a homeomorph of the Cantor set. Next observe that (iv) implies that

$$\bigcap\{\pi_1(f(V_{\varepsilon_1\varepsilon_2\ldots\varepsilon_k})) \colon (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in S_k\} \neq \emptyset \quad \text{for each } k \geqslant 1 .$$

It now follows that the set $\pi_1(f(C))$ contains exactly one point, say, x_0. Finally, conditions (i) and (iv) imply that $\pi_2 \circ f$, restricted to C, is one-one. So A^{x_0} contains a homeomorph of the Cantor set.

We now turn to the proof of the theorem. Assume that A^x is countable for each $x \in X$. Then, by Lemma 9, there is $\lambda < \omega_1$ such that $Z_\lambda = \emptyset$. Now Lemma 7 with $\alpha = \lambda$ and $B = \emptyset$ yields a set $H \in \mathcal{G}_\sigma$ such that $A = f(\Sigma) \subseteq H$. This completes the proof of the theorem.

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References

[1] C. Kuratowski, *Topologie*, Vol. 1, PWN, Warszawa 1958.
[2] N. Lusin, *Leçons sur Les Ensembles Analytiques et leurs Applications*, Gauthier-Villars, Paris 1930.
[3] J. Saint Raymond, *Boréliens à coupes K_σ*, Bull. Soc. Math. France 104 (1976), pp. 389–400.

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On pure semi-simple Grothendieck categories II

by

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Abstract. Given a pure semi-simple Grothendieck category $\mathcal{A}$, we construct a new pure-semi-simple functor category $I(\mathcal{A})$ such that gl. dim$\mathcal{A}$ = gl. dim$I(\mathcal{A})$. The map $\mathcal{A} \to I(\mathcal{A})$ defines a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. Applications of this result are given.

Introduction. In [13] the notion of a pure semi-simple Grothendieck category is introduced as a "pure" counterpart of semi-simple categories (cf. [9]). We recall that a Grothendieck category is pure semi-simple if each of its objects is a direct sum of finitely presented objects. Pure semi-simple Grothendieck categories are investigated in [11]–[17].

In the present paper we give two constructions of new pure semi-simple Grothendieck categories from a given pure semi-simple one. Given a pure semi-simple Grothendieck category $\mathcal{A}$ a pure semi-simple functor category $I(\mathcal{A})$ is constructed in a such a way that gl.dim$\mathcal{A}$ = gl.dim$I(\mathcal{A})$ and the category of all noetherian injective objects in $\mathcal{A}$ is equivalent to the category of all finitely generated projective objects in $I(\mathcal{A})$. Further, given a skeletally small additive category $\mathcal{C}$ such that the functor category $\mathcal{C}$-Mod is locally coherent and $\mathcal{C}^{\mathrm{op}}$-Mod is pure semi-simple, a pure semi-simple Grothendieck category $\tilde{\mathcal{C}}$ is constructed. The map $\mathcal{A} \mapsto I(\mathcal{A})$ is the inverse (with respect to an equivalence) of the map $\mathcal{C}^{\mathrm{op}}$-Mod $\mapsto \tilde{\mathcal{C}}$, and conversely. These maps define a one-one correspondence between equivalence classes of hereditary pure semi-simple Grothendieck categories and equivalence classes of hereditary pure semi-simple functor categories. In Section 2 we illustrate our constructions by simple examples.

In Section 1 we recall from [12]–[14] some background material on functor categories and pure semi-simple Grothendieck categories. An extension of Theorem A in [3] is given.

Section 2 contains the constructions and main results mentioned above. As a consequence of our general considerations we get the following two corollaries. Any injective noetherian object of a pure semi-simple Grothendieck category has a right pure semi-simple endomorphism ring. If $\mathcal{C}$ is a skeletally small abelian category such that the category $\mathcal{C}$-Mod is perfect, then $\mathcal{C}$-Mod is locally noetherian.

§ 1. **Notation and preliminaries.** We start by recalling the terminology and notation from [9] and [12]–[14]. If $\mathcal{A}$ is a locally finitely presented Grothendieck category, we denote by $\mathrm{fp}(\mathcal{A})$ the full subcategory of $\mathcal{A}$ consisting of all finitely presented objects. By an additive category we mean a category with an abelian group structure on each of its Hom sets.

Let $\mathcal{C}$ be a skeletally small additive category. We denote by $\mathcal{C}$-Mod the category of all covariant additive functors from $\mathcal{C}$ to the category $\mathcal{A}b$ of abelian groups. The objects of $\mathcal{C}$-Mod are called $\mathcal{C}$-modules. Given an object X in $\mathcal{C}$, the $\mathcal{C}$-module $(X, -)$ is denoted by h^X and the $\mathcal{C}^{\mathrm{op}}$-module $(-, X)$ is denoted by h_X. For convenience we will write $\mathrm{fp}_{\mathcal{C}}$ instead of $\mathrm{fp}(\mathcal{C}\text{-Mod})$. The category of finitely generated projective $\mathcal{C}$-modules will be denoted by $\mathrm{pr}_{\mathcal{C}}$.

