Equilateral triangles and continuous curves

by

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Abstract. If M is a simple closed curve in the plane, then for all, except perhaps two, points x of M we can find points y and z of M such that xyz is an equilateral triangle. The same result is true if M is a connected manifold, with or without boundary, of dimension at least two in E^n. Let T be a triod, an embedding of the letter "T", in E^n. Then one leg of T is such that for any point x of that leg, we can find points y and z of T such that xyz is an equilateral triangle. Given a triangle Δ there exist three points forming a similar triangle on every simple closed curve in every metric space if and only if Δ is isoceles with apex angle at most 60°.

0. Introduction. In the following, we prove several results closely related to the theorem: Every simple closed curve in the plane contains the vertices of an equilateral triangle. These results have such uncomplicated statements that they are of natural interest. In addition, we are motivated by the related unsolved conjecture: Every simple closed curve in the plane contains the vertices of a square. Results due to Emch [2], Schnirelmann [6], Jerrard [4], and Guggenheimer [3] have shown this conjecture to be valid provided various smoothness assumptions are made. These assumptions seem quite unnatural, both intuitively and in relation to the results of this paper.

The two continuous curves that hold our attention here are the simple closed curve and the triod (defined in Section 1). They are basic in the sense that a continuous curve which contains neither is an arc. Note that these are results about topological curves, not differentiable ones. Some remarks related to the smooth case are made in Section 4.

1. The plane. As motivation, we give an elementary proof of the theorem which we will later generalize.

THEOREM 1. *Every simple closed curve in the plane contains the vertices of an equilateral triangle.*

Proof. Let w be a point in the interior of a given simple closed curve, J. Let C be the smallest circle with center w which meets J. Let x be a point of $C \cap J$ and let y_0 and z_0 be on C so that x, y_0, z_0 are the vertices of an equilateral triangle. We will let y and z be variable points which will move about the plane under the restraint that x, y, z are always the vertices of an equilateral triangle. At first let $y = y_0$, $z = z_0$.

We now let y and z move as follows. Let y and z move radially outward from x through $y_0, z_0, \ldots$ (keeping $xy = xz$) until y or z meets J (if y or z already lie on J, no movement is needed). At this point we set $y_1 = y$, $z_1 = z$. Say $y_1 \in J$. Now let y move continuously along J until it reaches y_2, a point on J maximally distant from x.

As y moves from y_1 to y_2, z describes a continuous curve from z_1 to z_2 (actually z traverses the arc which is a 60° rotation of the arc described by y). Since $xy_2 = xz_2$, z_2 lies outside or on J. But z_1 lies inside or on J. Hence, at some stage we have x, y, and z, the vertices of an equilateral triangle, all lying on J. ∎

This result holds even if J does not lie in the plane. In fact, A. N. Milgram ([5, p. 25]) has proved (with, of course, a more complicated proof) that for any metric on J, there exists points x, y, and z in J such that $xy = yz = zx$. We will generalize Milgram's result in Section 3.

COROLLARY 1. *A point on a simple closed curve in the plane (say $x \in J$) is the vertex of an equilateral triangle with all vertices on J if and only if there are points y and z inside or on J so that x, y, and z are the vertices of an equilateral triangle.*

Proof. Clear from the proof of Theorem 1. ∎

A *triod*, T, is the union of three arcs, L_1, L_2, and L_3 (called *legs*), which all share a common endpoint, the *juncture*, so that any two of the legs meet only at the juncture. The other three endpoints of the legs, e_1, e_2 and e_3 respectively, are the *endpoints of the triod*. Equivalently, a triod is any homeomorph of the letter "T".

A triod also must contain the vertices of an equilateral triangle (as proved by Milgram, [5, p. 31]). This follows immediately from the new and stronger result below. Theorem 2 will also be a basic tool for further results.

THEOREM 2. *If T lies in the plane, then it contains the vertices of an equilateral triangle with e_1, e_2, or e_3 as a vertex.*

Before proving Theorem 2, we consider two lemmas.

