mathcal{C}) \). Then \( \mathcal{A} \) is pure semi-simple if and only if \( \mathcal{B} \) is such.

**Proof.** Apply Theorem 6.3 in [12] and the previous proposition.

We will also need the following result.

**Proposition 2.3.** Let \( \mathcal{A} \) be a locally noetherian Grothendieck category and suppose that each indecomposable injective object in \( \mathcal{A} \) is noetherian. Then there exist equivalences of categories

\[
\begin{align*}
\text{fp}(\mathcal{A}) & \cong \text{fp}^V(\mathcal{A}) \\
\text{fp}(\mathcal{A})(\text{i}(\mathcal{A})) & \cong \text{fp}^V(\mathcal{A})(\text{i}(\mathcal{A})) \\
\text{i}(\mathcal{A}) & \cong \text{pr}^V(\mathcal{A})
\end{align*}
\]

**Proof.** Let us consider a contravariant functor

\[ h^* : \mathcal{A} \to \text{i}(\mathcal{A})\text{-Mod} \]

defined by \( h^* = (\cdot, \cdot) \). Suppose \( X \) is an object in \( \text{fp}(\mathcal{A}) \). By our assumption there exists in \( \mathcal{A} \) an exact sequence

\[ 0 \to X \to Q_0 \to Q_1 \]

with \( Q_0 \) and \( Q_1 \) in \( \text{i}(\mathcal{A}) \). We derive the exact sequence in \( \text{i}(\mathcal{A})\text{-Mod} \)

\[ \mu^Q : h^Q \to h^Q \to h^Q \to 0 \]

and therefore \( h^* \) is a finitely presented \( \text{i}(\mathcal{A})\text{-Mod} \) module. Let \( h^*_0 : \text{fp}(\mathcal{A}) \to \text{fp}^V(\mathcal{A}) \) be the restriction of \( h^* \) to \( \text{fp}(\mathcal{A}) \). We claim that \( h^*_0 \) is an equivalence. First we point out that \( h^*_0 \) is faithful because our assumption every noetherian object in \( \mathcal{A} \) can be embedded into an injective object in \( \text{i}(\mathcal{A}) \). Now we prove that \( h^*_0 \) is full. Let \( f : M \to M' \) be a morphism in \( \text{fp}_0(\mathcal{A}) \). Then there exists a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\mu_1 \downarrow & \mu_2 \downarrow & \mu_3 \downarrow & \mu_4 \downarrow & \mu_5 \downarrow & \mu_6 \downarrow \\
M \to M' \\
\text{fp}_0(\mathcal{A}) \\
\text{fp}_0(\mathcal{A}) \\
\end{array}
\]

where \( \mu_1 : Q_0 \to Q_0 \) and \( \mu_2 : Q_1 \to Q_1 \) are in \( \text{i}(\mathcal{A}) \). We have a commutative diagram in \( \text{fp}(\mathcal{A}) \)

\[
\begin{array}{cccccc}
0 \to & \text{Ker} f & \to & Q_0 & \to & Q_1 \\
\downarrow & \downarrow f & \downarrow f & \downarrow & \downarrow & \downarrow \\
0 & \to & \text{Ker} f' & \to & Q_0' & \to & Q_1'
\end{array}
\]

It then follows that \( f = f_0 \) and therefore \( h^*_0 \) is an equivalence. The existence of equivalence (ii) is an immediate consequence of (i) and Proposition 1.1. Finally, equivalence (iii) is established by the Yoneda functor. The proposition is proved.

Let \( \mathcal{C} \) be an additive category. We recall from [14], p. 315, that a morphism \( f : C' \to C \) is said to be a pseudocokernel of a morphism \( g : C'' \to C' \) in \( \mathcal{C} \) if the sequence \( h'^{h} \to h''^{h} \to h'''^{h} \) is exact in \( \mathcal{C}_0\text{-Mod} \). It is well known that the following three statements are equivalent (14), (10):

\[
\begin{align*}
(\alpha) & \quad \mathcal{C} \text{ has pseudocokernels.} \\
(\beta) & \quad \text{fp}_\mathcal{C} \text{ is abelian.} \\
(\gamma) & \quad \text{\mathcal{C}_0\text{-Mod} \text{ is locally coherent.}}
\end{align*}
\]

Now let \( \mathcal{C} \) be a skeletally small additive category with pseudocokernels. We denote by

\[ \mathcal{C}_0 = \text{Lex}\text{fp}_\mathcal{C} \]

the category consisting of all left exact covariant additive functors from \( \text{fp}_\mathcal{C} \) to \( \mathcal{C}_0 \).