If $\mathcal{C}'$ is a full additive subcategory of the category $\mathcal{C}$, we denote by $(\mathcal{C}')$ the two-sided ideal in $\mathcal{C}$ generated by $\mathcal{C}'$. It is clear that $(\mathcal{C}')$ is the subfunctor of the two variable functor $(-, ?)$: $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{A}b$ defined as follows. Given two objects X and Y in $\mathcal{C}$, the group $(\mathcal{C}')(X, Y)$ is the subgroup of (X, Y) consisting of all finite sums of morphisms from X to Y in $\mathcal{C}$ which factor through objects in $\mathcal{C}'$.

We often use the following result proved in [4], § 6.

PROPOSITION 1.1. *If $\mathcal{C}$ is a skeletally small additive category, then there exists an equivalence* $\mathrm{fp}_{\mathcal{C}}/(\mathrm{pr}_{\mathcal{C}}) = \mathrm{fp}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}}/(\mathrm{pr}^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}})$.

We recall that $\mathcal{C}$-Mod is perfect (resp. semi-perfect) if every $\mathcal{C}$-module (resp. every finitely generated $\mathcal{C}$-module) has a projective cover (cf. [1], [12]). The following lemma results from Lemma 1.1 in [14] and in fact is proved in [14], p. 293.

LEMMA 1.2. *Let $\mathcal{C}$ be a skeletally small additive category and suppose that every object in $\mathcal{C}$ is a finite direct sum of objects having local endomorphism rings. Then $\mathcal{C}$-Mod is semi-perfect.*

For other basic notation and properties of pure semi-simple Grothendieck categories the reader is referred to [9] and [12]–[14].

We are now going to investigate pure semi-simple Grothendieck categories. The following result is an extension of a theorem of Auslander [3] from module categories over Artin algebras to arbitrary Grothendieck categories.

THEOREM 1.3. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. The following statements are equivalent*:

(a) *$\mathcal{A}$ is pure semi-simple.*

(b) *$\mathcal{A}$ is locally noetherian, and given any sequence*

$$X_1 \xrightarrow{f_1} X_2 \to \dots \to X_n \xrightarrow{f_n} X_{n+1} \to \dots$$

of monomorphisms between indecomposable noetherian objects in $\mathcal{A}$, there is an integer n such that f_i is an isomorphism for all $i \geqslant n$.

(c) *$\mathcal{A}$ is locally noetherian and every indecomposable object in $\mathcal{A}$ is noetherian.*

Proof. The implications (a)→(b) and (a)→(c) follow from Theorem 6.3 in [12]. The implications (b)→(a) and (c)→(a) may be proved by the method of Auslander [3]. For the convenience of the reader we sketch the proof. Let assume that $\mathcal{A}$ is locally noetherian and not pure semi-simple. It follows from Theorem 1.9 in [13] that the category $\mathrm{fp}(\mathcal{A})$-Mod is not semiartinian, i.e. that there exists an $\mathrm{fp}(\mathcal{A})$-module F: $\mathrm{fp}(\mathcal{A}) \to \mathcal{A}b$ which has no simple submodules, or equivalently, given an object A in $\mathrm{fp}(\mathcal{A})$ and a nonzero element x in $F(A)$ there is a morphism f: $A \to A'$ in $\mathrm{fp}(\mathcal{A})$ which is not a splittable monomorphism and such that $F(f)x \neq 0$ (use arguments from the proof of Proposition 2.9(a) in [2]). Since each object in $\mathrm{fp}(\mathcal{A})$ is noetherian, one can construct a sequence (see the proof of Theorem 1.5 in [3])

$$M_1 \xrightarrow{f_1} M_2 \to \dots \to M_n \xrightarrow{f_n} M_{n+1} \to \dots$$

in $\mathrm{fp}(\mathcal{A})$ such that M_i is indecomposable noetherian for each $i = 1, 2, \dots$, each f_i is a proper monomorphism and $\varinjlim M_i$ is indecomposable and not noetherian. Then the required implications follow and the theorem is proved.

Let us replace in condition (b) of Theorem 1.3 the word "monomorphisms" by the word "epimorphisms" and denote the new condition by (b′). It follows from Theorem 2.4 in [14] that (b) and (b′) are equivalent under the additional assumption that $\mathcal{A}$ has only a finite number of isomorphism classes of simple objects. Now we give an example showing that (b) and (b′) are not equivalent for $\mathcal{A}$ locally finite.