LEMMA 1. *Suppose Theorem 2 is false. Then there exists a triod T', also with endpoints e_1, e_2, and e_3, so that T' contains a segment with e_1 as one endpoint. Furthermore, T' does not contain the vertices of an equilateral triangle with e_1, e_2, or e_3 as a vertex.*

Proof of Lemma 1. Consider the plane containing T as the complex plane, so that $T \times T \subset C \times C = C^2$. Let E_j ($j = 1$, 2, or 3) be the union of the pair of one (complex) dimensional hyperplanes in C^2 defined by

$$E_j = \{(w, z) |\ z = k(w-e_j)+e_j \text{ or } z = k^{-1}(w-e_j)+e_j\}$$

where $k = e^{i\pi/3}$. Our assumption that Theorem 2 is false implies that $(T\times T) \cap E_j = (e_j, e_j)$, since for $(a, b) \in (T\times T) \cap E_j$, $b = k^{\varepsilon}(a-e_j)+e_j (\varepsilon = \pm 1)$; hence $(b-e_j) = k^{\varepsilon}(a-e_j)$ and for $a \neq e_j$ we have a, b, and e_j are the vertices of an equilateral triangle.

And so $L_1 \times T$ is a compact set disjoint from the closed set $E_2 \cup E_3$. Choose $\delta > 0$ sufficiently small so that if $d(a, L_1) \leqslant \delta$ and $d(b, T) \leqslant \delta$, then $(a, b) \notin E_2 \cup E_3$.

Further, insure that δ is small enough so that if $e \in T$ and $ee_1 \leqslant \delta$, then $e \in L_1$ (ee_1 is the distance from e to e_1).

As we travel along L_1 starting from the juncture, let c be the first point which is at a distance of δ from e_1. Let T' be the new triod formed by replacing the subarc of T from e_1 to c with the line segment, S, from e_1 to c. T' has endpoints e_1, e_2, and e_3. Note that for $e \in T'$, $ee_1 = \delta$ if and only if $e = c$.

We claim that T' does not contain the vertices of an equilateral triangle with e_1, e_2, or e_3 as one of the vertices. It suffices to show that $(T'\times T') \cap E_j = (e_j, e_j)$ for $j = 1$, 2, and 3. So suppose $(e, f) \in (T'\times T') \cap E_j$ for some j.

Case 1. $e, f \in T' \setminus S$. Then $(e, f) \in T\times T$, so $(e, f) = (e_j, e_j)$.

Case 2. $e \in S$, $f \in T' \setminus S$. Then $d(e, L_1) \leqslant \delta$, $f \in T$, so $(e, f) \notin E_2 \cup E_3$, i.e. $(e, f) \in E_1$. Hence $ee_1 = fe_1$. But $ee_1 \leqslant \delta < fe_1$ a contradiction. So Case 2 cannot occur.

Case 3. $e \in T' \setminus S$, $f \in S$. By symmetry with Case 2, Case 3 cannot occur.

Case 4. $e, f \in S$. Then $d(e, L_1) \leqslant ee_1 \leqslant \delta$ and $d(f, T) \leqslant fe_1 \leqslant \delta$, so $(e, f) \notin E_2 \cup E_3$. Hence $(e, f) \in E_1$. But then e and f lie on a segment (S) with endpoint e_1 and e_1, e, and f are either equal or non-colinear. It follows that $(e, f) = (e_1, e_1)$. ∎

Let A be a set of points in the (complex) plane. Let A_x be A rotated 60° about x in the counter-clockwise direction (specifically, $A_x = \{y \in C |$ for some $z \in A$, $y = k(z-x)+x\}$). If A is an oriented arc, we consider the arc A_x to be oriented in the obvious manner.

Let A be an oriented arc, and let ∂A be its endpoints. If $\partial A \cap A_x = \emptyset = A \cap (\partial A)_x$, then the intersection number, $i(A, A_x)$, is a well-defined integer. It can be defined as the intersection number of sufficiently close, piecewise linear, general position, approximations of A and A_x (see A. Dold [1, p. 197]).

Let Δ be an equilateral triangle. We write $xyz \sim \Delta$ if $xy = yz = zx$ (even if $x = y = z$). Let a_1 and a_2 be the endpoints of A. We give the following lemma without proof since it follows immediately from the preceeding definitions.