We know from [10] that \( \mathcal{C}_0 \) is a locally coherent Grothendieck category, \( \text{fp}_\mathcal{C} \) is abelian, there exists an equivalence

\[ \text{fp}(\mathcal{C}_0) = \text{fp}\mathcal{C}_0 \]

and the inclusion functor \( \text{fp}(\mathcal{C}_0) \to \mathcal{C}_0 \) is exact. If we denote by \( \text{inj}\mathcal{C}_0 \) the full subcategory of \( \text{fp}(\mathcal{C}_0) \) consisting of all injective objects, then we have equivalences

\[ \text{inj}\mathcal{C}_0 \cong \text{pr}^\mathcal{C}_0 \cong \text{pr}\mathcal{C}_0 \]

and using Proposition 1.1 we get the equivalence

\[ \text{fp}(\mathcal{C}_0)(\text{inj}\mathcal{C}_0) = \text{fp}(\mathcal{C}_0)\text{pr}\mathcal{C}_0 \]

We are now able to prove our main result.

**Theorem 2.4.** Let \( \mathcal{A} \) be a Grothendieck category and let \( \mathcal{C} \) be a skeletally small additive category with pseudocokernels.

\[
\begin{align*}
(\alpha) & \quad \text{If } \mathcal{A} \text{ is pure semi-simple, then } \text{i}(\mathcal{A}) \text{ is pure semi-simple, gl.dim } \\
& \quad \text{= gl.dim i}(\mathcal{A}) \text{ and there exists a natural equivalence } \mathcal{A} \cong \text{i}(\mathcal{A}) \\
(\beta) & \quad \text{If } \mathcal{A} \text{ is pure semi-simple and hereditary, then } \text{i}(\mathcal{A})\text{-Mod} \text{ is semihereditary and hence } \text{i}(\mathcal{A}) \text{ has pseudocokernels.}
\end{align*}
\]
Corollary 2.6. Let $\mathcal{A}$ be a locally noetherian Grothendieck category and suppose that every indecomposable injective object in $\mathcal{A}$ is noetherian. Then the following conditions are equivalent:

(a) $\mathcal{A}$ is pure semi-simple.

(b) $I(\mathcal{A})$ is pure semi-simple.

(c) $\mathcal{A}$ is locally noetherian.

Proof. By our assumption there exists a duality $(i)$ in Proposition 2.3. Then (a)$\implies$(b) is an immediate consequence of Theorem 2.4 and its proof. Furthermore, by Proposition 1.6 in [7] there exist equivalences

$$D(I(\mathcal{A})) = \mathcal{E}p_{i(\mathcal{A})}^\mathcal{A}_{\mathcal{M}}$$

Then (b)$\implies$(c) is a consequence of Theorem 1.9 in [13] and the corollary is proved.

Corollary 2.7. (a) Let $\mathcal{C}$ be a skeletally small additive category. If $\mathcal{C}$ is pure semi-simple, then for each finitely generated projective $\mathcal{C}$-module $P$ the ring $\text{End}_P$ is right pure semi-simple. If, in addition, $\mathcal{C}$ is hereditary, then $\text{End}_P$ is right hereditary.

(b) If $\mathcal{A}$ is a pure semi-simple Grothendieck category and $Q$ is a noetherian injective object in $\mathcal{A}$, then the ring $\text{End}_Q$ is left coherent and right pure semi-simple. If, in addition, $\mathcal{A}$ is hereditary then $\text{End}_Q$ is also hereditary.

Proof. The first part of statement (a) follows from Lemma 4.2 in [15] (see also the proof of Lemma 4.4 in [15]). In order to prove the second part of (a) we apply arguments from the proof of Proposition 1.4 in [5].

Now we prove (b). Suppose that $Q$ is a noetherian injective object in $\mathcal{A}$, i.e., $Q$ is in $i(\mathcal{A})$. Since $\mathcal{A}$ is pure semi-simple, by Theorem 2.6 the category $I(\mathcal{A})$ is pure semi-simple. Moreover, $h_Q = \text{Hom}_\mathcal{A}(\cdot, Q)$ is a finitely generated projective object in $I(\mathcal{A})$. Then by (a) of the corollary the ring $\text{End}_Q = \text{End}_hQ$ is right pure semi-simple.