EXAMPLE 1.4. Let K be a perfect field of finite characteristic $p>0$ and denote by $\mathcal{L}_1$ the category of all bicommutative graded connected primitively generated Hopf K-algebras which are generated by elements of degrees $2p^i$, $i = 0, 1, 2, \dots$ (see [8], [16], [17]). Let $\mathcal{L}_1^0$ be the full subcategory of $\mathcal{L}_1$ consisting of all directed unions of finite dimensional Hopf algebras in $\mathcal{L}_1$. It is clear that $\mathcal{L}_1$ is hereditary and by Theorem 5.3 in [16] it is pure semi-simple. It follows that $\mathcal{L}_1^0$ is also hereditary and pure semi-simple. Denote by $\mathcal{A}_0$ the category $\mathrm{Lex}\,\mathrm{fp}(\mathcal{L}_1^0)$ of all left exact covariant additive functors from $\mathrm{fp}(\mathcal{L}_1^0)$ to $\mathcal{A}b$. Since $\mathrm{fp}(\mathcal{A}_0) = \mathrm{fp}(\mathcal{L}_1^0)^{\mathrm{op}}$, then by Theorem 6.3 in [12] condition (b′) is satisfied for $\mathcal{A} = \mathcal{A}_0$ whereas (b) does not hold (see proof of Corollary 5.5 in [16]).

§ 2. **Categories $I(\mathcal{A})$ and $\widetilde{\mathcal{C}}$.** Throughout this section $\mathcal{A}$ will denote a Grothendieck category, and by $i(\mathcal{A})$ we denote the full subcategory of $\mathcal{A}$ consisting of finite direct sums of indecomposable injective objects. Let us consider the category

$$I(\mathcal{A}) = i(\mathcal{A})^{\mathrm{op}}\text{-Mod}.$$

We are going to prove that $I(\mathcal{A})$ is pure semi-simple if and only if $\mathcal{A}$ is such. Before proving our main results, we shall need some preliminary facts. We start with the following useful proposition.

PROPOSITION 2.1. *Suppose that $\mathcal{C}$ is a skeletally small additive category such that every finitely generated projective $\mathcal{C}$-module is a direct sum of indecomposable modules. Let $\mathcal{C}_0$ be a full additive subcategory of $\mathcal{C}$. Then $\mathcal{C}$-Mod is perfect if and only if $\mathcal{C}_0$-Mod and $\mathcal{C}/(\mathcal{C}_0)$-Mod are perfect.*

Proof. It follows from Proposition 3.1 in [1] that $\mathrm{pr}_{\mathcal{C}_0}$ can be considered as a full subcategory of $\mathrm{pr}_{\mathcal{C}}$. Since there are natural equivalences $\mathcal{C}\text{-Mod} = \mathrm{pr}^{\mathrm{op}}_{\mathcal{C}}\text{-Mod}$

and $\mathscr{C}_0$-Mod $= \mathrm{pr}^{\mathrm{op}}_{\mathscr{C}_0}$-Mod, we have reduced the proposition to the case where $\mathscr{C}$ and $\mathscr{C}_0$ are closed under direct sums. Now, using the same type of arguments as in the proof of Proposition 2.1 in [14], we get the required result.

COROLLARY 2.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be locally finitely presented Grothendieck categories such that each object in $\mathrm{fp}(\mathscr{A})$ and also each object in $\mathrm{fp}(\mathscr{B})$ are finite direct sums of indecomposable objects. Suppose that $\mathscr{C}$ is a full additive subcategory of $\mathrm{fp}(\mathscr{A})$ and $\mathscr{C}'$ is a full additive subcategory of $\mathrm{fp}(\mathscr{B})$ such that there are natural equivalences $\mathscr{C} \cong \mathscr{C}'$ and $\mathrm{fp}(\mathscr{A})/(\mathscr{C}) \cong \mathrm{fp}(\mathscr{B})/(\mathscr{C}')$. Then $\mathscr{A}$ is pure semi-simple if and only if $\mathscr{B}$ is such.*

Proof. Apply Theorem 6.3 in [12] and the previous proposition.

We will also need the following result.

PROPOSITION 2.3. *Let $\mathscr{A}$ be a locally noetherian Grothendieck category and suppose that each indecomposable injective object in $\mathscr{A}$ is noetherian. Then there exist equivalences of categories*

(i) $$\mathrm{fp}(\mathscr{A}) \cong \mathrm{fp}^{\mathrm{op}}_{i(\mathscr{A})},$$

(ii) $$\mathrm{fp}(\mathscr{A})/(i(\mathscr{A})) \cong \mathrm{fp}_{i(\mathscr{A})^{\mathrm{op}}}/(\mathrm{pr}_{i(\mathscr{A})^{\mathrm{op}}}),$$