LEMMA 2. a) *In its domain of definition, $i(A, A_x)$ is continuous in x. (In fact $\{x |\ i(A, A_x) = n\}$ is open for each n).*

b) *If $i(A, A_x)$ is undefined, then there exists $y \in A$ such that $xya_i \sim \Delta$ ($i = 1$ or 2).*

c) *If there does not exist $y, z \in A$ such that $xyz \sim \Delta$, then A and A_x are disjoint, and so $i(A, A_x) = 0$.*

d) *If B is a subarc of A with the same orientation and $A-B$ is disjoint from A_x and $A_x - B_x$ is disjoint from A, then $i(B, B_x) = i(A, A_x)$ provided the former is defined.*

e) *If S is a (closed) line segment and x lies on S, but is not on endpoint, then $i(S, S_x)$ is a non-zero integer.* ∎

We now proceed with:

Proof of Theorem 2. Suppose the theorem is false. Let T' and S be as in Lemma 1. Let L_j be the leg of T' determined by e_j and let A be an oriented arc with image $L_1 \cup L_2$. We wish to get a contradiction.

By Lemma 2b, we may conclude that $i(A, A_x)$ is defined for $x \in T' \setminus \{e_1, e_2\}$, and so (by Lemma 2a) is equal to a single integer on this set. By Lemma 2c, $i(A, A_{e_3}) = 0$. But A and A_{e_1} meet only at e_1. So $A \setminus S$ and A_{e_1} and also $A_{e_1} \setminus S_{e_1}$ and A are each a positive distance apart. And so for $x \in S$ sufficiently close to e_1, $A \setminus S$ and A_x and also $A_x \setminus S_x$ and A are disjoint. So by Lemma 2d) and e), $i(A, A_x) \neq 0$ for such an x. This is our contradiction. ■

Given a set G in the plane, call $x \in G$ a *vertex point* (of G) if there exists $y, z \in G$ such that x, y, z are the vertices of an equilateral triangle. (To extend this to any metric space, we just require $xy = yz = zx \neq 0$.) Theorem 1 then states that every simple closed curve (in the plane) contains at least three vertex points. Theorem 2 states that at least one of the endpoints of a triod is a vertex point. The following two corollaries show that triods and simple closed curves (both in the plane) contain many vertex points.

COROLLARY 2. *Let T be a triod (in the plane). Then every point of some leg, L, except perhaps the juncture, is a vertex point of T.*

Proof. Let L_1, L_2, and L_3 be the legs of T and j the juncture of T. Suppose the corollary is false. Then there exists $a_i \in L_i - j$ $(i = 1, 2, 3)$ such that for each i, a_i is not a vertex point of T. Let T' be the sub-triod of T with endpoints a_1, a_2, and a_3. Then by Theorem 2, a_k is a vertex point of T' for $k = 1, 2$, or 3. But $T' \subset T$, so a_k is a vertex point of T. This contradiction implies the corollary. ■

COROLLARY 3. *All but at most two points of any simple closed curve (in the plane) are vertex points for the simple closed curve.*

Proof. Suppose J is a simple closed curve in the plane with three non-vertex points, e_1, e_2, and e_3. Let T be a triod with these three points for endpoints and so that T is a subset of J together with the part of the plane interior to J. Such a triod exists by the Jordan Curve Theorem (or we could use the Schoenflies Theorem or the accessibility results of Whyburn [7], although they are not needed since we allow part of T to lie on J). Then by Theorem 2, e_k is a vertex point of T for $k = 1, 2$, or 3. But by Corollary 1, e_k is then a vertex point of J. ■

The above corollary is sharp, since for J an isoceles trangle with apex angle greater than 60°, the two base vertices are not vertex points.

Since a continuous curve containing neither a simple closed curve nor a triod is an arc, we may state without proof:

COROLLARY 4. *If a continuous curve in the plane is not an arc, then it contains an arc of vertex points.* ■

We mention one further result in the plane, although it does not deal with equilateral triangles.