In order to prove that $S = \text{End}_Q$ is left coherent denote by $\mathcal{A}_Q$ the full subcategory of $\mathcal{A}$ consisting of objects $A$ such that there exists an exact sequence in $\mathcal{A}$

$$0 \to A \to D_0 \to D_1$$

where $Q_0$ and $Q_1$ are finite direct sums of the object $Q$. Applying arguments from the proof of Proposition 2.3 one can prove that there exists an equivalence

$$\mathcal{A}_Q = \mathcal{A}^p_{\mathcal{A}_Q}.$$
Corollary 2.8. Let $\mathcal{A}$ be a skeletally small abelian category. If $q_{\text{sp}} \text{-Mod}$ is perfect, then it is locally noetherian.

Proof. We know from [10] that the category $\mathcal{A} = \text{Lex}_{q_{\text{sp}}}$ is locally coherent and $fp(\mathcal{A}) = \emptyset$. If $q_{\text{sp}} \text{-Mod}$ is perfect, then by Theorem 6.3 in [12] $\mathcal{A}$ is pure semi-simple. Now by Corollary 2.6 the category $q_{\text{sp}} \text{-Mod}$ is locally noetherian.

An immediate consequence of Theorem 2.4 is the following:

Corollary 2.9. The map $\mathcal{A} \mapsto I(\mathcal{A})$ establishes a one-one correspondence between equivalence classes of pure semi simple Grothendieck categories and equivalence classes of pure semi simple functor categories $\mathcal{A} \text{-Mod}$ where $\mathcal{A}$ has pseudo-co kernels. This map establishes also a one-one correspondence between equivalence classes of pure semi-simple hereditary Grothendieck categories and equivalence classes of pure semi-simple hereditary functor categories.

In [12] we have proved that a Grothendieck category $\mathcal{A}$ is pure semi-simple if and only if $fp(\mathcal{A}) \text{-Mod}$ is locally Artinian. We prove this fact by showing that $fp(\mathcal{A}) \text{-Mod}$ is locally Artinian if it is co-perfect (see [12], p. 111, (7)) (8)). Unfortunately this proof is not correct and therefore we do not know if Corollary 2.2 in [13] holds.

Now we give an example which illustrates the action of the map $\mathcal{A} \mapsto I(\mathcal{A})$.

Example 2.10. Let us consider the subcategory $\mathcal{L}^0_1$ of $\mathcal{L}_1$ in Example 1.4. We know from [8] that the category $I(\mathcal{L}^0_1)$ consists of all finite co-products of objects $K_{n,r} = \text{K}(X) \times \text{K}(X')$ where $\text{deg} X = 2$ and $r = 1, 2, 3, \ldots$. It is not difficult to observe that $I(\mathcal{L}^0_1)$ is the category $\mathcal{R}_0$ of all linear $K$-representations of the infinite quiver $\cdots \rightarrow \cdots \rightarrow \cdots$. Moreover, we observe that there is a natural equivalence $\mathcal{R}_0 \cong \mathcal{L}^0_1$. Further, the category $I(\mathcal{L}_1)$ consists of all finite co-products of objects $K_{n,r}$, $1 \leq r < \infty$, where $K_{n,r} = \text{K}(X)$, $\text{deg} X = 2$. Then it is not difficult to check that $I(\mathcal{L}_1)$ is the category $\mathcal{A}$ of all linear $K$-representations of the infinite quiver satisfying the designed commutativity conditions. Then we know from Theorem 2.4 that the category $\mathcal{A}$ is pure semi-simple and hereditary.

Remark 1. Suppose $\mathcal{A} = R \text{-Mod}$ is the category of all left $R$ modules over a ring $R$ of finite representation type. By Proposition 2.3 we have a duality $\mathcal{A}^{op} \cong \mathcal{A}^{op}$ where $\mathcal{A}$ is the endomorphism ring of the minimal injective cogenerator in $R \text{-Mod}$. The ring $\mathcal{A}$ is of finite representation type and we have $I(R \text{-Mod}) = R \text{-Mod}$, $R$. Continuing this procedure, we get a sequence of rings of finite representation type, $R, R_1, R_2, \ldots$ such that $I(R \text{-Mod}) = R \text{-Mod}$. We also point out that $R = R_1$, where $R_1$ is the endomorphism ring of the minimal injective cogenerator in $R \text{-Mod}$. Continuing this procedure, we get a second sequence $R, R_1, R_2, \ldots$ of rings of finite representation type.

It would be interesting to know how many of the rings $R$ and $\mathcal{A}$ are not pairwise Morita equivalent. If $R$ is an Artin algebra, then all those rings are Morita equivalent.