(iii) $$i(\mathscr{A}) \cong \mathrm{pr}_{i(\mathscr{A})^{\mathrm{op}}}.$$

Proof. Let us consider a contravariant functor

$$h^{\cdot}\colon \mathscr{A} \to i(\mathscr{A})\text{-Mod}$$

defined by $h^{\bullet} = (.,-)$. Suppose X is an object in $\mathrm{fp}(\mathscr{A})$. By our assumption there exists in $\mathscr{A}$ an exact sequence

$$0 \to X \to Q_0 \xrightarrow{j} Q_1$$

with Q_0 and Q_1 in $i(\mathscr{A})$. We derive the exact sequence in $i(\mathscr{A})$-Mod

$$h^{Q_1} \xrightarrow{h^j} h^{Q_0} \to h^X \to 0$$

and therefore h^X is a finitely presented $i(\mathscr{A})$-module. Let $h_0^{\bullet}\colon \mathrm{fp}(\mathscr{A}) \to \mathrm{fp}_{i(\mathscr{A})}$ be the restriction of $h^{\bullet}$ to $\mathrm{fp}(\mathscr{A})$. We claim that $h_0^{\bullet}$ is an equivalence. First we point out that $h_0^{\bullet}$ is faithful because by our assumption every noetherian object in $\mathscr{A}$ can be embedded into an injective object in $i(\mathscr{A})$. Now we prove that $h_0^{\bullet}$ is full. Let $t\colon M \to M'$ be a morphism in $\mathrm{fp}_{i(\mathscr{A})}$. Then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccc} h^{Q_1} & \xrightarrow{h^j} & h^{Q_0} & \longrightarrow & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle h^{i_1}} & & \downarrow{\scriptstyle h^{i_0}} & & \downarrow{\scriptstyle t} & & \\ h^{Q_1'} & \xrightarrow{h^{j'}} & h^{Q_0'} & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

where $i_0\colon Q_0' \to Q_0$ and $i_1\colon Q_1' \to Q_1$ are in $i(\mathscr{A})$. We have a commutative diagram in $\mathrm{fp}(\mathscr{A})$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}\, j & \longrightarrow & Q_0 & \xrightarrow{j} & Q_1 \\ & & \uparrow{\scriptstyle f} & & \uparrow{\scriptstyle i_0} & & \uparrow{\scriptstyle i_1} \\ 0 & \longrightarrow & \mathrm{Ker}\, j' & \longrightarrow & Q_0' & \xrightarrow{j'} & Q_1' \end{array}$$

It then follows that $t = h_0^f$ and therefore $h_0^{\cdot}$ is an equivalence. The existence of equivalence (ii) is an immediate consequence of (i) and Proposition 1.1. Finally, equivalence (iii) is established by the Yoneda functor. The proposition is proved.

Let $\mathscr{C}$ be an additive category. We recall from [4], p. 315, that a morphism $f\colon C' \to C$ is said to be a *pseudocokernel of a morphism* $g\colon C'' \to C'$ in $\mathscr{C}$ if the sequence $h^{\mathscr{C}} \xrightarrow{h^f} h^{\mathscr{C}'} \xrightarrow{h^g} h^{\mathscr{C}''}$ is exact in $\mathscr{C}$-Mod. It is well known that the following three statements are equivalent ([4], [10])

(a) $\mathscr{C}$ has pseudocokernels.

(b) $\mathrm{fp}_{\mathscr{C}}$ is abelian.

(c) $\mathscr{C}$-Mod is locally coherent.

Now let $\mathscr{C}$ be a skeletally small additive category with pseudocokernels. We denote by

$$\widetilde{\mathscr{C}} = \mathrm{Lex}\,\mathrm{fp}_{\mathscr{C}}$$

the category consisting of all left exact covariant additive functors from $\mathrm{fp}_{\mathscr{C}}$ to $\mathscr{A}b$. We know from [10] that $\widetilde{\mathscr{C}}$ is a locally coherent Grothendieck category, $\mathrm{fp}_{\mathscr{C}}$ is abelian, there exists an equivalence

(i′) $$\mathrm{fp}(\widetilde{\mathscr{C}}) = \mathrm{fp}^{\mathrm{op}}_{\mathscr{C}}$$

and the inclusion functor $\mathrm{fp}(\widetilde{\mathscr{C}}) \subset \widetilde{\mathscr{C}}$ is exact. If we denote by $\mathrm{inj}\,\widetilde{\mathscr{C}}$ the full subcategory of $\mathrm{fp}(\widetilde{\mathscr{C}})$ consisting of all injective objects, then we have equivalences

(ii′) $$\mathrm{inj}\,\widetilde{\mathscr{C}} = \mathrm{pr}^{\mathrm{op}}_{\mathscr{C}} = \mathrm{pr}_{\mathscr{C}^{\mathrm{op}}}$$

and using Proposition 1.1 we get the equivalence

(iii′) $$\mathrm{fp}(\widetilde{\mathscr{C}})/(\mathrm{inj}\,\widetilde{\mathscr{C}}) = \mathrm{fp}_{\mathscr{C}^{\mathrm{op}}}/(\mathrm{pr}_{\mathscr{C}^{\mathrm{op}}}).$$

We are now able to prove our main result.