THEOREM 3. *Let Δ be a fixed arbitrary triangle. Then every simple closed curve in the plane contains the vertices of a similar triangle.*

Proof. We note the few changes in the proof of Theorem 1 needed to prove Theorem 3. The variable points x, y, and z are required to form a triangle similar to Δ with x corresponding to the maximal angle of Δ. We let x_3 and y_2 be two maximally distant points on J. Then after moving y to y_2 and z to z_2 $(x = x_2)$ we move x to x_3 while holding y fixed. So z describes a continuous curve from z_1 to z_2 to z_3. Since $y_2 z_3$ corresponds to the longest side of Δ, z_3 must lie on or outside J. So the result follows as before. ■

2. Euclidean space. In this section we will generalize some of our results to n-dimensional Euclidean space, E^n, for any integer n greater than one. The definition of vertex point applies to E^n without change and we still write $xyz \sim \Delta$ if $xy = yz = zx$ (allowing $x = y = z$).

Let T be a triod in E^n with endpoints e_1, e_2, and e_3 and corresponding legs L_1, L_2, and L_3. Then the following theorem generalizes Theorem 2.

THEOREM 4. *One of the endpoints of T is a vertex point.* ■

The proof of Theorem 2 will prove Theorem 4 provided we properly generalize Lemmas 1 and 2. The proof of Lemma 1 works in higher dimensions (where we suppose Theorem 4 is false) by replacing the first paragraph of the proof by the following paragraph:

Let $E_j = \{(w, z) |\ w, z \in E^n \text{ and } e_j wz \sim \Delta\}$ for $j = 1, 2$, or 3. Note that each E_j is a closed set in E^{2n}. The assumption that Theorem 4 is false is equivalent to: $(T \times T) \cap E_j = (e_j, e_j)$ for $j = 1, 2$, and 3.

To generalize Lemma 2, we need to extend the definition of A_x to higher dimensions. For A a set in E^n, let A_x be the set formed by rotating the points of A through an angle of 60°, about x, in all directions. In other words, let $A_x = \{y \in E^n |$ there exists $z \in A$ with $xyz \sim \Delta\}$.

From now on we will let A represent both a continuous map of an oriented interval, I, into E^n and the range of that map: $A: I \to A$. A_x is then, in a natural way, the continuous image of $I \times S^{n-2}$ (and we identify A_x with this map). S^{n-2} is the standard unit sphere in E^{n-1}.

We make this precise with the following lemma. For α a vector in E^n, α_i will represent the ith coordinate of α.

LEMMA 3. *A_x is the image of $I \times S^{n-2}$ under a continuous map, A_x, such that for any t in I, A_x restricted to $t \times S^{n-2}$ is a uniform contraction or expansion onto $A(t)_x$. Further, if $A(t) \neq x$, the line from x to $A(t)$ (extended indefinitely) links $A(t)_x$ positively. In other words, the ordered orthogonal basis of E^n consisting of first the vector from x to $A(t)$, and then the vectors from the center of $A(t)_x$ (which is also the midpoint of the segment from x to $A(t)$) to the images, in order, of the standard basis elements in $S^{n-2} \subset E^{n-1}$, is a positively oriented basis for E^n.*

Proof. Suppose we have chosen r and s in I and $t_0 = r$ or s, such that $r < s$ and $x \notin A([r, s])$. Suppose that $\{\alpha^i(t_0)\}_{i=1}^n$ is a given positively oriented orthonormal subset of S^{n-1}, with $\alpha^1(t_0)$ equal to the unit vector with direction from x to $A(t_0)$. For $r \leqslant t \leqslant s$, let $\alpha(t)$ be the unit vector with direction from x to $A(t)$ (so $\alpha(t_0) = \alpha^1(t_0)$). Suppose further that for $r \leqslant t \leqslant s$, the distance from $\alpha(t)$ to $\alpha(t_0)$ is less than $\sqrt{2}$ (by choice of r and s).