Remark 2. It would be interesting to know if the pure-semi-simplicity of the category $q_{\text{sp}} \text{-Mod}$ implies that $\mathcal{A} \text{-Mod}$ is locally Artinian.

Remark 3. Applying Corollaries 2.6 and 2.7, one can prove that given a finitely presented indecomposable non-projective object $A$ in an arbitrary pure semi-simple Grothendieck category $\mathcal{A}$, there exists a unique right almost split epimorphism $B \rightarrow A$ in $fp(\mathcal{A})$ (see [6]).

Remark 4. Suppose that $\mathcal{A}$ is a pure semi-simple Grothendieck category and has the property that the endomorphism ring of any injective noetherian object in $\mathcal{A}$ is an Artin algebra i.e., is a finitely generated algebra over a commutative noetherian ring. Observe that for any objects $M_1, \ldots, M_n$ in $fp(\mathcal{A})$ there exists an Artin algebra $S$ of finite representation type and a pair of additive functors $\mathcal{A}^{op} \rightarrow S$ with $\mathcal{A}^{op} \cong S$ such that $T\mathcal{A} \cong \text{id}$ and $T\mathcal{A}(M_i) \cong M_i$ for $i = 1, 2, \ldots, n$. To prove this fix an exact sequence $0 \rightarrow M_1 \rightarrow Q_1 \rightarrow Q_1'$ with $Q_1$ and $Q_1'$ injective and noetherian for every $i = 1, 2, \ldots, n$. If we put $Q = Q_1 \oplus \cdots \oplus Q_n \oplus Q_1' \oplus \cdots \oplus Q_1'$ then by Corollary 2.7(b) and Theorem A in [3] the Artin Algebra $S = \text{End} \mathcal{Q}$ is of finite representation type. Given an object $N$ in $fp(\mathcal{A})$ we set $T\mathcal{A}(N) = (N, Q)$. The functor $T$ is defined as the composition of the natural equivalence $\mathcal{A}^{op} \cong S$ and the natural inclusion $\mathcal{A} \cong S$ (see proof of Corollary 2.7).

The property of $\mathcal{A}$ we remarked above can be considered as a local approximation of the category $fp(\mathcal{A})$ by module categories $\mathcal{A}^{op} \cong S$ with $S$ of finite representation type.

References

Irreducible continua with degenerate end-tranches and arcwise accessibility in hyperspaces

by

J. Grisolakis and E. D. Tymchatyn* (Saskatoon, Sask.)

Abstract. In a 1960 paper G. W. Henderson proved that every hereditarily decomposable chainable continuum with a degenerate end-tranche. In this paper some other classes of hereditarily decomposable continua which also have this property are investigated. In particular it is proved that in a rational continuum of finite rim-type every point is a degenerate tranche of some continuum. An example of a hereditarily decomposable chainable continuum such that no subcontinuum has a cut-point is presented. Hence the degenerate tranches guaranteed by Henderson’s construction are end-tranches. These results are used to answer several questions of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. In 1960, G. W. Henderson [2] proved that every hereditarily decomposable chainable continuum contains an irreducible subcontinuum with a degenerate end-tranche. In 1967, W. Mahavier asked whether Henderson’s theorem is true for any hereditarily decomposable continuum. In Theorem 3.2, we give another class of continua with the property that they contain irreducible subcontinua with a degenerate end-tranche. In particular, we prove that hereditarily decomposable continua which contain subcontinua of finite rim-type at some point have this property. In 5.1 we give an example of a hereditarily decomposable chainable continuum with the property that no subcontinuum has a cut-point, and hence, no subcontinuum has a degenerate tranche other than an end-tranche.

In Section 4, we prove that the existence of irreducible continua with degenerate end-tranches implies the arcwise accessibility of points in hyperspaces (see Theorem 4.1), and we resolve several problems raised by Nadler in [5] and [6].

2. Preliminaries. Throughout this paper by a continuum we mean a connected, compact, metric space and by a mapping we mean a continuous function. A continuum $X$ is said to be chainable (or arc-like or snake-like) provided for each $e > 0$ there exists a finite open cover $\{U_i : 1 \leq i \leq n\}$ consisting of open sets with diameter less than $e$ and such that $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A mapping $f : X \to Y$...

* The first author was supported by a University of Saskatchewan Postdoctoral Fellowship and the second author was supported partially by a NRC (Canada) grant No. A5616.