THEOREM 2.4. *Let $\mathscr{A}$ be a Grothendieck category and let $\mathscr{C}$ be a skeletally small additive category with pseudocokernels.*

(a) *If $\mathscr{A}$ is pure semi-simple, then $I(\mathscr{A})$ is pure semi-simple, $\mathrm{gl.dim}\,\mathscr{A} = \mathrm{gl.dim}\, I(\mathscr{A})$ and there exists a natural equivalence $\mathscr{A} = \widetilde{i(\mathscr{A})}$.*

(b) *If $\mathscr{A}$ is pure semi-simple and hereditary, then $i(\mathscr{A})$-Mod is semihereditary and hence $i(\mathscr{A})$ has pseudocokernels.*

(c) *If* $\mathscr{C}^{op}$-Mod *is pure semi-simple, then* $\mathscr{C}$ *is pure semi-simple,* $\mathrm{gl.dim}\mathscr{C} = \mathrm{gl.dim}\mathscr{C}^{op}$-Mod *and there exists an equivalence* $\mathscr{C}^{op}\text{-Mod} \simeq I(\mathscr{C})$.

(d) *If* $\mathscr{C}^{op}$-Mod *is pure semi-simple and hereditary, then* $\mathscr{C}$-Mod *is semihereditary and hence* $\mathscr{C}$ *has pseudocokernels.*

Proof. (a) Suppose that $\mathscr{A}$ is pure semi-simple. We know from Theorem 6.3 in [12] that $\mathscr{A}$ is locally noetherian, every indecomposable injective object in $\mathscr{A}$ is noetherian and the Jacobson radical of $i(\mathscr{A})$ is right T-nilpotent. Then by Proposition 2.3 there exist equivalences (i)-(iii) and by Theorem 5.4 in [12] the category $I(\mathscr{A})$ is perfect. Then the assumptions in Corollary 2.2 are satisfied for $\mathscr{A}$ and $\mathscr{B} = I(\mathscr{A})$, and we conclude that $I(\mathscr{A})$ is pure semi-simple. Moreover, in view of equivalence (i) we have equivalences

$$\overrightarrow{i(\mathscr{A})} \simeq \mathrm{Lex}\,\mathrm{fp}_{i(\mathscr{A})} \simeq \mathrm{Lex}\,\mathrm{fp}(\mathscr{A})^{op} \simeq \mathscr{A}.$$

Finally, since we know that categories $\mathscr{A}$ and $I(\mathscr{A})$ are locally noetherian and $i(\mathscr{A})$-Mod is locally coherent, we get

$$\begin{aligned}\mathrm{gl.dim}\mathscr{A} &= \mathrm{gl.dim}\,\mathrm{fp}(\mathscr{A})^{op} = \mathrm{gl.dim}\,\mathrm{fp}_{i(\mathscr{A})} = \mathrm{w.gl.dim}\,i(\mathscr{A})\text{-Mod}\\ &= \mathrm{w.gl.dim}\,i(\mathscr{A})^{op}\text{-Mod} = \mathrm{gl.dim}\,I(\mathscr{A}).\end{aligned}$$

(b) Suppose that $\mathscr{A}$ is hereditary and pure semi-simple. The first three equalities above yield $\mathrm{w.gl.dim}\,i(\mathscr{A})\text{-Mod} \leqslant 1$. We know from (a) that $I(\mathscr{A})$ is pure semi-simple and therefore by Theorem 5.6 in [12] the category $i(\mathscr{A})$-Mod is semi-perfect. Hence any finitely generated submodule N of a projective $\mathscr{C}$-module is flat and has a projective cover. Then, applying the arguments of Mares (see [12], p. 109), we prove that N is projective and therefore $i(\mathscr{A})$-Mod is semi-hereditary.

(c) Suppose that $\mathscr{C}^{op}$-Mod is pure semi-simple. It follows that $\mathscr{C}$-Mod is semi-perfect and therefore any finitely presented $\mathscr{C}$-module is a finite direct sum of indecomposable modules. Applying Corollary 2.2 and equivalences (i')-(iii'), we conclude that the category $\mathscr{C}$ is pure semi-simple. Now it follows that $i(\mathscr{C}) = \mathrm{inj}\mathscr{C}$ and in view of (i') and the obvious equivalence $\mathrm{pr}_{\mathscr{C}}^{op} \simeq \mathrm{pr}_{\mathscr{C}^{op}}$ we get the following equivalences:

$$I(\mathscr{C}) \simeq i(\mathscr{C})^{op}\text{-Mod} \simeq \mathrm{pr}_{\mathscr{C}^{op}}^{op}\text{-Mod} \simeq \mathscr{C}^{op}\text{-Mod}.$$

Now applying arguments from the proof of statement (a), we get $\mathrm{gl.dim}\mathscr{C} = \mathrm{gl.dim}\mathscr{C}^{op}$-Mod.

Since statement (d) can be proved similarly to (b), the proof of the theorem is complete.