Let $\{\alpha^i(t_0)\}_{i=1}^n$ be the coordinate system $\{e_i\}_{i=1}^n$ for E^n. Note that $\alpha = \alpha(t)$ lies in S^{n-1} and that $\alpha_1 > 0$. For $2 \leqslant m \leqslant n$, let $\beta^m = e_m - \alpha_m e_1/\alpha_1$. Since $\sum_{m=2}^{n} \alpha_m \beta^m = \alpha - |\alpha|^2 e_1/\alpha_1$, we can write each e_i $(1 \leqslant i \leqslant n)$ in terms of $\{\alpha, \beta^m\}_{m=2}^n$, which must therefore span E^n. And since $\alpha \cdot \beta^m = 0$, $\{\beta^m\}_{m=2}^n$ forms a basis for the tangent space to S^{n-1} at α. Using the Gram–Schmidt process, inductively define $\alpha^2, \ldots, \alpha^n$ by letting $\gamma^m = \beta^m - \sum_{i=2}^{m-1} (\beta^m \cdot \alpha^i)\alpha^i$, and then letting $\alpha^m = \gamma^m/|\gamma^m|$ $(2 \leqslant m \leqslant n)$. Then $\{\alpha^i\}_{i=1}^n$ (where $\alpha^1 = \alpha$) is an orthonormal set in E^n and each α^i varies continuously in α (for $\alpha_1 > 0$). (An alternative way of stating this is that there is a continuous cross-section of the bundle of orthonormal $(n-1)$-frames over the open upper hemisphere of S^{n-1}). It now follows that each α^i is continuous in t, and since $\{\alpha^i(t)\}_{i=1}^n$ is positively oriented for $t = t_0$, it is a positively oriented orthonormal basis for $r \leqslant t \leqslant s$.

We can now define $A_x|[r, s]$ as follows. For $r \leqslant t \leqslant s$, $A(t)_x$ is a geometric $(n-2)$-sphere lying in an $(n-1)$-hyperplane orthogonal to $\alpha^1(t)$. The vectors $\{\alpha^i(t)\}_{i=2}^n$ uniquely determine an $(n-1)$-dimensional coordinate system containing $A(t)_x$ (with origin the center of $A(t)_x$), and we define $A_x|t \times S^{n-2}$ by first mapping (t, e_{i-1}) to the center of $A(t)_x$ plus $\alpha^i(t)$ $(2 \leqslant i \leqslant n)$ and extending linearly, and then multiplying by the radius of $A(t)_x$.

By subdividing I into intervals of the type $[r, s]$ we can define A_x on $(I \setminus A^{-1}(x)) \times S^{n-2}$. But since the image of $t \times S^{n-2}$ lies within $|A(t) - x|$ of x, we can extend A_x continuously by letting $A_x(t \times S^{n-2}) = x$ if $A(t) = x$.

Finally, note that the vectors mentioned in the statement of the lemma, when normalized to be unit length, give us $\{\alpha^i(t)\}_{i=1}^n$. ■

For A an oriented curve we can now define the usual intersection number, $i(A, A_x)$, as long as $A(\partial I) \cap A_x(I \times S^{n-2}) = \emptyset = A(I) \cap A_x(\partial I \times S^{n-2})$, where ∂I is the set of endpoints of I (see Dold [1, p. 197]). One can now check that with these definitions, Lemma 2 still holds. Note that our new definition of A_x is slightly different, for $n = 2$, than the earlier definition. According to the new definition for $n = 2$, A_x consists of two curves (corresponding to either a positive or negative rotation of 60° about x) rather than a single curve. The orientation is such that for S a line segment with interior point x we now have $i(S, S_x) = 2$ instead of 1 (see Lemma 2e).

To see that, in general, $i(S, S_x) = 2$, let S: $[0, 1] \to S$ be linear and let A: $E^1 \to E^n$ be linear so that $S(t) = A(t)$ for $0 \leqslant t \leqslant 1$. Then $A(E^1 - S)$ is disjoint from S_x, so $i(S, S_x)$ equals the linking number of A and $S_x|1 \times S^{n-2} - S_x|0 \times S^{n-2}$. By Lemma 3, the line through x and $S(t)$ (in that order) links $S_x|t \times S^{n-2}$ with linking number 1 for $t = 0$ or 1. But then A links $S_x|t \times S^{n-2}$ with linking number $(-1)^{t+1}$ for $t = 0$ or 1, and so $i(S, S_x) = 2$.

Theorem 4 can now be proven exactly as Theorem 2 was. Also, the proof of Corollary 2 may now be used to prove Corollary 5:

COROLLARY 5. *Let T be a triod in any Euclidean space. Then every point of some leg, except perhaps the juncture, is a vertex point.* ■

We close this section with:

COROLLARY 6. *Let M be any connected manifold (with or without boundary) of dimension at least two, embedded in E^n (for any n). Then all but at most two points of M are vertex points.*

Proof. Suppose there exist three non-vertex points, e_1, e_2, and e_3. Since M is connected and of dimension at least two, there is a triod T in M with endpoints e_1, e_2, and e_3. But then, by Theorem 4, some e_i is a vertex point of T, and hence of M; a contradiction. ■

3. Metric spaces and thin isoceles triangles. We now consider simple closed curve J in arbitrary metric spaces. Equivalently, let J be the unit circle in E^2, but with arbitrary metrics on it. We then have:

THEOREM 5. *Let Δ be a fixed, arbitrary triangle with sides $a \leqslant b \leqslant c$. Then for every metrized simple closed curve J, there exist distinct points x, y, and z on J forming a similar triangle (i.e., $xy : yz : zx = a : b : c$) if and only if $b = c$ (i.e., Δ is isoceles with apex ongle at most 60°).* ■

This theorem is the special case where $n = 2$ of the following theorem. The metric used in the necessity part of the proof was suggested to the author by Ralph Alexander. The sufficiency of this theorem, for all $r_i = 1$, is the n-Lattice Theorem of Arthur N. Milgram ([5, p. 31]).

THEOREM 6. *For any positive integer n, let r_i be such that $0 < r_i \leqslant 1$ for $i = 1, 2, \ldots, n$. For every metrized simple closed curve J there exist $n+1$ points $q_0, q_1, \ldots, q_n$ in cyclic order on J such that $q_{i-1}q_i = r_i(q_n, q_0)$ (for all $i = 1, 2, \ldots, n$) if and only if at least one $r_i = 1$.*

Proof (Necessity). Suppose $r_i < 1$ for all i. Let r and m be such that $r_i < r < 1$ for all i and $r^m < 1/n$ (m a positive integer). Then $\sum_{i=1}^{n} (r_i^m) < 1$. Now let (J, d) be the standard unit circle in the plane with E^2 metric d. Let d' be the new metric defined by $d'(x, y) = (d(x, y))^{1/m}$. (The triangle inequality is satisfied by d' since for a, b and c non-negative, $a + b \geqslant c$ implies that $a + ma^{(m-1)/m}b^{1/m} + \ldots + b \geqslant c$, which implies that $a^{1/m} + b^{1/m} \geqslant c^{1/m}$). Suppose $q_0, q_1, \ldots, q_n$ were the desired points on (J, d'), i.e., $d'(q_{i-1}, q_i) = r_i d'(q_n, q_0)$ for $i = 1, 2, \ldots, n$. So $d(q_{i-1}, q_i) = r_i^m d(q_n, q_0)$ for $i = 1, 2, \ldots, n$. But $d(q_0, q_1) + d(q_1, q_2) + \ldots + d(q_{n-1}, q_n) \geqslant d(q_n, q_0)$, so $\sum_{i=1}^{n} (r_i^m) \geqslant 1$, a contradiction.

(Sufficiency). We make the following notational conventions. The standard $(n-1)$-dimensional simplex is $\Delta^{n-1} \subset E^n$ with points given by coordinates $(x_1, \ldots, x_n)$, $x_i \geqslant 0$, $\sum x_i = 1$. The set of points of Δ^{n-1} with at least one coordinate zero is denoted $\partial\Delta^{n-1}$. The unit cube in E^n is denoted I^n and its boundary is denoted ∂I^n.

The origin is 0 and the interval $[-1, 1]$ is I^*. The points of E^n with all coordinates positive is E^{n+}, its closure is E^{n*}, and its boundary is ∂E^{n+}. Let J be any fixed metrized simple closed curve. Assume that at least one $r_i = 1$.