We point out that we have in fact proved the following fact:

COROLLARY 2.5. *Let* $\mathscr{C}$ *be a skeletally small additive category with pseudocokernels. Then* $\mathscr{C}^{op}$-Mod *is pure semi-simple if and only if* $\mathscr{C}$ *is such.*

Further consequences of Theorem 2.4 are included in the following two corollaries:

COROLLARY 2.6. *Let* $\mathscr{A}$ *be a locally noetherian Grothendieck category and suppose that every indecomposable injective object in* $\mathscr{A}$ *is noetherian. Then the following conditions are equivalent:*

(a) $\mathscr{A}$ *is pure semi-simple.*

(b) $I(\mathscr{A})$ *is pure semi-simple.*

(c) $\mathrm{fp}(\mathscr{A})^{op}$-Mod *is locally noetherian.*

Proof. By our assuption there exists a duality (i) in Proposition 2.3. Then (a)$\leftrightarrow$(b) is an immediate consequence of Theorem 2.4 and its proof. Furthermore, by Proposition 1.6 in [7] there exist equivalences

$$D(I(\mathscr{A})) \simeq \mathrm{fp}_{i(\mathscr{A})}\text{-Mod} \simeq \mathrm{fp}(\mathscr{A})^{op}\text{-Mod}.$$

Then (b)$\leftrightarrow$(c) is a consequence of Theorem 1.9 in [13] and the corollary is proved.

COROLLARY 2.7. (a) *Let* $\mathscr{C}$ *be a skeletally small additive category. If* $\mathscr{C}$-Mod *is pure semi-simple, then for each finitely generated projective* $\mathscr{C}$*-module* P *the ring* $\mathrm{End}P$ *is right pure semi-simple. If, in addition,* $\mathscr{C}$-Mod *is hereditary, then* $\mathrm{End}P$ *is right hereditary.*

(b) *If* $\mathscr{A}$ *is a pure semi-simple Grothendieck category and* Q *is a noetherian injective object in* $\mathscr{A}$, *then the ring* $\mathrm{End}\,Q$ *is left coherent and right pure semi-simple. If, in addition,* $\mathscr{A}$ *is hereditary then* $\mathrm{End}\,Q$ *is also hereditary.*

Proof. The first part of statement (a) follows from Lemma 4.2 in [15] (see also the proof of Lemma 4.4 in [15]). In order to prove the second part of (a) we apply arguments from the proof of Proposition 1.4 in [5].

Now we prove (b). Suppose that Q is a noetherian injective object in $\mathscr{A}$, i.e. Q is in $i(\mathscr{A})$. Since $\mathscr{A}$ is pure semi-simple, by Theorem 2.4 the category $I(\mathscr{A})$ is pure semi-simple. Moreover $h_Q = (-, Q)$ is a finitely generated projective object in $I(\mathscr{A})$. Then by part (a) of the corollary the ring $\mathrm{End}\,Q \cong \mathrm{End}\,h_Q$ is right pure semi-simple.

In order to prove that $S = \mathrm{End}\,Q$ is left coherent denote by $\mathscr{A}_Q$ the full subcategory of $\mathscr{A}$ consisting of objects A such that there exists an exact sequence in $\mathscr{A}$

$$0 \to A \to Q_0 \to Q_1$$

where Q_0 and Q_1 are finite direct sums of the object Q. Applying arguments from the proof of Proposition 2.3 one can prove that there exists an equivalence

$$\mathscr{A}_Q \simeq {}_S\mathrm{fp}^{op}.$$

Since Q is noetherian, every object in $\mathscr{A}_Q$ is noetherian and therefore every object in ${}_S\mathrm{fp}$ is artinian. In particular S is left coherent as required.

Finally, if $\mathscr{A}$ is hereditary and pure semi-simple, then by Theorem 2.4 $I(\mathscr{A})$ is hereditary and pure semi-simple. Now by (a) the ring $\mathrm{End}\,Q \cong \mathrm{End}\,h_Q$ is hereditary and the proof is complete.

As a consequence of Corollary 2.6 we get the following result, communicated to the author by Professor H. Lenzing.

COROLLARY 2.8. *Let $\mathscr{C}$ be a skeletally small abelian category. If $\mathscr{C}^{op}$-Mod is perfect, then it is locally noetherian.*

Proof. We know from [10] that the category $\mathscr{A} = \mathrm{Lex}\,\mathscr{C}^{op}$ is locally coherent and $\mathrm{fp}(\mathscr{A}) = \mathscr{C}$. If $\mathscr{C}^{op}$-Mod is perfect, then by Theorem 6.3 in [12] $\mathscr{A}$ is pure semi-simple. Now by Corollary 2.6 the category $\mathscr{C}^{op}$-Mod is locally noetherian.