In [5, pp. 32–33] Milgram constructs a map $F: \Delta^{n-1}\times I^* \to E^{n*}\setminus 0$ such that:

(i) For $0 \leqslant t \leqslant 1$, $F(p, t) = g\circ h_t(p)$ where g and h_t are maps and:

a) $h_t(x_1, \ldots, x_n)$ is a cyclically ordered set $(q_0, \ldots, q_n)$ on C, and

b) $g(q_0, \ldots, q_n) = (q_0 q_1/q_0 q_n, q_1 q_2/q_0 q_n, \ldots, q_{n-1} q_n/q_0 q_n)$,

(ii) $F(\Delta^{n-1}\times 1)\subset E^{n*}\setminus \operatorname{Int} I^n$,

(iii) $F(\partial\Delta^{n-1}\times I^*)\subset \partial E^{n+}$,

(iv) $F(\cdot, -1)$ is the identity map on Δ^{n-1},

(v) for $p\in \Delta^{n-1}\setminus\partial\Delta^{n-1}$ and $-1\leqslant t<0$, $F(p, t)\in \operatorname{Int} I^n$.

Now define $P\in E^n$ by $P = (r_1, r_2, \ldots, r_n)$. P lies in E^{n+} since all $r_i>0$, and lies in ∂I^n since all $r_i\leqslant 1$ and some $r_i = 1$. We shall show that there exists $p\in\Delta^{n-1}$ and t, $0\leqslant t\leqslant 1$, such that $F(p, t) = P$. Hence by (i), $g(h_t(p)) = P$ and for $h_t(p) = (q_0, \ldots, q_n)$, $q_{i-1}q_i = r_i(q_0 q_n)$ and the theorem will be established.

Let L be the ray starting at P and passing through the origin. If $P\in F(\Delta^{n-1}\times 1)$ we are done, so we may assume $P\notin F(\Delta^{n-1}\times 1)$. By (ii), L does not meet $F(\Delta^{n-1}\times 1)$. From (iii) it follows that L does not meet $F(\partial\Delta^{n-1}\times I^*)$. But by (iv), L meets $F(\Delta^{n-1}\times(-1))$ transversally in the one interior point of Δ^{n-1}, $P/\sum_{i=1}^{n} r_i$. The intersection number of P and $F(\Delta^{n-1}\times I^*)$ must equal the intersection number of L and the singular chain of F restricted to the boundary of $\Delta^{n-1}\times I^*$. This is ± 1, and so $P\in F(\Delta^{n-1}\times I^*)$. But then by (v) and (iii), we are done. ∎

4. Some comments on smoothness. Many of the results of this paper would be easier to prove if we had insisted on some appropriate smoothness condition. We give an example below of a generalization of our results under a weak smoothness condition.

Let Δ be a fixed arbitrary triangle with a distinguished vertex with angle θ. Call a single closed curve, J, in E^n θ-smooth at a point x in J, if J has a tangent from each direction at x, and the angle between the tangents is greater that θ. (A ray is a tangent, if for arbitrarily acute solid cones with the ray as the part of the axis inside the cone, some beginning part of the arc lies in the cone.) Call a point x a *Δ-vertex point* (of set G) if there are distinct points y and z in G such that $xyz\sim\Delta$ and, in addition, x corresponds to the distinguished vertex of Δ.

THEOREM 7. *Every θ-smooth point of a simple closed curve in E^n is a Δ-vertex point.*

Proof. Let $r\leqslant 1$ be the ratio of the sides of Δ adjacent to the distinguished vertex. When we write $xyz\sim\Delta$ we now mean that xyz is similar to Δ with x corresponding to the distinguished vertex, or $x = y = z$. We redefine A_x to be the set formed by rotating the points of A about x through an angle of θ and then contracting them toward x by a factor of r. Equivalently, $A_x = \{y|$ there exists $z\in A$ with $xyz\sim\Delta$ and $xy\leqslant xz\}$. As in Section 2, A_x will also represent a natural continuous map from $I\times S^{n-2}$ onto A_x. Note that if $A\cap A_x - x\neq\varnothing$, then x is a Δ-vertex point. Also note that Lemma 2 holds with this definition.

Now let x be a θ-smooth point of simple closed curve J. The smoothness condition insures that there is a small subarc, A, containing x such that $i(A, A_x) = 2$. Both J and J_x have empty boundary, and so $i(J, J_x) = 0$ (see Dold [1, p. 199]). Hence J and J_x must meet at a point other than x and so x is a Δ-vertex point. ∎

Note that we could further relax the definition of smoothness as follows. Call J Δ-smooth at x if for some subarc A containing x internally, $i(A, A_x)\neq 0$. Then x is a Δ-vertex of J if and only if J is Δ-smooth at x.

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