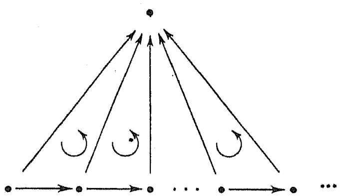
An immediate consequence of Theorem 2.4 is the following:

COROLLARY 2.9. *The map $\mathscr{A} \mapsto I(\mathscr{A})$ establishes a one-one correspondence between equivalence classes of pure semi simple Grothendieck categories and equivalence classes of pure semi simple functor categories $\mathscr{C}$-Mod where $\mathscr{C}$ has pseudo-cokernels. This map establishes also a one-one correspondence between equivalence classes of pure semi-simple hereditary Grothendieck categories and equivalence classes of pure semi-simple hereditary functor categories.*

In [12] we have proved that a Grothendieck category $\mathscr{A}$ is pure semi-simple if and only if $\mathrm{fp}(\mathscr{A})$-Mod is locally artinian. We prove this fact by showing that $\mathrm{fp}(\mathscr{A})$-Mod is locally artinian if it is co-perfect (see [12], p. 111, (7)→(8)). Unfortunately this proof is not correct and therefore we do not know if Corollary 2.2 in [13] holds.

Now we give an example which illustrates the action of the map $\mathscr{A} \mapsto I(\mathscr{A})$.

Example 2.10. Let us consider the subcategory $\mathscr{L}_1^0$ of $\mathscr{L}_1$ in Example 1.4. We know from [8] that the category $i(\mathscr{L}_1^0)$ consists of all finite co-products of objects $K_{0,r} = K[X]/(X^{p^r})$ where $\deg X = 2$ and $r = 1, 2, 3, \ldots$ It is not difficult to observe that $I(\mathscr{L}_1^0)$ is the category $\mathscr{R}_0$ of all linear K-representations of the infinite quiver $\cdot\to\cdot\to\cdot\to\cdot\ldots$ Moreover, we observe that there is a natural equivalence $\mathscr{R}_0 = \mathscr{L}_1$. Further, the category $i(\mathscr{L}_1)$ consists of all finite co-products of objects $K_{0,r}$, $1 \leqslant r \leqslant \infty$, where $K_{0,\infty} = K[X]$, $\deg X = 2$. Then it is not diffícult to check that $I(\mathscr{L}_1)$ is the category $\mathscr{R}$ of all linear K-representations of the infinite quiver satisfying the



designed commutativity conditions. Then we know from Theorem 2.4 that the category $\mathscr{R}$ is pure semi-simple and hereditary.

Remark 1. Suppose $\mathscr{A} = R$-Mod is the category of all left R modules over a ring R of finite representation type. By Proposition 2.3 we have a duality ${}_R\mathrm{fp} = {}_{{}_1R}\mathrm{fp}^{op}$ where ${}_1R$ is the endomorphism ring of the minimal injective cogenerator in R-Mod. The ring ${}_1R$ is of finite representation type and we have $I(R\text{-Mod}) = \text{Mod-}{}_1R$. Continuing this procedure, we get a sequence of rings of finite representation type ${}_1R, {}_2R, {}_3R, \ldots$ such that $I(\text{Mod-}{}_nR) = \text{Mod-}{}_{n+1}R$. We also point out that $\bar{R} = \text{Mod-}R_1$ where R_1 is the endomorphism ring of the minimal injective cogenerator in Mod-R. Continuing this procedure, we get a second sequence $R, R_1, R_2, \ldots$ of rings of finite representation type.

It would be interesting to know how many of the rings R_i and ${}_jR$ are not pairwise Morita equivalent. If R is an Artin algebra, then all those rings are Morita equivalent.

Remark 2. It would be interesting to know if the pure semi-simplicity of the category $\mathscr{C}^{op}$-Mod implies that $\mathscr{C}$-Mod is locally coherent.

Remark 3. Applying Corollaries 2.6 and 2.7, one can prove that given a finitely presented indecomposable nonprojective object A in an arbitrary pure semi-simple Grothendieck category $\mathscr{A}$, there exists a unique right almost split epimorphism $B \to A$ in $\mathrm{fp}(\mathscr{A})$ (see [6]).

Remark 4. Suppose that $\mathscr{A}$ is a pure semi-simple Grothendieck category and has the property that the endomorphism ring of any injective noetherian object in $\mathscr{A}$ is an Artin algebra i.e., is a finitely generated algebra over a commutative artinian ring. Observe that for any objects $M_1, \ldots, M_n$ in $\mathrm{fp}(\mathscr{A})$ there exists an Artin algebra S of finite representation type and a pair of additive functors

$$ {}_S\mathrm{fp}^{op} \xrightarrow{T} \mathrm{fp}(\mathscr{A}) \xrightarrow{F} {}_S\mathrm{fp}^{op} $$

such that $FT \cong \mathrm{id}$ and $TF(M_i) \cong M_i$ for $i = 1, 2, \ldots, n$. To prove this fix an exact sequence

$$ 0 \to M_i \to Q_i \to Q_i' $$

with Q_i and Q_i' injective and noetherian for every $i = 1, 2, \ldots, n$. If we put $Q = Q_1 \oplus \ldots \oplus Q_n \oplus Q_1' \oplus \ldots \oplus Q_n'$ then by Corollary 2.7(b) and Theorem A in [3] the Artin Algebra $S = \mathrm{End}\,Q$ is of finite representation type. Given an object N in $\mathrm{fp}(\mathscr{A})$ we set $F(N) = (N, Q)$. The functor T is defined as the composition of the natural equivalence ${}_S\mathrm{fp}^{op} = \mathscr{A}_Q$ and the natural inclusion $\mathscr{A}_Q \hookrightarrow \mathrm{fp}(\mathscr{A})$ (see proof of Corollary 2.7).

The property of $\mathscr{A}$ we remarked above can be considered as a local approximation of the category $\mathrm{fp}(A)$ by module categories ${}_S\mathrm{fp}^{op} \cong \mathrm{fp}_S$ with S of finite representation type.

References

[1] M. Auslander, *Representation theory of Artin algebras I*, Comm. in Algebra 1 (1974), pp. 177–268.

[2] — *Representation theory of Artin algebras II*, ibid., pp. 269–310.

[3] — *Large modules over Artin algebras*, Algebra, Topology and Category Theory, A Collection of Papers in Honor of Samuel Eilenberg, Academic Press 1976, pp. 3–17.

[4] — and I. Reiten, *Stable equivalence of dualizing R-varieties*, Advances in Math. 12 (1974), pp. 306–366.

[5] M. Auslander and I. Reiten, *Stable equivalence of dualizing R-varieties II: Hereditary dualizing R-varieties*, ibid. 17 (1975), pp. 93–121.
[6] — *Representation theory of Artin algebras IV: Invariants given by almost split sequences*, Comm. in Algebra 5 (1977), pp. 443–518.
[7] C. U. Jensen and D. Simson, *Purity and generalized chain conditions*, J. Pure Appl. Algebra 14 (1979), pp. 297–305.
[8] J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965), pp. 211–264.
[9] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), pp. 1–161.
[10] J. E. Roos, *Locally noetherian categories and generalized linearly compact rings. Applications*, Lecture Notes in Math. 92 (1969), pp. 197–277.
[11] D. Simson, *On colimits of injectives in Grothendieck categories*, Arkiv för Matematik 12 (1974), pp. 161–165.
[12] — *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. 96 (1977), pp. 91–116.
[13] D. Simson, *On pure semi-simple Grothendieck categories I*, Fund. Math. 100 (1978), pp. 211–222.
[14] — *Pure semi-simple categories and rings of finite representation type*, J. Algebra 48 (1977), pp. 290–296.
[15] — *Categories of representations of species*, J. Pure Appl. Algebra 14 (1979), pp. 101–114.
[16] — and A. Skowroński, *On the category of commutative connected graded Hopf algebras over a perfect field*, Fund. Math. 101 (1978), pp. 137–149.
[17] A. Skowroński, *On the category of abelian Hopf algebras over a nonperfect field*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (1976), pp. 675–682.
[18] — *The category of abelian Hopf algebras*, Fund. Math. 107 (1980), pp. 167–193.

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Irreducible continua with degenerate end-tranches and arcwise accessibility in hyperspaces

by

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Abstract. In a 1960 paper G. W. Henderson proved that every hereditarily decomposable chainable continuum has a subcontinuum with a degenerate tranche. In this paper some other classes of hereditarily decomposable continua which also have this property are investigated. In particular it is proved that in a rational continuum of finite rim-type every point is a degenerate tranche of some continuum. An example of a hereditarily decomposable chainable continuum such that no subcontinuum has a cut-point is presented. Hence the degenerate tranches guaranteed by Henderson's construction are end-tranches. These results are used to answer several questions of Nadler concerning arcwise accessibility in hyperspaces.

1. Introduction. In 1960, G. W. Henderson [2] proved that every hereditarily decomposable chainable continuum contains an irreducible subcontinuum with a degenerate end-tranche. In 1967, W. Mahavier asked whether Henderson's theorem is true for any hereditarily decomposable continuum. In Theorem 3.2, we give another class of continua with the property that they contain irreducible subcontinua with a degenerate end-tranche. In particular, we prove that hereditarily decomposable continua which contain subcontinua of finite rim-type at some point have this property. In 5.1 we give an example of a hereditarily decomposable chainable continuum with the property that no subcontinuum has a cut-point, and hence, no subcontinuum has a degenerate tranche other than an end-tranche.

In Section 4, we prove that the existence of irreducible continua with degenerate end-tranches implies the arcwise accessibility of points in hyperspaces (see Theorem 4.1), and we resolve several problems raised by Nadler in [5] and [6].

2. Preliminaries. Throughout this paper by a *continuum* we mean a connected, compact, metric space and by a *mapping* we mean a continuous function. A continuum X is said to be *chainable* (or *arc-like* or *snake-like*) provided for each $\varepsilon>0$ there exists a finite open cover $\{U_1, \ldots, U_n\}$ consisting of open sets with diameter less than ε and such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. A mapping $f\colon X \to Y$

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